

## GENERALIZED WEIGHTED COMPOSITION OPERATORS ON WEIGHTED HARDY SPACES

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*Abstract.* In this paper, we investigate the complex symmetric structure of generalized weighted composition operators  $D_{\psi, \varphi}^m$  on the weighted Hardy space  $H^2(\beta)$ . We obtain explicit conditions for  $D_{\psi, \varphi}^m$  to be complex symmetric with the conjugation  $J_w$ . Under the assumption that  $D_{\psi, \varphi}^m$  is  $J_w$ -symmetric, some sufficient and necessary conditions for  $D_{\psi, \varphi}^m$  to be Hermitian and normal are given.

### 1. Introduction

We denote by  $\mathbb{D}$  the open unit disc and by  $H(\mathbb{D})$  the space of all analytic functions in  $\mathbb{D}$ . Let  $\{\beta(n)\}$  be a sequence of positive number such that  $\beta(0) = 1$  and  $\liminf \beta(n)^{1/n} \geq 1$ . The weighted Hardy space  $H^2(\beta)$  consists of all  $f \in H(\mathbb{D})$  given by  $f(z) = \sum_{n=0}^{\infty} a_n z^n$ , such that

$$\|f\|^2 = \sum_{n=0}^{\infty} |a_n|^2 \beta(n)^2 < \infty.$$

Every weighted Hardy space  $H^2(\beta)$  is a Hilbert space. The weighted sequence for  $H^2(\beta)$  is written as  $\beta(n) = \|z^n\|^2$ . The set  $\{e_n(z) = \frac{z^n}{\beta(n)}\}_{n \geq 0}$  forms an orthonormal basis for the space  $H^2(\beta)$ . For  $f(z) = \sum_{n=0}^{\infty} a_n z^n$  and  $g(z) = \sum_{n=0}^{\infty} c_n z^n$  in  $H^2(\beta)$ , the inner product on  $H^2(\beta)$  is given by  $\langle f, g \rangle = \sum_{n=0}^{\infty} a_n \bar{c}_n \beta(n)^2$ .  $H^2(\beta)$  is a reproducing kernel Hilbert space of analytic functions which means that the point evaluations of functions on  $H^2(\beta)$  are bounded linear functions. For any point  $\alpha$  in  $\mathbb{D}$ , define

$$K_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{\bar{\alpha}^n z^n}{\beta(n)^2}, \quad z \in \mathbb{D}.$$

Obviously,  $K_{\alpha}$  is the reproducing kernel function for  $H^2(\beta)$ , i.e.,  $\langle f, K_{\alpha} \rangle = f(\alpha)$  for any  $f$  in  $H^2(\beta)$ . For each point  $\alpha$  in  $\mathbb{D}$  and positive integer  $m$ , evaluation of the  $m^{\text{th}}$

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derivative of functions in  $H^2(\beta)$  at  $\alpha$  is a bounded linear functional and  $f^{(m)}(\alpha) = \langle f, K_\alpha^{[m]} \rangle$  (see [1]), where

$$K_\alpha^{[m]}(z) = \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} \frac{\overline{\alpha}^{n-m} z^n}{\beta(n)^2}.$$

Clearly, the Hardy space  $H^2$ , the Bergman space  $A^2$ , the Dirichlet space  $\mathcal{D}$  and the derivative Hardy space  $\mathcal{S}^2$  are the weighted Hardy spaces which are identified with the weighted sequences  $\beta(n) = 1$ ,  $\beta(n) = (n+1)^{-\frac{1}{2}}$ ,  $\beta(n) = n^{\frac{1}{2}}$  and  $\beta(n) = n$ , respectively.

Let  $m \in \mathbb{N}$ ,  $\psi \in H(\mathbb{D})$  and  $\varphi$  be an analytic self-map of  $\mathbb{D}$ . The generalized weighted composition operator  $D_{\psi, \varphi}^m$  (see [29, 30, 31]) is defined by

$$D_{\psi, \varphi}^m f(z) = \psi(z) f^{(m)}(\varphi(z)), f \in H(\mathbb{D}), z \in \mathbb{D}.$$

If  $m = 0$ , the operator  $D_{\psi, \varphi}^m$  becomes the weighted composition operator, which is always denoted by  $\psi C_\varphi$ . If  $\psi = 1$  and  $m = 0$ , the operator  $D_{\psi, \varphi}^m$  is the composition operator  $C_\varphi$ . When  $\psi = 1$  and  $m = 1$ , the operator  $D_{\psi, \varphi}^m$  is called the composition-differentiation operator and denoted by  $D_\varphi$ . When  $m = 1$ , the operator  $D_{\psi, \varphi}^m$  is called the weighted composition-differentiation operator and denoted by  $\psi D_\varphi$ . In [5], Fatehi and Hammond obtained the adjoint, norm and spectrum of  $D_\varphi$  on the Hardy space  $H^2$ . Some properties of weighted composition-differentiation operators were investigated in [6, 15, 16, 20]. See [2, 21, 27, 29, 30, 31, 32, 33] for more results on generalized weighted composition operators on analytic function spaces.

An operator  $C$  is called a conjugation on complex Hilbert space  $\mathcal{H}$  if it satisfies the following conditions:

- (i) conjugate-linear or anti-linear:  $C(ax + by) = \overline{a}C(x) + \overline{b}C(y)$ , for any  $x, y \in \mathcal{H}$  and  $a, b \in \mathbb{C}$ ;
- (ii) isometric:  $\|Cx\| = \|x\|$ , for any  $x \in \mathcal{H}$ ;
- (iii) involutive:  $C^2 = I$ , where  $I$  is an identity operator.

The operator  $J$ , defined as  $Jf(z) = \overline{f(\overline{z})}$ , is a standard conjugation. In this paper, we consider a generalized conjugation  $J_w$ , which is defined as follows:

$$J_w f(z) = \overline{f(w\overline{z})}, z \in \mathbb{D},$$

where  $f \in H^2(\beta)$  and  $w \in \mathbb{C}$  with  $|w| = 1$ .

A bounded linear operator  $T$  is said to be complex symmetric (complex symmetric with  $C$  or  $C$ -symmetric) if there is a conjugation  $C$  on a Hilbert space  $\mathcal{H}$  such that

$$T = CT^*C.$$

It follows from [9] that the operator  $T$  is complex symmetric if and only if  $T$  has a self-transpose matrix representation with respect to an orthonormal basis. Complex

symmetric operators can be regarded as a generalization of complex symmetric matrices. In [10, 11, 12, 13], Garcia, Putinar and Wogen initiated the general study of complex symmetric operators. Examples of complex symmetric operators include normal operators, binormal operators, Hermitian operators, compressed Toeplitz operators and Hankel operators. In recent decades, complex symmetric composition operators and weighted composition operators acting on some Hilbert spaces of analytic functions have been studied considerably. See [3, 4, 5, 6, 14, 8, 10, 11, 12, 13, 7, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 28] for more results on complex symmetric operators.

Garcia and Hammond in [8] gave several classes of  $J$ -symmetric composition operators and weighted composition operators on  $H^2(\beta)$ . In [22], Malhotra and Gupta characterized complex symmetric weighted composition operators on  $H^2(\beta)$ . Complex symmetric weighted composition-differentiation operators on the Hardy space  $H^2$  were investigated by Han and Wang in [15]. Complex symmetric weighted composition-differentiation operators  $\psi D_\phi$  on the weighted Bergman space  $A^2_\alpha$  and the derivative Hardy space were characterized in [20]. In [16], Han and Wang studied complex symmetric generalized weighted composition operators on the Bergman space  $A^2$ .

In this paper, we investigate the symbols  $\psi$  and  $\phi$  give rise to  $J_w$ -symmetric generalized weighted composition operator  $D^m_{\psi,\phi}$  on  $H^2(\beta)$ . As an application, we give some necessary and sufficient conditions for  $J_w$ -symmetric operator  $D^m_{\psi,\phi}$  to be Hermitian and normal.

### 2. Main results and proofs

In this section, we state and prove our main results in this paper. For this purpose, we need the following lemma, which will be used in proving our main result.

LEMMA 1. *Let  $m \in \mathbb{N}$ ,  $\phi$  be an analytic self-map of  $\mathbb{D}$  and  $\psi \in H(\mathbb{D})$  such that  $D^m_{\psi,\phi}$  is bounded on  $H^2(\beta)$ . Then for any  $\alpha \in \mathbb{D}$ ,*

$$(D^m_{\psi,\phi})^* K_\alpha(z) = \overline{\psi(\alpha)} K_{\phi(\alpha)}^{[m]}(z), \quad z \in \mathbb{D}.$$

*Proof.* For any  $f \in H^2(\beta)$ , we have

$$\begin{aligned} \langle f, (D^m_{\psi,\phi})^* K_\alpha \rangle &= \langle D^m_{\psi,\phi} f, K_\alpha \rangle = \psi(\alpha) f^{(m)}(\phi(\alpha)) \\ &= \psi(\alpha) \langle f, K_{\phi(\alpha)}^{[m]} \rangle = \langle f, \overline{\psi(\alpha)} K_{\phi(\alpha)}^{[m]} \rangle, \end{aligned}$$

which implies the desired result.  $\square$

The following theorem gives the characterization of  $\psi$  and  $\phi$  such that the operator  $D^m_{\psi,\phi}$  is  $J_w$ -symmetric on  $H^2(\beta)$ .

THEOREM 1. *Let  $m \in \mathbb{N}$ ,  $\phi$  be an analytic self-map of  $\mathbb{D}$  and  $\psi \in H(\mathbb{D})$  be not identically zero such that  $D^m_{\psi,\phi}$  is bounded on  $H^2(\beta)$ . If  $D^m_{\psi,\phi}$  is  $J_w$ -symmetric on*

$H^2(\beta)$ , then

$$\varphi(z) = a_0 + \frac{\beta(m+1)^2 a_1 q(z)}{(m+1)\overline{w}^{n+1} \beta(m)^2 p(z)} \tag{1}$$

and

$$\psi(z) = \frac{\beta(m)^2 a_2}{(m!)^2} K_{w\overline{a_0}}^{[m]}(z), \tag{2}$$

where  $a_0 = \varphi(0)$ ,  $a_1 = \varphi'(0)$ ,  $a_2 = \psi^{(m)}(0)$ ,

$$p(z) = \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} \frac{(\overline{w}a_0 z)^{n-m}}{\beta(n)^2} \tag{3}$$

and

$$q(z) = \sum_{n=m+1}^{\infty} \frac{n!}{(n-m-1)!} \frac{\overline{w}^n a_0^{n-m-1} z^{n-m}}{\beta(n)^2}. \tag{4}$$

Conversely, let  $a_0, a_1 \in \mathbb{D}$  and  $a_2 \in \mathbb{C}$ . If  $\varphi$  and  $\psi$  are analytic maps of  $\mathbb{D}$ , defined as in equations (1) and (2), then  $D_{\psi, \varphi}^m$  is  $J_w$ -symmetric on  $H^2(\beta)$  only if  $a_0 = 0$  or  $a_1 = 0$ .

*Proof.* Assume that  $D_{\psi, \varphi}^m$  is  $J_w$ -symmetric on  $H^2(\beta)$ . Then for any  $z, \alpha \in \mathbb{D}$ ,

$$J_w(D_{\psi, \varphi}^m)^* K_{\alpha}(z) = D_{\psi, \varphi}^m J_w K_{\alpha}(z). \tag{5}$$

Lemma 1 yields that

$$\begin{aligned} J_w(D_{\psi, \varphi}^m)^* K_{\alpha}(z) &= J_w \overline{\psi(\alpha)} K_{\varphi(\alpha)}^{[m]}(z) = J_w \overline{\psi(\alpha)} \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} \frac{\overline{\varphi(\alpha)^{n-m} z^n}}{\beta(n)^2} \\ &= \psi(\alpha) \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} \frac{\varphi(\alpha)^{n-m} (\overline{w}z)^n}{\beta(n)^2} \end{aligned}$$

and

$$\begin{aligned} D_{\psi, \varphi}^m J_w K_{\alpha}(z) &= D_{\psi, \varphi}^m J_w \sum_{n=0}^{\infty} \frac{(\overline{\alpha}z)^n}{\beta(n)^2} = D_{\psi, \varphi}^m \sum_{n=0}^{\infty} \frac{(\alpha \overline{w}z)^n}{\beta(n)^2} \\ &= D_{\psi, \varphi}^m K_{w\overline{\alpha}}(z) = \psi(z) K_{w\overline{\alpha}}^{(m)}(\varphi(z)) = \psi(z) \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} \frac{\varphi(z)^{n-m} (\overline{w}\alpha)^n}{\beta(n)^2} \end{aligned}$$

for any  $z, \alpha \in \mathbb{D}$ . Hence, equation (5) is equivalent to

$$\psi(\alpha) \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} \frac{\varphi(\alpha)^{n-m} (\overline{w}z)^n}{\beta(n)^2} = \psi(z) \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} \frac{\varphi(z)^{n-m} (\overline{w}\alpha)^n}{\beta(n)^2} \tag{6}$$

for any  $z, \alpha \in \mathbb{D}$ . Let  $\alpha = 0$  in (6). We obtain that

$$\psi(0) \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} \frac{\varphi(0)^{n-m} (\overline{wz})^n}{\beta(n)^2} = 0$$

for any  $z \in \mathbb{D}$ , which means that  $\psi(0) = 0$ .

Let  $\psi(z) = z^k h(z)$ , where  $k$  is a positive integer and  $h$  is analytic on  $\mathbb{D}$  with  $h(0) \neq 0$ . Next we claim that  $k = m$ . If  $k > m$ , equation (6) is equivalent to

$$\alpha^{k-m} h(\alpha) \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} \frac{\varphi(\alpha)^{n-m} \overline{w}^n z^{n-m}}{\beta(n)^2} = z^{k-m} h(z) \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} \frac{\varphi(z)^{n-m} \overline{w}^n \alpha^{n-m}}{\beta(n)^2}$$

for any  $z, \alpha \in \mathbb{D}$ . Setting  $\alpha = 0$ , we have that  $h \equiv 0$ , which contradicts with  $h(0) \neq 0$ . If  $k < m$ , the equation (6) is equivalent to

$$z^{m-k} h(\alpha) \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} \frac{\varphi(\alpha)^{n-m} \overline{w}^n z^{n-m}}{\beta(n)^2} = \alpha^{m-k} h(z) \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} \frac{\varphi(z)^{n-m} \overline{w}^n \alpha^{n-m}}{\beta(n)^2}.$$

Setting  $\alpha = 0$ , we have that  $h(0) = 0$ , which contradicts with  $h(0) \neq 0$ . Thus  $k = m$  and the equation (6) becomes

$$h(\alpha) \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} \frac{\varphi(\alpha)^{n-m} \overline{w}^n z^{n-m}}{\beta(n)^2} = h(z) \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} \frac{\varphi(z)^{n-m} \overline{w}^n \alpha^{n-m}}{\beta(n)^2} \tag{7}$$

for any  $z, \alpha \in \mathbb{D}$ . Let  $\alpha = 0$  in (7). We get

$$h(0) \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} \frac{\varphi(0)^{n-m} \overline{w}^n z^{n-m}}{\beta(n)^2} = h(z) \frac{m! \overline{w}^m}{\beta(m)^2},$$

that is

$$h(z) = \frac{h(0) \beta(m)^2}{m!} \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} \frac{\varphi(0)^{n-m} \overline{w}^{n-m} z^{n-m}}{\beta(n)^2}.$$

Therefore,

$$\begin{aligned} \psi(z) = z^m h(z) &= \frac{h(0) \beta(m)^2}{m!} \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} \frac{\varphi(0)^{n-m} \overline{w}^{n-m} z^n}{\beta(n)^2} \\ &= \frac{\psi^{(m)}(0) \beta(m)^2}{(m!)^2} \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} \frac{\varphi(0)^{n-m} \overline{w}^{n-m} z^n}{\beta(n)^2} \\ &= \frac{\psi^{(m)}(0) \beta(m)^2}{(m!)^2} K_{w\varphi(0)}^{[m]}(z), \end{aligned} \tag{8}$$

where  $\psi^{(m)}(0) = m! h(0) \neq 0$ . Substituting  $\psi(z)$  in (6), we obtain that

$$K_{w\varphi(0)}^{[m]}(\alpha) \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} \frac{\varphi(\alpha)^{n-m} (\overline{wz})^n}{\beta(n)^2} = K_{w\varphi(0)}^{[m]}(z) \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} \frac{\varphi(z)^{n-m} (\overline{w\alpha})^n}{\beta(n)^2} \tag{9}$$

for any  $z, \alpha \in \mathbb{D}$ . Let

$$F_1(z) = \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} \frac{\varphi(\alpha)^{n-m} (\bar{w}z)^n}{\beta(n)^2}$$

and

$$F_2(z) = \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} \frac{\varphi(z)^{n-m} (\bar{w}\alpha)^n}{\beta(n)^2}.$$

It is clear that  $N^{th}$  derivative of  $K_{w\varphi(0)}^{[m]}$  is equal to 0 at  $z = 0$ , that is,  $\left(K_{w\varphi(0)}^{[m]}\right)^{(N)}(0) = 0$ , where  $N = 1, 2, \dots, m - 1$ . In addition, we have

$$\left(K_{w\varphi(0)}^{[m]}(z)\right)^{(m)} = \sum_{n=m}^{\infty} \frac{(n!)^2}{[(n-m)!]^2} \frac{\varphi(0)^{n-m} \bar{w}^{n-m} z^{n-m}}{\beta(n)^2},$$

$$\left(K_{w\varphi(0)}^{[m]}(z)\right)^{(m+1)} = \sum_{n=m+1}^{\infty} \frac{(n!)^2}{(n-m)!(n-m-1)!} \frac{\varphi(0)^{n-m} \bar{w}^{n-m} z^{n-m-1}}{\beta(n)^2},$$

$$F_1^{(m+1)} = \sum_{n=m+1}^{\infty} \frac{(n!)^2}{(n-m)!(n-m-1)!} \frac{\bar{w}^n \varphi(\alpha)^{n-m} z^{n-m-1}}{\beta(n)^2}$$

and

$$F_2'(z) = \sum_{n=m}^{\infty} \frac{n!}{(n-m-1)!} \frac{\varphi(z)^{n-m-1} (\bar{w}\alpha)^n \varphi'(z)}{\beta(n)^2}.$$

Therefore, differentiating the equation (9)  $(m + 1)$  times with respect to  $z$ , we have

$$\begin{aligned} & \sum_{i=0}^{m+1} \binom{m+1}{i} \left(K_{w\varphi(0)}^{[m]}(z)\right)^{(i)} F_2(z)^{(m+1-i)} \\ &= \sum_{n=m+1}^{\infty} \frac{(n!)^2}{(n-m)!(n-m-1)!} \frac{\varphi(0)^{n-m} \bar{w}^{n-m} z^{n-m-1}}{\beta(n)^2} \cdot \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} \frac{\varphi(z)^{n-m} (\bar{w}\alpha)^n}{\beta(n)^2} \\ & \quad + \sum_{i=0}^{m-1} \binom{m+1}{i} \left(K_{w\varphi(0)}^{[m]}(z)\right)^{(i)} F_2^{(m+1-i)}(z) \\ & \quad + (m+1) \sum_{n=m}^{\infty} \frac{(n!)^2}{[(n-m)!]^2} \frac{\varphi(0)^{n-m} \bar{w}^{n-m} z^{n-m}}{\beta(n)^2} \\ & \quad + \sum_{n=m}^{\infty} \frac{n!}{(n-m-1)!} \frac{\varphi(z)^{n-m-1} (\bar{w}\alpha)^n \varphi'(z)}{\beta(n)^2} \\ &= K_{w\varphi(0)}^{[m]}(\alpha) \sum_{n=m+1}^{\infty} \frac{(n!)^2}{(n-m)!(n-m-1)!} \frac{\bar{w}^n \varphi(\alpha)^{n-m} z^{n-m-1}}{\beta(n)^2}. \end{aligned} \tag{10}$$

Let  $z = 0$  in (10). We get

$$\begin{aligned} & \frac{[(m+1)!]^2 \bar{w} \varphi(0)}{\beta(m+1)^2} \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} \frac{\varphi(0)^{n-m} (\bar{w} \alpha)^n}{\beta(n)^2} \\ & + \frac{(m+1)(m!)^2}{\beta(m)^2} \sum_{n=m}^{\infty} \frac{n!}{(n-m-1)!} \frac{\varphi(0)^{n-m-1} (\bar{w} \alpha)^n \varphi'(0)}{\beta(n)^2} \\ & = \frac{[(m+1)!]^2 \bar{w}^{m+1} \varphi(\alpha)}{\beta(m+1)^2} K_{w\varphi(0)}^{[m]}(\alpha) \end{aligned}$$

for any  $\alpha \in \mathbb{D}$ . Thus

$$\begin{aligned} & \frac{[(m+1)!]^2 \bar{w} \varphi(0)}{\beta(m+1)^2} \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} \frac{\varphi(0)^{n-m} \bar{w}^n \alpha^{n-m}}{\beta(n)^2} \\ & + \frac{(m+1)(m!)^2}{\beta(m)^2} \sum_{n=m}^{\infty} \frac{n!}{(n-m-1)!} \frac{\varphi(0)^{n-m-1} \bar{w}^n \alpha^{n-m} \varphi'(0)}{\beta(n)^2} \tag{11} \\ & = \frac{[(m+1)!]^2 \bar{w}^{m+1} \varphi(\alpha)}{\beta(m+1)^2} \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} \frac{\varphi(0)^{n-m} \bar{w}^{n-m} \alpha^{n-m}}{\beta(n)^2} \end{aligned}$$

for any  $\alpha \in \mathbb{D}$ . Hence, (11) deduces that

$$\varphi(\alpha) = \varphi(0) + \frac{\beta(m+1)^2 \varphi'(0) q(\alpha)}{(m+1) \bar{w}^{m+1} \beta(m)^2 p(\alpha)},$$

where

$$p(\alpha) = \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} \frac{(\bar{w} a_0 \alpha)^{n-m}}{\beta(n)^2}$$

and

$$q(\alpha) = \sum_{n=m+1}^{\infty} \frac{n!}{(n-m-1)!} \frac{\bar{w}^n a_0^{n-m-1} \alpha^{n-m}}{\beta(n)^2}.$$

Conversely, let  $a_0, a_1 \in \mathbb{D}$  and  $a_2 \in \mathbb{C}$ ,

$$\varphi(z) = a_0 + \frac{\beta(m+1)^2 a_1 q(z)}{(m+1) \bar{w}^{m+1} \beta(m)^2 p(z)} \text{ and } \psi(z) = \frac{\beta(m)^2 a_2}{(m!)^2} K_{w a_0}^{[m]}(z),$$

where  $p(z)$  and  $q(z)$  are defined as (3) and (4). Then for  $J_w$ -symmetric operator  $D_{\psi, \varphi}^m$ , equation (6) must hold. This is equivalent to

$$\begin{aligned} & \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} \frac{a_0^{n-m} \bar{w}^{n-m} \alpha^n}{\beta(n)^2} \left( \sum_{n=m}^{\infty} \frac{n! \bar{w}^n z^n}{(n-m)! \beta(n)^2} \left( a_0 + \frac{\beta(m+1)^2 a_1 q(\alpha)}{(m+1) \bar{w}^{m+1} \beta(m)^2 p(\alpha)} \right)^{n-m} \right) \\ & = \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} \frac{a_0^{n-m} \bar{w}^{n-m} z^n}{\beta(n)^2} \left( \sum_{n=m}^{\infty} \frac{n! \bar{w}^n \alpha^n}{(n-m)! \beta(n)^2} \left( a_0 + \frac{\beta(m+1)^2 a_1 q(z)}{(m+1) \bar{w}^{m+1} \beta(m)^2 p(z)} \right)^{n-m} \right). \tag{12} \end{aligned}$$

For any  $\alpha, z \in \mathbb{D}$ ,  $\frac{q(z)}{p(z)}$  is analytic and  $q(0) = 0$ . Thus  $\frac{q(z)}{p(z)}$  can be written as

$$\frac{q(z)}{p(z)} = \sum_{i=1}^{\infty} c_i \bar{w}^{i+m} a_0^{i-1} z^i, \tag{13}$$

where  $c_1 = 1$  and  $c_i \in \mathbb{R}$ ,  $i = 2, 3, \dots$ . Therefore, (12) is equivalent to

$$\begin{aligned} & \sum_{n=m}^{\infty} \frac{n! \bar{w}^{n-m} a_0^{n-m} \alpha^n}{(n-m)! \beta(n)^2} \sum_{l=m}^{\infty} \frac{l! \bar{w}^l z^l}{(l-m)! \beta(l)^2} \\ & \cdot \sum_{k=0}^{l-m} \binom{l-m}{k} a_0^k \left( \sum_{i=1}^{\infty} \frac{\beta(m+1)^2 a_1}{(m+1) \bar{w}^{m+1} \beta(m)^2} c_i \bar{w}^{i+m} a_0^{i-1} \alpha^i \right)^{l-m-k} \\ = & \sum_{n=m}^{\infty} \frac{n! \bar{w}^{n-m} a_0^{n-m} z^n}{(n-m)! \beta(n)^2} \sum_{l=m}^{\infty} \frac{l! \bar{w}^l \alpha^l}{(l-m)! \beta(l)^2} \\ & \cdot \sum_{k=0}^{l-m} \binom{l-m}{k} a_0^k \left( \sum_{i=1}^{\infty} \frac{\beta(m+1)^2 a_1}{(m+1) \bar{w}^{m+1} \beta(m)^2} c_i \bar{w}^{i+m} a_0^{i-1} z^i \right)^{l-m-k} \end{aligned}$$

for any  $\alpha, z \in \mathbb{D}$ . Considering the coefficient of  $z^{m+2} \alpha^{m+1}$ , we obtain that

$$\begin{aligned} & \frac{(m+1)! \bar{w}^{m+1}}{\beta(m+1)^2} \left( \frac{m! \beta(m+1)^2}{(m+1) \beta(m)^4} c_2 \bar{w} a_0 a_1 + \frac{m!}{\beta(m)^2} c_1 \bar{w} a_0 a_1 + \frac{(m+2)! \bar{w}^2}{2 \beta(m+2)^2} a_0^3 \right) \\ = & \frac{(m+2)! \bar{w}^{m+2}}{2 \beta(m+2)^2} \left( \frac{(m+1)! \bar{w}}{\beta(m+1)^2} a_0^3 + \frac{2m! \beta(m+1)^2}{(m+1) \beta(m)^4} c_1 a_0 a_1 \right). \end{aligned} \tag{14}$$

Therefore, equation (14) holds only if  $a_0 = 0$  or  $a_1 = 0$ . Next, we consider the following two cases:

Case 1.  $a_0 = 0$ . In this case,

$$\varphi(z) = a_1 z \quad \text{and} \quad \psi(z) = \frac{a_2}{m!} z^m.$$

Then

$$J_w(D_{\psi, \varphi}^m)^* K_{\alpha}(z) = \frac{a_2}{m!} \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} \frac{a_1^{n-m} (\bar{w} \alpha z)^n}{\beta(n)^2} = D_{\psi, \varphi}^m J_w K_{\alpha}(z).$$

Case 2.  $a_1 = 0$ . In this case,

$$\varphi(z) = a_0 \quad \text{and} \quad \psi(z) = \frac{\beta(m)^2 a_2}{(m!)^2} K_{w a_0}^{[m]}(z).$$

Then

$$\begin{aligned} J_w(D_{\psi, \varphi}^m)^* K_{\alpha}(z) &= \frac{\beta(m)^2 \bar{w}^m a_2}{(m!)^2} \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} \frac{a_0^{n-m} \bar{w}^{n-m} z^n}{\beta(n)^2} \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} \frac{a_0^{n-m} \bar{w}^{n-m} \alpha^n}{\beta(n)^2} \\ &= D_{\psi, \varphi}^m J_w K_{\alpha}(z). \end{aligned}$$



The proof is complete.  $\square$

The following result obtains the condition on  $\varphi$  so that  $\bar{\varphi}$  is an automorphism on  $\mathbb{D}$  and  $D_{\psi, \varphi}^m$  is  $J_w$ -symmetric on  $H^2(\beta)$ .

**THEOREM 2.** *Let  $m \in \mathbb{N}$ ,  $\varphi$  be an automorphism on  $\mathbb{D}$  and  $\psi \in H(\mathbb{D})$  be not identically zero such that  $D_{\psi, \varphi}^m$  is  $J_w$ -symmetric on  $H^2(\beta)$ . Then one of the following statements holds:*

(i)  $\varphi(z) = -\lambda z$  with  $|\lambda| = 1$  for some  $\lambda \in \mathbb{C}$ .

(ii)

$$\varphi(z) = \frac{\bar{a}\beta(m+1)^2\beta(m+2)^2}{a\bar{w}[(m+2)\beta(m+1)^4 - (m+1)\beta(m)^2\beta(m+2)^2]} \cdot \frac{a-z}{1-\bar{a}z}$$

for some  $a \in \mathbb{D} \setminus \{0\}$ .

*Proof.* Since  $D_{\psi, \varphi}^m$  is  $J_w$ -symmetric on  $H^2(\beta)$ , Theorem 1 yields that

$$\varphi(z) = a_0 + \frac{\beta(m+1)^2 a_1 q(z)}{(m+1)\bar{w}^{m+1}\beta(m)^2 p(z)},$$

where  $a_0 = \varphi(0)$ ,  $a_1 = \varphi'(0)$ ,  $p(z)$  and  $q(z)$  are defined as Theorem 1. Since  $\varphi$  is an automorphism on  $\mathbb{D}$ , then there are  $a \in \mathbb{D}$  and  $\lambda \in \mathbb{C}$  with  $|\lambda| = 1$  such that for any  $z \in \mathbb{D}$ ,

$$a_0 + \frac{\beta(m+1)^2 a_1 q(z)}{(m+1)\bar{w}^{m+1}\beta(m)^2 p(z)} = \lambda \frac{a-z}{1-\bar{a}z},$$

which is equivalent to

$$\begin{aligned} & (m+1)a_0\beta(m)^2\bar{w}^{m+1}p(z) - (m+1)a_0\bar{a}\beta(m)^2\bar{w}^{m+1}zp(z) \\ & + a_1\beta(m+1)^2q(z) - a_1\bar{a}\beta(m+1)^2zq(z) \\ & = (m+1)\lambda a\beta(m)^2\bar{w}^{m+1}p(z) - (m+1)\lambda\beta(m)^2\bar{w}^{m+1}zp(z) \end{aligned} \tag{15}$$

for any  $z \in \mathbb{D}$ . Considering the constant in (15), we get

$$a_0 = \lambda a.$$

Similarly, considering the coefficients of  $z$  and  $z^2$ , we get

$$\begin{aligned} & \frac{(m+1)\beta(m)^2}{\beta(m+1)^2}\bar{w}^{m+2}a_0^2 - \bar{a}\bar{w}^{m+1}a_0 + \bar{w}^{m+1}a_1 \\ & = \frac{(m+1)\beta(m)^2}{\beta(m+1)^2}\bar{w}^{m+2}\lambda a a_0 - \lambda\bar{w}^{m+1} \end{aligned} \tag{16}$$

and

$$\begin{aligned} & \frac{(m+1)(m+2)!\beta(m)^2}{2\beta(m+2)^2} \bar{w}^{m+3} a_0^3 - \frac{(m+1)(m+1)!\beta(m)^2}{\beta(m+1)^2} \bar{w}^{m+2} \bar{a} a_0^2 \\ & + \frac{(m+2)!\beta(m+1)^2}{\beta(m+2)^2} \bar{w}^{m+2} a_0 a_1 - (m+1)! \bar{w}^{n+1} \bar{a} a_1 \tag{17} \\ & = \frac{(m+1)(m+2)!\beta(m)^2}{2\beta(m+2)^2} \bar{w}^{m+3} \lambda a a_0^2 - \frac{(m+1)(m+1)!\beta(m)^2}{\beta(m+1)^2} \bar{w}^{m+2} \lambda a_0 \end{aligned}$$

for any  $w \in \mathbb{D}$ .

If  $a = 0$ , then  $a_0 = \lambda a = 0$ . Therefore, (16) deduces that  $a_1 = -\lambda$ , which implies that

$$p(z) = \frac{m!}{\beta(m)^2} \quad \text{and} \quad q(z) = \frac{(m+1)! \bar{w}^{n+1} z}{\beta(m+1)^2}.$$

Hence,  $\varphi(z) = -\lambda z$  with  $|\lambda| = 1$ .

If  $a \neq 0$ ,  $a_0 = \lambda a$  and (17) give that

$$a_1 = \frac{(m+1)\beta(m)^2\beta(m+2)^2\bar{w}\lambda^2 a(|a|^2 - 1)}{\beta(m+1)^2[(m+2)\beta(m+1)^2\bar{w}\lambda a - \beta(m+2)^2\bar{a}]},$$

which with (16) yields that

$$\lambda = \frac{\bar{a}\beta(m+1)^2\beta(m+2)^2}{\bar{a}\bar{w}[(m+2)\beta(m+1)^4 - (m+1)\beta(m)^2\beta(m+2)^2]}.$$

The proof is complete.  $\square$

As an application of Theorem 1, we investigate the necessary and sufficient conditions for  $J_w$ -symmetric operator  $D_{\psi,\varphi}^m$  to be Hermitian and normal. Recall that a bounded linear operator  $T$  is Hermitian if  $T = T^*$ . An operator  $T$  on  $\mathcal{H}$  is normal if and only if  $TT^* = T^*T$ , or for any  $x \in \mathcal{H}$ ,  $\|Tx\| = \|T^*x\|$ .

**THEOREM 3.** *Let  $m \in \mathbb{N}$ ,  $\varphi$  be an analytic self-map of  $\mathbb{D}$  and  $\psi \in H(\mathbb{D})$  be not identically zero such that  $D_{\psi,\varphi}^m$  is bounded and  $J_w$ -symmetric on  $H^2(\beta)$ . Then  $D_{\psi,\varphi}^m$  is Hermitian if and only if*

$$\psi^{(m)}(0), \varphi'(0) \in \mathbb{R} \quad \text{and} \quad \overline{\varphi(0)} = \bar{w}\varphi(0).$$

*Proof.* It is clear that  $D_{\psi,\varphi}^m$  is Hermitian if and only if  $D_{\psi,\varphi}^m K_\alpha(z) = (D_{\psi,\varphi}^m)^* K_\alpha(z)$  for any  $z, \alpha \in \mathbb{D}$ . Since  $D_{\psi,\varphi}^m$  is  $J_w$ -symmetric, then for any  $z, \alpha \in \mathbb{D}$ ,

$$J_w(D_{\psi,\varphi}^m)^* K_\alpha(z) = D_{\psi,\varphi}^m J_w K_\alpha(z).$$

Therefore,  $D_{\psi,\varphi}^m$  is Hermitian if and only if for any  $z, \alpha \in \mathbb{D}$ ,

$$J_w(D_{\psi,\varphi}^m)^* K_\alpha(z) = J_w D_{\psi,\varphi}^m K_\alpha(z) = D_{\psi,\varphi}^m J_w K_\alpha(z). \tag{18}$$

Since

$$\begin{aligned}
 J_w D_{\psi, \varphi}^m K_\alpha(z) &= J_w \psi(z) K_\alpha^{(m)}(\varphi(z)) \\
 &= J_w \psi(z) \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} \frac{\overline{\alpha}^n \varphi(z)^{n-m}}{\beta(n)^2} \\
 &= \overline{\psi(w\bar{z})} \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} \frac{\alpha^n \overline{\varphi(w\bar{z})}^{n-m}}{\beta(n)^2}
 \end{aligned}$$

and

$$D_{\psi, \varphi}^m J_w K_\alpha(z) = \psi(z) \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} \frac{\varphi(z)^{n-m} (\overline{w\alpha})^n}{\beta(n)^2}$$

for any  $z, \alpha \in \mathbb{D}$ , then (18) is equivalent to

$$\overline{\psi(w\bar{z})} \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} \frac{\alpha^n \overline{\varphi(w\bar{z})}^{n-m}}{\beta(n)^2} = \psi(z) \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} \frac{\varphi(z)^{n-m} (\overline{w\alpha})^n}{\beta(n)^2} \tag{19}$$

for any  $z, \alpha \in \mathbb{D}$ . Considering the coefficients of  $\alpha^m$  and  $\alpha^{m+1}$  respectively, we obtain that  $\overline{\psi(w\bar{z})} = \overline{w}^m \psi(z)$  and  $\overline{\psi(w\bar{z})} \varphi(w\bar{z}) = \overline{w}^{m+1} \psi(z) \varphi(z)$ , which means that

$$\overline{\varphi(w\bar{z})} = \overline{w} \varphi(z).$$

Therefore,

$$\begin{aligned}
 \overline{p(w\bar{z})} &= \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} \frac{w^{n-m} \overline{\varphi(0)}^{n-m} (\overline{wz})^{n-m}}{\beta(n)^2} \\
 &= \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} \frac{w^{n-m} \overline{w}^{n-m} \varphi(0)^{n-m} (\overline{wz})^{n-m}}{\beta(n)^2} \\
 &= \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} \frac{(\varphi(0) \overline{wz})^{n-m}}{\beta(n)^2} = p(z)
 \end{aligned} \tag{20}$$

and

$$\begin{aligned}
 \overline{q(w\bar{z})} &= \sum_{n=m+1}^{\infty} \frac{n!}{(n-m-1)!} \frac{w^n \overline{\varphi(0)}^{n-m-1} (\overline{wz})^{n-m}}{\beta(n)^2} \\
 &= \sum_{n=m+1}^{\infty} \frac{n!}{(n-m-1)!} \frac{w^{m+n} \overline{w}^{n-m-1} \varphi(0)^{n-m-1} \overline{w}^n z^{n-m}}{\beta(n)^2} \\
 &= \sum_{n=m+1}^{\infty} \frac{n!}{(n-m-1)!} \frac{w^{m+n} \overline{w}^{n-m-1} \overline{w}^{m+1} w^{m+1} \varphi(0)^{n-m-1} \overline{w}^n z^{n-m}}{\beta(n)^2} \\
 &= w^{2m+1} q(z).
 \end{aligned} \tag{21}$$

Then

$$\begin{aligned}
 \overline{\varphi(0)} + \frac{\beta(m+1)^2 \overline{\varphi'(0)}}{(m+1)w^{m+1}\beta(m)^2} \frac{\overline{q(w\bar{z})}}{p(w\bar{z})} &= \overline{w}\varphi(0) + \frac{\beta(m+1)^2 \overline{\varphi'(0)}}{(m+1)w^{m+1}\beta(m)^2} \frac{w^{2m+1}q(z)}{p(z)} \\
 &= \overline{w}\varphi(0) + \frac{\beta(m+1)^2 \overline{\varphi'(0)} w^m \overline{w}^{m+1}}{(m+1)\overline{w}^{m+1}\beta(m)^2} \frac{q(z)}{p(z)} \\
 &= \overline{w} \left( \varphi(0) + \frac{\beta(m+1)^2 \overline{\varphi'(0)}}{(m+1)\overline{w}^{m+1}\beta(m)^2} \frac{q(z)}{p(z)} \right) \\
 &= \overline{w} \left( \varphi(0) + \frac{\beta(m+1)^2 \overline{\varphi'(0)}}{(m+1)\overline{w}^{m+1}\beta(m)^2} \frac{q(z)}{p(z)} \right),
 \end{aligned} \tag{22}$$

which implies that  $\varphi'(0) \in \mathbb{R}$ . Since  $D_{\psi, \varphi}^m$  is  $J_w$ -symmetric, Theorem 1 yields that

$$\begin{aligned}
 &\frac{\overline{\psi^{(m)}(0)}\beta(m)^2}{(m!)^2} \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} \frac{w^{n-m} \overline{\varphi(0)}^{n-m} (\overline{wz})^n}{\beta(n)^2} \\
 &= \frac{\overline{\psi^{(m)}(0)}\beta(m)^2}{(m!)^2} \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} \frac{\overline{w}^n \varphi(0)^{n-m} z^n}{\beta(n)^2} \\
 &= \frac{\psi^{(m)}(0)\beta(m)^2}{(m!)^2} \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} \frac{\overline{w}^n \varphi(0)^{n-m} z^n}{\beta(n)^2},
 \end{aligned}$$

which implies  $\overline{\psi^{(m)}(0)} = \psi^{(m)}(0)$ .

Conversely, assume that  $\psi^{(m)}(0), \varphi'(0) \in \mathbb{R}$  and  $\overline{\varphi(0)} = \overline{w}\varphi(0)$ . Obviously, it is sufficient to verify that equation (19) holds. Since  $\overline{\varphi(0)} = \overline{w}\varphi(0)$  and

$$\begin{aligned}
 \overline{\psi(w\bar{z})} &= \frac{\beta(m)^2 \overline{\psi^{(m)}(0)}}{(m!)^2} \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} \frac{w^{n-m} \overline{\varphi(0)}^{n-m} (\overline{wz})^n}{\beta(n)^2} \\
 &= \frac{\beta(m)^2 \psi^{(m)}(0)}{(m!)^2} \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} \frac{\varphi(0)^{n-m} (\overline{wz})^n}{\beta(n)^2} = \overline{w}^m \psi(z),
 \end{aligned}$$

we see that (20) and (21) hold. Thus from (22), we obtain that  $\overline{\varphi(w\bar{z})} = \overline{w}\varphi(z)$ . Therefore,

$$\begin{aligned}
 \overline{\psi(w\bar{z})} \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} \frac{\alpha^n \overline{\varphi(w\bar{z})}^{n-m}}{\beta(n)^2} &= \overline{w}^m \psi(z) \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} \frac{\alpha^n \overline{w}^{n-m} \varphi(z)^{n-m}}{\beta(n)^2} \\
 &= \psi(z) \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} \frac{\alpha^n \overline{w}^n \varphi(z)^{n-m}}{\beta(n)^2}
 \end{aligned}$$

for any  $z, \alpha \in \mathbb{D}$ . The proof is complete.  $\square$

If  $\varphi(0) = 0$ , the following result implies that every  $J_w$ -symmetric operator  $D_{\psi, \varphi}^m$  is normal.

**THEOREM 4.** *Let  $m \in \mathbb{N}$ ,  $\varphi$  be an analytic self-map of  $\mathbb{D}$  with  $\varphi(0) = 0$  and  $\psi \in H(\mathbb{D})$  be not identically zero such that  $D_{\psi, \varphi}^m$  is bounded and  $J_w$ -symmetric on  $H^2(\beta)$ . Then  $D_{\psi, \varphi}^m$  is normal.*

*Proof.* Obviously,  $\varphi(0) = 0$  gives

$$p(z) = \frac{m!}{\beta(m)^2} \quad \text{and} \quad q(z) = \frac{(m+1)! \bar{w}^{m+1} z}{\beta(m+1)^2}.$$

Since  $D_{\psi, \varphi}^m$  is  $J_w$ -symmetric, Theorem 1 yields that

$$\varphi(z) = a_1 z \quad \text{and} \quad \psi(z) = \frac{a_2}{m!} z^m,$$

where  $a_1 = \varphi'(0)$  and  $a_2 = \psi^{(m)}(0)$ . Then for  $j \in \mathbb{N}^+$ ,

$$\begin{aligned} \|D_{\psi, \varphi}^m e_j\|^2 &= \sum_{n=0}^{\infty} |\langle D_{\psi, \varphi}^m e_j, e_n \rangle|^2 = \sum_{n=0}^{\infty} |\langle \psi e_j^{(m)} \circ \varphi, \frac{z^n}{\beta(n)} \rangle|^2 \\ &= \sum_{n=0}^{\infty} \left| \left\langle \frac{j! a_2 a_1^{j-m} z^j}{m!(j-m)! \beta(j)}, \frac{z^n}{\beta(n)} \right\rangle \right|^2 \end{aligned}$$

and

$$\begin{aligned} \|(D_{\psi, \varphi}^m)^* e_j\|^2 &= \sum_{n=0}^{\infty} |\langle (D_{\psi, \varphi}^m)^* e_j, e_n \rangle|^2 = \sum_{n=0}^{\infty} |\langle e_j, D_{\psi, \varphi}^m e_n \rangle|^2 \\ &= \sum_{n=0}^{\infty} \left| \left\langle \frac{z^j}{\beta(j)}, \psi e_n^{(m)} \circ \varphi \right\rangle \right|^2 = \sum_{n=0}^{\infty} \left| \left\langle \frac{z^j}{\beta(j)}, \frac{n! a_2 a_1^{n-m} z^n}{m!(n-m)! \beta(n)} \right\rangle \right|^2. \end{aligned}$$

Therefore, for  $j \in \mathbb{N}^+$ ,

$$\|D_{\psi, \varphi}^m e_j\|^2 = \|(D_{\psi, \varphi}^m)^* e_j\|^2 = |a_2 a_1^{j-m}|^2 \left( \frac{j!}{m!(j-m)!} \right)^2.$$

Hence  $D_{\psi, \varphi}^m$  is normal. The proof is complete.  $\square$

The following result finds a necessary and sufficient condition for a  $J_w$ -symmetric operator  $D_{\psi, \varphi}^m$  to be normal.

**THEOREM 5.** *Let  $m \in \mathbb{N}$ ,  $\varphi$  be an analytic self-map of  $\mathbb{D}$  with  $\varphi'(0) = 0$  and  $\psi \in H(\mathbb{D})$  be not identically zero such that  $D_{\psi, \varphi}^m$  is bounded and  $J_w$ -symmetric on  $H^2(\beta)$ . Then  $D_{\psi, \varphi}^m$  is normal if and only if  $\overline{\varphi(0)} = \bar{w}\varphi(0)$ .*

*Proof.* Since  $D_{\psi, \varphi}^m$  is  $J_w$ -symmetric and  $\varphi'(0) = 0$ , Theorem 1 deduces that

$$\varphi(z) = \varphi(0) \quad \text{and} \quad \psi(z) = \frac{\psi^{(m)}(0)\beta(m)^2}{(m!)^2} K_{w\varphi(0)}^{[m]}(z).$$

Since for any  $f \in H^2(\beta)$ , we have

$$\langle f, (D_{\psi,\varphi}^m)^* K_\alpha^{[m]} \rangle = \langle D_{\psi,\varphi}^m f, K_\alpha^{[m]} \rangle = \psi^{(m)}(\alpha) f^{(m)}(\varphi(\alpha)) = \langle f, \overline{\psi^{(m)}(\alpha)} K_{\varphi(\alpha)}^{[m]} \rangle.$$

Then

$$(D_{\psi,\varphi}^m)^* K_\alpha^{[m]} = \overline{\psi^{(m)}(\alpha)} K_{\varphi(\alpha)}^{[m]}$$

for any  $\alpha \in \mathbb{D}$ . Hence, for any  $\alpha, z \in \mathbb{D}$ ,

$$\begin{aligned} (D_{\psi,\varphi}^m)^* D_{\psi,\varphi}^m K_\alpha(z) &= (D_{\psi,\varphi}^m)^* \psi(z) K_\alpha^{(m)}(\varphi(z)) \\ &= \frac{\psi^{(m)}(0)\beta(m)^2}{(m!)^2} \sum_{n=m}^\infty \frac{n!}{(n-m)!} \frac{\overline{\alpha}^n \varphi(0)^{n-m}}{\beta(n)^2} (D_{\psi,\varphi}^m)^* \psi(z) K_{w\varphi(0)}^{[m]}(z) \\ &= \frac{\psi^{(m)}(0)\beta(m)^2}{(m!)^2} \sum_{n=m}^\infty \frac{n!}{(n-m)!} \frac{\overline{\alpha}^n \varphi(0)^{n-m}}{\beta(n)^2} \overline{\psi^{(m)}(w\overline{\varphi(0)})} K_{\varphi(0)}^{[m]}(z) \\ &= \frac{|\psi^{(m)}(0)|^2 \beta(m)^4}{(m!)^4} \sum_{n=m}^\infty \frac{n!}{(n-m)!} \frac{\overline{\alpha}^n \varphi(0)^{n-m}}{\beta(n)^2} \\ &\quad \cdot \sum_{n=m}^\infty \frac{(n!)^2}{[(n-m)!]^2} \frac{|\varphi(0)|^{2(n-m)}}{\beta(n)^2} \sum_{n=m}^\infty \frac{n!}{(n-m)!} \frac{\overline{\varphi(0)}^{n-m} z^n}{\beta(n)^2} \end{aligned}$$

and

$$\begin{aligned} D_{\psi,\varphi}^m (D_{\psi,\varphi}^m)^* K_\alpha(z) &= D_{\psi,\varphi}^m \overline{\psi(\alpha)} K_{\varphi(\alpha)}^{[m]}(z) = \overline{\psi(\alpha)} \psi(z) \left( K_{\varphi(\alpha)}^{[m]}(z) \right)^{(m)} \circ \varphi(z) \\ &= \frac{|\psi^{(m)}(0)|^2 \beta(m)^4}{(m!)^4} \overline{K_{w\varphi(0)}^{[m]}(\alpha)} K_{w\varphi(0)}^{[m]}(z) \left( K_{\varphi(\alpha)}^{[m]}(z) \right)^{(m)} \circ \varphi(z) \\ &= \frac{|\psi^{(m)}(0)|^2 \beta(m)^4}{(m!)^4} \sum_{n=m}^\infty \frac{n!}{(n-m)!} \frac{\overline{w}^{n-m} \varphi(0)^{n-m} z^n}{\beta(n)^2} \\ &\quad \cdot \sum_{n=m}^\infty \frac{n!}{(n-m)!} \frac{w^{n-m} \overline{\varphi(0)}^{n-m} \overline{\alpha}^n}{\beta(n)^2} \sum_{n=m}^\infty \frac{(n!)^2}{[(n-m)!]^2} \frac{|\varphi(0)|^{2(n-m)}}{\beta(n)^2}. \end{aligned}$$

Therefore,  $D_{\psi,\varphi}^m$  is normal if and only if

$$\begin{aligned} &\sum_{n=m}^\infty \frac{n!}{(n-m)!} \frac{\overline{\alpha}^n \varphi(0)^{n-m}}{\beta(n)^2} \sum_{n=m}^\infty \frac{n!}{(n-m)!} \frac{\overline{\varphi(0)}^{n-m} z^n}{\beta(n)^2} \\ &= \sum_{n=m}^\infty \frac{n!}{(n-m)!} \frac{\overline{w}^{n-m} \varphi(0)^{n-m} z^n}{\beta(n)^2} \sum_{n=m}^\infty \frac{n!}{(n-m)!} \frac{w^{n-m} \overline{\varphi(0)}^{n-m} \overline{\alpha}^n}{\beta(n)^2} \end{aligned} \tag{23}$$

for any  $z, \alpha \in \mathbb{D}$ . Considering the coefficient of  $\overline{\alpha}^m z^{m+1}$  in (23), we have

$$\overline{\varphi(0)} = \overline{w} \varphi(0).$$

Conversely, assume that  $\overline{\varphi(0)} = \overline{w} \varphi(0)$ . By a simple calculation, equation (23) holds. The proof is complete.  $\square$

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#### REFERENCES

- [1] C. COWEN AND B. MACCLUER, *Composition Operators on Spaces of Analytic Functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1995.
- [2] G. DATT, M. JAIN AND N. OHRI, *On weighted generalized composition operators on weighted Hardy spaces*, *Filomat* **34** (5) (2020), 1689–1700.
- [3] T. EKLUND, M. LINDSTRÖM AND P. MLECZKO, *A note on complex symmetric composition operators on the Bergman space  $A^2(\mathbb{D})$* , *Funct. Approx. Comment. Math.* **59** (1) (2018), 129–139.
- [4] M. FATEHI, *Complex symmetric weighted composition operators*, *Complex Var. Elliptic Equ.* **64** (4) (2019), 710–720.
- [5] M. FATEHI AND C. HAMMOND, *Composition-differentiation operators on the Hardy space*, *Proc. Amer. Math. Soc.* **148** (7) (2020), 2893–2900.
- [6] M. FATEHI AND C. HAMMOND, *Normality and self-adjointness of weighted composition-differentiation operators*, *Complex Anal. Oper. Theory* **15** (1) (2021), 1–13.
- [7] Y. GAO AND Z. ZHOU, *Complex symmetric composition operators induced by linear fractional maps*, *Indiana Univ. Math. J.* **69** (2) (2020), 367–384.
- [8] S. GARCIA AND C. HAMMOND, *Which weighted composition operators are complex symmetric?*, *Concrete operators, spectral theory, operators in harmonic analysis and approximation*, 171–179, *Oper. Theory Adv. Appl.*, 236, Birkhäuser/Springer, Basel.
- [9] S. GARCIA, E. PRODAN AND M. PUTINAR, *Mathematical and physical aspects of complex symmetric operators*, *J. Phys. A* **47** (2014), 533–538.
- [10] S. GARCIA AND M. PUTINAR, *Complex symmetric operators and applications*, *Trans. Amer. Math. Soc.* **358** (3) (2006), 1285–1315.
- [11] S. GARCIA AND M. PUTINAR, *Complex symmetric operators and applications, II*, *Trans. Amer. Math. Soc.* **359** (8) (2007), 3913–3931.
- [12] S. GARCIA AND W. WOGEN, *Complex symmetric partial isometries*, *J. Funct. Anal.* **257** (4) (2009), 1251–1260.
- [13] S. GARCIA AND W. WOGEN, *Some new classes of complex symmetric operators*, *Trans. Amer. Math. Soc.* **362** (11) (2010), 6065–6077.
- [14] A. GUPTA AND A. MALHOTRA, *Complex symmetric weighted composition operators on the space  $H^2_p(\mathbb{D})$* , *Complex Var. Elliptic Equ.* **65** (9) (2020), 1488–1500.
- [15] K. HAN AND M. WANG, *Weighted composition-differentiation operators on the Hardy space*, *Banach J. Math. Anal.* **15** (3) (2021), 1–18.
- [16] K. HAN AND M. WANG, *Weighted composition-differentiation operators on the Bergman space*, *Complex Anal. Oper. Theory* **15** (5) (2021), Paper No. 89, 17 pp.
- [17] C. JIANG, S. HAN AND Z. ZHOU, *Complex symmetric weighted composition operators on the Hardy space*, *Czechoslovak Math. J.* **70** (3) (2020), 817–831.
- [18] S. JUNG, Y. KIM, E. KO AND J. LEE, *Complex symmetric weighted composition operators on  $H^2(\mathbb{D})$* , *J. Funct. Anal.* **267** (2014), 323–351.
- [19] R. LIM AND L. KHOI, *Complex symmetric weighted composition operators on  $H_\gamma(\mathbb{D})$* , *J. Math. Anal. Appl.* **464** (1) (2018), 101–118.
- [20] J. LIU, S. PONNUSAMY AND H. XIE, *Complex symmetric weighted composition-differentiation operators*, *Linear Mult. Algebra* (2022), <https://doi.org/10.1080/03081087.2022.2043816>.

- [21] A. MALHOTRA AND A. GUPTA, *Complex symmetry of generalized weighted composition operators on Fock space*, J. Math. Anal. Appl. **495** (2) (2021), Paper No. 124740, 12 pp.
- [22] A. MALHOTRA AND A. GUPTA, *Complex symmetric weighted composition operators on weighted Hardy space*, Adv. Pure Appl. Math. **13** (1) (2022), 39–49.
- [23] S. NARAYAN, D. SIEVEWRIGHT AND D. THOMPSON, *Complex symmetric composition operators on  $H^2$* , J. Math. Anal. Appl. **443** (1) (2016), 625–630.
- [24] S. NARAYAN, D. SIEVEWRIGHT AND M. TJANI, *Complex symmetric composition operators on weighted Hardy spaces*, Proc. Amer. Math. Soc. **148** (5) (2020), 2117–2127.
- [25] D. THOMPSON, T. MCCLATCHEY AND C. HOLLEMAN, *Binormal, complex symmetric operators*, Linear Mult. Algebra **69** (2021), 1705–1715.
- [26] X. YAO, *Complex symmetric composition operators on a Hilbert space of Dirichlet series*, J. Math. Anal. Appl. **452** (2017), 1413–1419.
- [27] Y. ZHANG, *Essential norm of generalized weighted composition operators from  $H^\infty$  to the logarithmic Bloch space*, J. Integral Equations Appl. **31** (1) (2019), 131–147.
- [28] H. ZHOU AND Z. ZHOU, *Normal complex symmetric weighted composition operators on the Hardy space*, J. Korean Math. Soc. **58** (4) (2021), 799–817.
- [29] X. ZHU, *Products of differentiation, composition and multiplication from Bergman type spaces to Bers type space*, Integ. Tran. Spec. Funct. **18** (2007), 223–231.
- [30] X. ZHU, *Generalized weighted composition operators on weighted Bergman spaces*, Numer. Funct. Anal. Opt. **30** (2009), 881–893.
- [31] X. ZHU, *Generalized weighted composition operators on Bloch-type spaces*, J. Ineq. Appl. **2015** (2015), 59–68.
- [32] X. ZHU, *Essential norm of generalized weighted composition operators on Bloch-type spaces*, Appl. Math. Comput. **274** (2016), 133–142.
- [33] X. ZHU, *Generalized weighted composition operators on weighted Bergman spaces, II*, Math. Inequal. Appl. **22** (3) (2019), 1055–1066.

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