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SOME RESULTS ON MATRICES WITH RESPECT TO RESISTANCE DISTANCE

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Abstract. The resistance matrix $R = R(G)$ of G is a matrix whose (i,j)-th entry is equal to the resistance distance $r_G(v_i,v_j)$. The resistance $Re(v_i)$ of a vertex v_i is defined to be the sum of the resistance from v_i to all other vertices in G, i.e., $Re(v_i) = \sum_{j=1}^{n} r_G(v_i,v_j)$. The resistance signless Laplacian matrix of a connected graph G is defined to be $\mathcal{R}^Q = diag(Re)+R$, where $diag(Re)$ is the diagonal matrix of the vertex resistances in G. In this paper, we obtain upper bounds on the minimal and maximal entries of the principal eigenvector of $R(G)$ and $\mathcal{R}^Q$, respectively, and characterize the corresponding extremal graphs. In addition, a lower bound of the resistance (resp. resistance signless Laplacian) spectral radius of graphs with n vertices and independence number α is obtained, the corresponding extremal graph is also characterized.

1. Introduction

All graphs considered in this paper are simple and connected. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. For graph-theoretical terms that are not defined here, we refer to Bollobás's book [1].

The *distance* $d_G(v_i,v_j)$ between two vertices v_i and v_j of G is defined as the length of a shortest (v_i,v_j)-path in G. Although this graph parameter has great important effect on many problems with respect to graphs, the use of shortest path has some obvious drawbacks. In many cases, shortest paths form a small subset of all paths between two vertices; it follows that paths even slightly longer than the shortest one are not considered at all in researching of some problems. Furthermore, the distance between the vertices does not consider the actual number of (shortest) paths that lie among the two vertices: two vertices that are separated by a single path have the same distance of two vertices that are separated by many paths of the same length.

To overcome these limitations, an alternative notion-*resistance distance*-of distance between two vertices v_i and v_j of G has been proposed [3, 4], denoted by $r_G(v_i,v_j)$, which is defined as the effective resistance between the two vertices, with unit resistors taken over any edge of G. It has the following characteristics: (1) The existence of multiple paths between two vertices reduces the distance; (2) Two vertices separated by a set of edge independent paths–paths that taken in pairs do not share

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edges–are closer than two vertices divided by redundant paths–paths that taken in pairs share some edges; (3) Two vertices separated by a shorter path–a path with less edges–are closer than two vertices set apart by a longer path. As E. Bozzo and M. Franceschet said in [5] that resistance distance is an interesting, but underestimated, notion of distance on graphs. Indeed in many applications, such as in electrical network and in a social network, paths longer than the shortest ones are also relevant, hence the research on resistance distance has both theoretical and practical importance. For an acyclic graph G, $d_G(v_i,v_j)=r_G(v_i,v_j)$ for any $v_i,v_j\in V(G)$ and therefore the resistance distances are mainly of interest in the case of cycle-containing graphs.

The *resistance matrix* $R=R(G)$ of G is a matrix whose (i,j)-th entry is equal to the resistance distance $r_G(v_i,v_j)$. The resistance $Re(v_i)$ of a vertex v_i is defined to be the sum of the resistance from v_i to all other vertices in G, i.e., $Re(v_i)=\sum_{j=1}^{n} r_G(v_i,v_j)$. Let $T=\max\{Re(v_i),v_i\in V(G)\}$ and $t=\min\{Re(v_i),v_i\in V(G)\}$. If $Re(v_1)=Re(v_2)=\cdots=Re(v_n)$, then G is called resistance-regular. In [7], the authors asked that which graphs are resistance-regular? Obviously, complete graph K_n and k-regular bipartite graph are resistance-regular. Similar to the signless Laplacian matrix, we can define the resistance signless Laplacian matrix of a connected graph G as $\mathcal{R}^Q=\mathcal{R}^Q(G)=diag(Re)+R$, where $diag(Re)$ is the diagonal matrix of the vertex resistances in G. For a matrix M, $det(M)$ denotes the determinant of M.

Let $\vartheta_1\geqslant\vartheta_2\geqslant\cdots\geqslant\vartheta_n$ (resp. $\vartheta_1^Q\geqslant\vartheta_2^Q\geqslant\cdots\geqslant\vartheta_n^Q$) be the spectrum of R (resp. $\mathcal{R}^Q$). We call it the resistance (resp. resistance signless Laplacian) spectrum of the graph G. The spectral radius $\vartheta=\vartheta(G)$ of R is defined as $\max\{|\vartheta_1|,|\vartheta_2|,\ldots,|\vartheta_n|\}$. From [9], we know that $\vartheta_1>0>\vartheta_2\geqslant\cdots\geqslant\vartheta_n$ and $\vartheta_1+\vartheta_2+\cdots+\vartheta_n=0$, then $\vartheta=\vartheta_1$. Similarly, the spectral radius $\vartheta^Q=\vartheta^Q(G)$ of $\mathcal{R}^Q$ is defined as $\max\{|\vartheta_1^Q|,|\vartheta_2^Q|,\ldots,|\vartheta_n^Q|\}$. Note that $\mathcal{R}^Q$ is real nonnegative irreducible matrix, by the Perron-Frobenius theorem [8], ϑ^Q is a simple eigenvalue of $\mathcal{R}^Q$, then $\vartheta^Q=\vartheta_1^Q$.

By the Perron-Frobenius theorem [8], fixed p $(1\leqslant p<\infty)$, each of R and $\mathcal{R}^Q$ has a unique eigenvector $X=(x_1,x_2,\ldots,x_n)^T$, positive and unitary ($\|X\|_p=1$) associated with its spectral radius. This eigenvector is called the p-normalized principal eigenvector of the matrix. Denote $x^{\min_p}$ (resp. $x^{\max_p}$) be the minimal (resp. maximal) entry of the p-normalized principal eigenvector of x. The study on principal eigenvector may be referred to [2] and references therein.

Let $\mathcal{G}(n,\alpha)$ denote the set of all connected graphs with n vertices and independence number α, where $1\leqslant\alpha\leqslant n-1$. Obviously, if $\alpha=1$, there is only one graph K_n in $\mathcal{G}(n,\alpha)$. In this paper, we will study the properties of the spectral radius and the corresponding p-normalized principal eigenvector of R and $\mathcal{R}^Q$, respectively, in $\mathcal{G}(n,\alpha)$.

Given an $n\times n$ matrix M, denote the submatrix of M the $M(i_1,\ldots,i_k)$, yield from the deletion of the i_1-th, $\ldots$, i_k-th rows and columns. The following results are useful for our main results.

LEMMA 1.1. [12] *Let G be a connected graph on n vertices, $n\geqslant 3$, and $1\leqslant i\neq j\leqslant n$. Let $L(i)$ and $L(i,j)$ be the above defined submatices of the Laplacian matrix of the graph G. Then* $r_G(v_i,v_j)=\frac{detL(i,j)}{detL(i)}$.

Similar to the proof of Lemma 2.4 in [10], we have the following result.

LEMMA 1.2. *Let G be a connected graph with η being an automorphism of G, and x the p-normalized principal eigenvector of R (resp. $\mathcal{R}^Q$). Then for $v_i, v_j \in V(G)$, $\eta(v_i) = v_j$ implies that $x_i = x_j$.*

LEMMA 1.3. *Let G be a connected graph. Then*

(i) *$t \leqslant \vartheta(G) \leqslant T$ and $2t \leqslant \vartheta^Q(G) \leqslant 2T$. Moreover, any of the equalities occurs if and only if G is resistance-regular.*
(ii) *$\vartheta^Q(G) \geqslant T$.*

Proof. (i) Note that for a nonnegative matrix A with spectral radius $\lambda(A)$ and row sums $r_1, r_2, \cdots, r_n$, it has $\min_{1 \leqslant i \leqslant n} r_i \leqslant \lambda(A) \leqslant \max_{1 \leqslant i \leqslant n} r_i$ [11]. Obviously, (i) holds.

(ii) Setting x to be the normalized column vector with a unique non-zero component 1 corresponding to a vertex of maximal vertex resistance in G, by Rayleigh's principle, we have $\vartheta^Q(G) \geqslant x^T \mathcal{R}^Q x = T$. □

Recall that the spectral radius of a nonnegative irreducible matrix increases when an entry increases [11]. Since R and $\mathcal{R}^Q$ are nonnegative irreducible matrices for any connected graphs, we have the following lemma.

LEMMA 1.4. *Let G be a connected graph with two nonadjacent vertices $u, v \in V(G)$. Then $\vartheta(G+uv) < \vartheta(G)$ and $\vartheta^Q(G+uv) < \vartheta^Q(G)$.*

LEMMA 1.5. *Let $f(x) = \frac{1}{\alpha+(n-\alpha)x^p}$ and $g(x) = \frac{1}{\alpha x^p+(n-\alpha)}$ $(\alpha < n)$. Then $f(x)$ and $g(x)$ are decreasing functions on positive real number set.*

Proof. Since $\frac{df(x)}{dx} = -\frac{p(n-\alpha)x^{p-1}}{[\alpha+(n-\alpha)x^p]^2} < 0$ and $\frac{dg(x)}{dx} = -\frac{p\alpha x^{p-1}}{[\alpha x^p+(n-\alpha)]^2} < 0$. Then we have our desirable results. □

LEMMA 1.6. [6] *For graph $G = (V, E)$, $\forall v_i, v_j \in V (i \neq j)$, $\frac{1}{2}(\frac{1}{d_i} + \frac{1}{d_j}) \leqslant r_G(v_i, v_j) \leqslant 2|E|D(\frac{1}{d_i} + \frac{1}{d_j})$, where d_s is the degree of vertex v_s, D is the diameter of G.*

2. A lower bound of spectral radius for R (resp. $\mathcal{R}^Q$)

In this section, we consider a lower bound of spectral radius for R (resp. $\mathcal{R}^Q$). Let $G_1 \vee G_2$ be the join of the graphs G_1 and G_2.

LEMMA 2.1. *Let $G = \overline{K_\alpha} \vee K_{n-\alpha}$, $V(\overline{K_\alpha}) = \{1, 2, \ldots, \alpha\}$ and $V(K_{n-\alpha}) = \{\alpha + 1, \alpha+2, \ldots, n\}$. Then*

$$r_G(i,j) = \begin{cases} \frac{2}{n-\alpha}, & \text{if } 1 \leqslant i \neq j \leqslant \alpha; \\ \frac{2}{n}, & \text{if } \alpha+1 \leqslant i \neq j \leqslant n; \\ \frac{2n-\alpha-1}{n(n-\alpha)}, & \text{if } 1 \leqslant i \leqslant \alpha \ \ \alpha+1 \leqslant j \leqslant n. \end{cases}$$

Proof. Let $J_{s\times t}$ be the $s\times t$ all-ones matrix. Then the Laplacian matrix of $\overline{K_\alpha}\vee K_{n-\alpha}$ can be written as

$$L=\begin{pmatrix}(n-\alpha)I_{\alpha\times\alpha} & -J_{\alpha\times(n-\alpha)}\\ -J_{(n-\alpha)\times\alpha} & nI_{(n-\alpha)\times(n-\alpha)}-J_{(n-\alpha)\times(n-\alpha)}\end{pmatrix}.$$

By direct calculation, we have the following results.

$$\begin{aligned}\det L(i)&=(n-\alpha)^{\alpha-1}n^{n-\alpha-1},\ \ 1\leqslant i\leqslant n,\\ \det L(i,j)&=2(n-\alpha)^{\alpha-2}n^{n-\alpha-1},\ \ 1\leqslant i\neq j\leqslant\alpha,\\ \det L(i,j)&=2(n-\alpha)^{\alpha-2}n^{n-\alpha-2},\ \ \alpha+1\leqslant i\neq j\leqslant n,\\ \det L(i,j)&=(2n-\alpha-1)(n-\alpha)^{\alpha-2}n^{n-\alpha-2},\ \ 1\leqslant i\leqslant\alpha,\ \ \alpha+1\leqslant j\leqslant n.\end{aligned}$$

Further by Lemma 1.1, we have our desirable results. □

LEMMA 2.2. *Let* $G=\overline{K_\alpha}\vee K_{n-\alpha}$ *with* $V(\overline{K_\alpha})=\{1,2,\dots,\alpha\}$ *and* $V(K_{n-\alpha})=\{\alpha+1,\alpha+2,\dots,n\}$*, and* ϑ *the spectral radius of* G*. Then*

(i) $\vartheta=\frac{1}{n(n-\alpha)}\{n^2-n\alpha-2n+\alpha^2+\alpha+[(n^2-n\alpha-2n+\alpha^2+\alpha)^2-4n(\alpha-1)(n-\alpha)(n-\alpha-1)+\alpha(n-\alpha)(2n-\alpha-1)^2]^{\frac{1}{2}}\}$.

(ii) *the eigencomponents of the* p*-normalized principal eigenvector corresponding to the eigenvalue* ϑ *are*

$$x_i=\left(\frac{[\vartheta n^2-\vartheta n\alpha-2(n-\alpha)(n-\alpha-1)]^p}{\alpha[\vartheta n^2-\vartheta n\alpha-2(n-\alpha)(n-\alpha-1)]^p+(n-\alpha)\alpha^p(2n-\alpha-1)^p}\right)^{\frac{1}{p}}=w_1,$$

for $1\leqslant i\leqslant\alpha$,

$$x_j=\left(\frac{(\vartheta n^2-\vartheta n\alpha-2n\alpha+2n)^p}{\alpha(2n-\alpha-1)^p(n-\alpha)^p+(n-\alpha)(\vartheta n^2-\vartheta n\alpha-2n\alpha+2n)^p}\right)^{\frac{1}{p}}=w_2,$$

for $\alpha+1\leqslant i\leqslant n$.

Proof. Let $x=(x_1,x_2,\dots,x_n)^T$ be p-normalized principal eigenvector corresponding to the eigenvalue ϑ. By Lemma 1.2, we have $x_i=x_1$ when $1\leqslant i\leqslant\alpha$, and $x_i=x_{\alpha+1}$ when $\alpha+1\leqslant i\leqslant n$. From $Rx=\vartheta x$, it is easy to that

$$\begin{aligned}\vartheta x_1&=\frac{2(\alpha-1)}{n-\alpha}x_1+\frac{2n-\alpha-1}{n}x_{\alpha+1},\\ \vartheta x_{\alpha+1}&=\frac{\alpha(2n-\alpha-1)}{n(n-\alpha)}x_1+\frac{2(n-\alpha-1)}{n}x_{\alpha+1}.\end{aligned}$$

Note that $\sum_{k=1}^n x_k^p=1$. By direct calculation, we have our desirable results. □

LEMMA 2.3. *Let* $G=\overline{K_\alpha}\vee K_{n-\alpha}$, $V(\overline{K_\alpha})=\{1,2,\dots,\alpha\}$ *and* $V(K_{n-\alpha})=\{\alpha+1,\alpha+2,\dots,n\}$*. Then*

(i) $\vartheta^Q = \frac{1}{2n(n-\alpha)}\Big(6n^2-9n-5n\alpha+4\alpha^2+4\alpha+[(6n^2-9n-5n\alpha+4\alpha^2+4\alpha)^2 - 4[4n(\alpha-1)+(n-\alpha)(2n-\alpha-1)][\alpha(2n-\alpha-1)+4(n-\alpha)(n-\alpha-1)] + 4\alpha(n-\alpha)(2n-\alpha-1)^2]^{\frac{1}{2}}\Big)$

(ii) *the components of the p-normalized principal eigenvector corresponding to the eigenvalue $\vartheta^Q(G)$ are*

$$x_i = \left(\frac{(\vartheta^Q n^2-\vartheta^Q n\alpha-4n^2+6n\alpha+4n-3\alpha^2-3\alpha)^p}{\alpha(\vartheta^Q n^2-\vartheta^Q n\alpha-4n^2+6n\alpha+4n-3\alpha^2-3\alpha)^p+(n-\alpha)\alpha^p(2n-\alpha-1)^p}\right)^{\frac{1}{p}} = u_1, \qquad \text{for } 1\leqslant i\leqslant \alpha,$$

$$x_j = \left(\frac{(\vartheta^Q n^2-\vartheta^Q n\alpha-2n^2-\alpha n+5n-\alpha^2-\alpha)^p}{\alpha(n-\alpha)^p(2n-\alpha-1)^p+(n-\alpha)(\vartheta^Q n^2-\vartheta^Q n\alpha-2n^2-\alpha n+5n-\alpha^2-\alpha)^p}\right)^{\frac{1}{p}} = u_2, \qquad \text{for } \alpha+1\leqslant i\leqslant n.$$

Proof. Let $x=(x_1,x_2,\ldots,x_n)^T$ be p-normalized principal eigenvector corresponding to the eigenvalue $\vartheta(G)^Q$. By Lemma 1.2, we have $x_i=x_1$ when $1\leqslant i\leqslant\alpha$, and $x_i=x_{\alpha+1}$ when $\alpha+1\leqslant i\leqslant n$. From $\mathcal{R}^Q x=\vartheta^Q(G)x$, it is easy to that

$$\vartheta^Q x_1 = \left(\frac{2(\alpha-1)}{n-\alpha}+\frac{2n-\alpha-1}{n}\right)x_1+\frac{2(\alpha-1)}{n-\alpha}x_1+\frac{2n-\alpha-1}{n}x_{\alpha+1},$$
$$\vartheta^Q x_{\alpha+1} = \left(\frac{\alpha(2n-\alpha-1)}{n(n-\alpha)}+\frac{2(n-\alpha-1)}{n}\right)x_{\alpha+1}+\frac{\alpha(2n-\alpha-1)}{n(n-\alpha)}x_1+\frac{2(n-\alpha-1)}{n}x_{\alpha+1}$$

Note that $\sum_{k=1}^n x_k^p=1$. By direct calculation, we have our desirable results. □

For a graph $G\in\mathcal{G}(n,\alpha)$, it can be obtain from $\overline{K_\alpha}\vee K_{n-\alpha}$ by removing some edges. By Lemma 1.4, we can obtain a lower bound of spectral radius for ϑ (resp. ϑ^Q) in $\mathcal{G}(n,\alpha)$.

THEOREM 2.4. *Let $G\in\mathcal{G}(n,\alpha)$, then*

(i) $\vartheta(G)\geqslant\frac{1}{n(n-\alpha)}\left\{n^2-3n+\alpha^2+\alpha+\left[(n^2-3n+\alpha^2+\alpha)^2-4n(\alpha-1)(n-\alpha)(n-\alpha-1)+(n-\alpha)(2n-\alpha-1)^2\right]^{\frac{1}{2}}\right\}$. *The equality holds if and only if $G\cong\overline{K_\alpha}\vee K_{n-\alpha}$.*

(ii) $\vartheta^Q(G)\geqslant\frac{1}{2n(n-\alpha)}\Big(6n^2-9n-5n\alpha+4\alpha^2+4\alpha+[(6n^2-9n-5n\alpha+4\alpha^2+4\alpha)^2-4[4n(\alpha-1)+(n-\alpha)(2n-\alpha-1)][\alpha(2n-\alpha-1)+4(n-\alpha)(n-\alpha-1)]+4\alpha(n-\alpha)(2n-\alpha-1)^2]^{\frac{1}{2}}\Big)$. *The equality holds if and only if $G\cong\overline{K_\alpha}\vee K_{n-\alpha}$.*

3. Upper bounds on the minimal entries of the p-normalized principal eigenvector

In this section, we consider lower bounds on the entries of the p-normalized principal eigenvector for R (resp. $\mathcal{R}^Q$). Denote $x^{\min_p}$ the minimum entry of the p-normalized principal eigenvector in the following.

3.1. Upper bounds on the minimal entries of the p-normalized principal eigenvector of R

THEOREM 3.1. *Let $G \in \mathcal{G}(n,\alpha)$. Fix $p \geqslant 1$ and let $x = (x_1, x_2, \ldots, x_n)$ be the p-normalized principal eigenvector corresponding to the spectral radius ϑ of R and w_1, w_2 be the numbers as shown in Lemma 2.2. Then $x^{\min_p} \leqslant \min\{w_1, w_2\}$. The equality holds if and only if $G \cong \overline{K_\alpha} \vee K_{n-\alpha}$.*

Proof. From $G \in \mathcal{G}(n,\alpha)$, we know that $V(G)$ can be partitioned into two disjoint subsets A and B such that $V(G) = A \cup B$ with $A = \{1,2,\ldots,\alpha\}$ and $B = \{\alpha+1, \alpha+2, \ldots, n\}$ and A is an independent set. Let $i \in A$, $j \in B$ such that $x_i \leqslant x_k$ for $k \in A$ and $x_j \leqslant x_k$ for $k \in B$. By Lemma ref 1.4, $r_G(i,k) \geqslant r_{\overline{K_\alpha} \vee K_{n-\alpha}}(i,k) = \frac{2}{n-\alpha}$ for $k \in A$, and $r_G(i,k) \geqslant r_{\overline{K_\alpha} \vee K_{n-\alpha}}(i,k) = \frac{2n-\alpha-1}{n(n-\alpha)}$ for $k \in B$, By $Rx = \vartheta x$, we have

$$\begin{aligned} \vartheta x_i &= \sum_{k=1,k\neq i}^{\alpha} r_G(i,k)x_k + \sum_{k=\alpha+1}^{n} r_G(i,k)x_k \\ &\geqslant \sum_{k=1,k\neq i}^{\alpha} \frac{2}{n-\alpha}x_i + \sum_{k=\alpha+1}^{n} \frac{2n-\alpha-1}{n(n-\alpha)}x_j \\ &= \frac{2(\alpha-1)}{n-\alpha}x_i + \frac{2n-\alpha-1}{n}x_j, \end{aligned} \tag{3.1}$$

$$\begin{aligned} \vartheta x_j &= \sum_{k=1}^{\alpha} r_G(j,k)x_k + \sum_{k=\alpha+1,k\neq j}^{n} r_G(j,k)x_k \\ &\geqslant \sum_{k=1}^{\alpha} \frac{2n-\alpha-1}{n(n-\alpha)}x_i + \sum_{k=\alpha+1,k\neq j}^{n} \frac{2}{n}x_j \\ &= \frac{\alpha(2n-\alpha-1)}{n(n-\alpha)}x_i + \frac{2(n-\alpha-1)}{n}x_j, \end{aligned} \tag{3.2}$$

thus

$$x_i \geqslant \frac{(2n-\alpha-1)(n-\alpha)}{n(\vartheta n - \vartheta\alpha - 2\alpha + 2)}x_j, \tag{3.3}$$

$$x_j \geqslant \frac{\alpha(2n-\alpha-1)}{n(\vartheta n - \vartheta\alpha) - 2(n-\alpha)(n-\alpha-1)}x_i. \tag{3.4}$$

Note that x is p-normalized, we have

$$\alpha x_i^p + (n-\alpha)x_j^p \leqslant \sum_{i=1}^{n} x_i^p = 1. \tag{3.5}$$

By (3.3)–(3.5), we have $x^{\min_p} \leqslant \min\{w_1, w_2\}$. The first part of the proof is done.

If $G \cong \overline{K_\alpha} \vee K_{n-\alpha}$, by Lemma 2.2, we have $x^{\min_p} = \min\{w_1, w_2\}$.

If $x^{\min_p} = \min\{w_1, w_2\}$, then all inequalities in (3.1) and (3.2) must be equalities. Further we have $r_G(i,k) = \frac{2}{n-\alpha}$ for $k \in A$, and $r_G(i,k) = \frac{2n-\alpha-1}{n(n-\alpha)}$ for $k \in B$. By Lemma 1.4, it must have $G \cong \overline{K_\alpha} \vee K_{n-\alpha}$. □

THEOREM 3.2. *Let $G \in \mathcal{G}(n,\alpha)$. Fix $p \geqslant 1$ and let $x = (x_1, x_2, \ldots, x_n)$ be the p-normalized principal eigenvector corresponding to the spectral radius ϑ of R. Then $x^{\min_p} \leqslant \min\{\overline{w_1}, \overline{w_2}\}$, where*

$$\overline{w_1} = \left(\frac{[\vartheta n - 2(n-\alpha-1)]^p}{\alpha[\vartheta n - 2(n-\alpha-1)]^p + (n-\alpha)[nt - 2(n-\alpha-1)]^p} \right)^{\frac{1}{p}},$$

$$\overline{w_2} = \left(\frac{(\vartheta n - \vartheta\alpha - 2\alpha + 2)^p}{\alpha(nt - t\alpha - 2\alpha + 2)^p + (n-\alpha)(\vartheta n - \vartheta\alpha - 2\alpha + 2)^p} \right)^{\frac{1}{p}}.$$

The equality holds if and only if $G \cong K_n$.

Proof. From $G \in \mathcal{G}(n,\alpha)$, we know that $V(G)$ can be partitioned into two disjoint subsets A and B such that $V(G) = A \cup B$ with $A = \{1, 2, \ldots, \alpha\}$ and $B = \{\alpha+1, \alpha+2, \ldots, n\}$ and A is an independent set. Let $i \in A, j \in B$ such that $x_i \leqslant x_k$ for $k \in A$, and $x_j \leqslant x_k$ for $k \in B$. By Lemma 1.4, $r_G(i,k) \geqslant r_{\overline{K_\alpha} \vee K_{n-\alpha}}(i,k) = \frac{2}{n-\alpha}$ for $k \in A (k \neq i)$, and $r_G(i,k) \geqslant r_{\overline{K_\alpha} \vee K_{n-\alpha}}(i,k) = \frac{2n-\alpha-1}{n(n-\alpha)}$ for $k \in B$, By $Rx = \vartheta x$, we have

$$\begin{aligned} \vartheta x_i &= \sum_{k=1, k\neq i}^{\alpha} r_G(i,k) x_k + \sum_{k=\alpha+1}^{n} r_G(i,k) x_k \\ &\geqslant \sum_{k=1, k\neq i}^{\alpha} \frac{2}{n-\alpha} x_i + \sum_{k=\alpha+1}^{n} r_G(i,k) x_j \\ &= \frac{2(\alpha-1)}{n-\alpha} x_i + \left(Re(i) - \frac{2(\alpha-1)}{n-\alpha} \right) x_j \\ &\geqslant \frac{2(\alpha-1)}{n-\alpha} x_i + \left(t - \frac{2(\alpha-1)}{n-\alpha} \right) x_j, \end{aligned} \tag{3.6}$$

$$\begin{aligned} \vartheta x_j &= \sum_{k=1}^{\alpha} r_G(j,k) x_k + \sum_{k=\alpha+1, k\neq j}^{n} r_G(j,k) x_k \\ &\geqslant \sum_{k=1}^{\alpha} r_G(j,k) x_i + \sum_{k=\alpha+1, k\neq j}^{n} \frac{2}{n} x_j \\ &= \left(Re(j) - \frac{2(n-\alpha-1)}{n} \right) x_i + \frac{2(n-\alpha-1)}{n} x_j \\ &\geqslant \left(t - \frac{2(n-\alpha-1)}{n} \right) x_i + \frac{2(n-\alpha-1)}{n} x_j. \end{aligned} \tag{3.7}$$

Then

$$x_i \geqslant \frac{nt - t\alpha - 2\alpha + 2}{\vartheta n - \vartheta\alpha - 2\alpha + 2} x_j, \tag{3.8}$$

$$x_j \geqslant \frac{nt - 2(n-\alpha-1)}{\vartheta n - 2(n-\alpha-1)} x_i. \tag{3.9}$$

By equations (3.8) and (3.9), we have $x^{\min_p} \leqslant \min\{\overline{w_1}, \overline{w_2}\}$. Then the first part of the proof is done.

If $G \cong K_n$, then $\alpha = 1$. Then $A = \{1\}$ and $B = \{2, \dots, n\}$ and $\vartheta = \frac{2(n-1)}{n}$, $x = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})^T$. By direct calculation, we have $\overline{w_1} = \overline{w_2} = (\frac{1}{n})^{\frac{1}{p}}$. That is $x^{\min_p} = \min\{\overline{w_1}, \overline{w_2}\} = \overline{w_1} = \overline{w_2}$.

If $x^{\min_p} = \min\{\overline{w_1}, \overline{w_2}\}$, each of in (3.6) and (3.7) the equality holds. Obviously, it must have $Re(i) = Re(j) = t$, and $t = \frac{2(\alpha-1)}{n-\alpha} + \frac{2n-\alpha-1}{n} = \frac{2(n-\alpha-1)}{n} + \frac{\alpha(2n-\alpha-1)}{n(n-\alpha)}$, then $\alpha = 1$. So $G \cong K_n$. □

COROLLARY 3.3. *Let $G \in \mathcal{G}(n, \alpha)$. Fix $p \geqslant 1$, let $x = (x_1, x_2, \dots, x_n)$ be the p-normalized principal eigenvector corresponding to the spectral radius ϑ of R and $w_1, w_2, \overline{w_1}, \overline{w_2}$ be the numbers in theorems 3.1 and 3.2, respectively. Then*

(i) $t \geqslant \frac{2n^2 - n\alpha - 3n + \alpha^2 + \alpha}{n^2 - n\alpha}$, $x^{\min_p} \leqslant \min\{\overline{w_1}, \overline{w_2}\}$. *The equality holds if and only if* $G \cong K_n$.

(ii) $\frac{2n^2 - 2n\alpha - 2n + \alpha^2 + \alpha}{n^2 - n\alpha} < t < \frac{2n^2 - n\alpha - 3n + \alpha^2 + \alpha}{n^2 - n\alpha}$, $x^{\min_p} \leqslant \min\{\overline{w_1}, w_2\}$.

(iii) $t \leqslant \frac{2n^2 - 2n\alpha - 2n + \alpha^2 + \alpha}{n^2 - n\alpha}$, $x^{\min_p} \leqslant \min\{w_1, w_2\}$. *The equality holds if and only if* $G \cong \overline{K_\alpha} \vee K_{n-\alpha}$.

Proof. If $t > \frac{2n^2 - 2n\alpha - 2n + \alpha^2 + \alpha}{n^2 - n\alpha}$, we have $\frac{nt - 2(n-\alpha-1)}{\vartheta n - 2(n-\alpha-1)} > \frac{\alpha(2n-\alpha-1)}{n(\vartheta n - \vartheta\alpha) - 2(n-\alpha)(n-\alpha-1)}$. Then by Lemma 1.5, we have $\overline{w_1} = f\left(\frac{nt - 2(n-\alpha-1)}{\vartheta n - 2(n-\alpha-1)}\right) < f\left(\frac{\alpha(2n-\alpha-1)}{n(\vartheta n - \vartheta\alpha) - 2(n-\alpha)(n-\alpha-1)}\right)$ $= w_1$.

If $t > \frac{2n^2 - n\alpha - 3n + \alpha^2 + \alpha}{n^2 - n\alpha}$, we have $\frac{nt - t\alpha - 2\alpha + 2}{\vartheta n - \vartheta\alpha - 2\alpha + 2} > \frac{(2n-\alpha-1)(n-\alpha)}{\vartheta n^2 - \vartheta n\alpha - 2n\alpha + 2n}$. Then also by Lemma 1.5, we have $\overline{w_2} = g\left(\frac{nt - t\alpha - 2\alpha + 2}{\vartheta n - \vartheta\alpha - 2\alpha + 2}\right) < g\left(\frac{(2n-\alpha-1)(n-\alpha)}{\vartheta n^2 - \vartheta n\alpha - 2n\alpha + 2n}\right) = w_2$.

Note that $\frac{2n^2 - n\alpha - 3n + \alpha^2 + \alpha}{n^2 - n\alpha} > \frac{2n^2 - 2n\alpha - 2n + \alpha^2 + \alpha}{n^2 - n\alpha}$. It is easy to obtain our desirable results. □

3.2. Upper bounds on the minimal entries of the p-normalized principal eigenvector of ϑ^Q

THEOREM 3.4. *Let $G \in \mathcal{G}(n, \alpha)$. Fix $p \geqslant 1$ and let $x = (x_1, x_2, \dots, x_n)$ be the p-normalized principal eigenvector corresponding to the spectral radius ϑ^Q of $\mathcal{R}^Q$ and u_1, u_2 be the numbers as shown in Lemma 2.3. Then $x^{\min_p} \leqslant \min\{u_1, u_2\}$. The equality holds if and only if $G \cong \overline{K_\alpha} \vee K_{n-\alpha}$.*

Proof. From $G \in \mathcal{G}(n,\alpha)$, we know that $V(G)$ can be partitioned into two disjoint subsets A and B such that $V(G) = A \cup B$ with $A = \{1,2,\dots,\alpha\}$ and $B = \{\alpha+1,\alpha+2,\dots,n\}$ and A is an independent set. Let $i \in A, j \in B$ such that $x_i \leqslant x_k$ for $k \in A$, and $x_j \leqslant x_k$ for $k \in B$. By Lemma 1.4, $r_G(i,k) \geqslant r_{\overline{K_\alpha} \vee K_{n-\alpha}}(i,k) = \frac{2}{n-\alpha}$ for $k \in A$, and $r_G(i,k) \geqslant r_{\overline{K_\alpha} \vee K_{n-\alpha}}(i,k) = \frac{2n-\alpha-1}{n(n-\alpha)}$ for $k \in B$. Then $Re(i) \geqslant \frac{2n^2-(\alpha+3)n+\alpha^2+\alpha}{n(n-\alpha)}$ and $Re(j) \geqslant \frac{2n^2-(\alpha+2)n+\alpha^2+\alpha}{n(n-\alpha)}$. By $\vartheta^Q x = \vartheta^Q x$, we have

$$\begin{aligned}
\vartheta^Q x_i &= Re(i)x_i + \sum_{k=1,k\neq i}^{\alpha} r_G(i,k)x_k + \sum_{h=\alpha+1}^{n} r_G(i,k)x_k \\
&\geqslant Re(i)x_i + \sum_{k=1,k\neq i}^{\alpha} \frac{2}{n-\alpha}x_i + \sum_{h=\alpha+1}^{n} \frac{2n-\alpha-1}{n(n-\alpha)}x_j \\
&\geqslant \frac{2n^2-(\alpha+3)n+\alpha^2+\alpha}{n(n-\alpha)}x_i + \frac{2(\alpha-1)}{n-\alpha}x_i + \frac{2n-\alpha-1}{n}x_j, \qquad (3.10)\\
\vartheta^Q x_j &= Re(j)x_j + \sum_{k=1}^{\alpha} r_G(j,k)x_k + \sum_{k=\alpha+1,k\neq j}^{n} r_G(j,k)x_k \\
&\geqslant Re(j)x_j + \sum_{k=1}^{\alpha} \frac{2n-\alpha-1}{n(n-\alpha)}x_i + \sum_{h=\alpha+1,k\neq j}^{n} \frac{2}{n}x_j \\
&\geqslant \frac{2n^2-(\alpha+2)n+\alpha^2+\alpha}{n(n-\alpha)}x_j + \frac{\alpha(2n-\alpha-1)}{n(n-\alpha)}x_i + \frac{2(n-\alpha-1)}{n}x_j, \qquad (3.11)
\end{aligned}$$

thus

$$x_i \geqslant \frac{(2n-\alpha-1)(n-\alpha)}{\vartheta^Q n^2 - \vartheta^Q n\alpha - 2n^2 - (\alpha-5)n - \alpha^2 - \alpha}x_j, \qquad (3.12)$$

$$x_j \geqslant \frac{\alpha(2n-\alpha-1)}{\vartheta^Q n(n-\alpha) - 4n^2 + 6n\alpha + 4n - 3\alpha^2 - 3\alpha}x_i. \qquad (3.13)$$

By (3.5) and (3.12)–(3.13), we have $x^{\min_p} \leqslant \min\{u_1,u_2\}$. This is proved the first part of the theorem.

If $G \cong \overline{K_\alpha} \vee K_{n-\alpha}$, by Lemma 2.3, we have $x^{\min_p} = \min\{u_1,u_2\}$.

If $x^{\min_p} = \min\{u_1,u_2\}$, then all inequalities in (3.10) and (3.11) must be equalities. Further we have $r_G(i,k) = \frac{2}{n-\alpha}$ for $k \in A$, and $r_G(i,k) = \frac{2n-\alpha-1}{n(n-\alpha)}$ for $k \in B$. By Lemma 1.4, it must have $G \cong \overline{K_\alpha} \vee K_{n-\alpha}$. □

THEOREM 3.5. *Let $G \in \mathcal{G}(n,\alpha)$. Fix $p \geqslant 1$ and let $x = (x_1,x_2,\dots,x_n)$ be the p-normalized principal eigenvector corresponding to the spectral radius ϑ^Q of $\mathcal{R}^Q$. Then $x^{\min_p} \leqslant \min\{\overline{u_1},\overline{u_2}\}$, where*

$$\overline{u_1} = \left(\frac{[(\vartheta^Q-t)n(n-\alpha)-2(n-\alpha-1)(n-\alpha)]^p}{\alpha[(\vartheta^Q-t)n(n-\alpha)-2(n-\alpha-1)(n-\alpha)]^p+(n-\alpha)\alpha^p(2n-\alpha-1)^p}\right)^{\frac{1}{p}},$$

$$\overline{u_2} = \left(\frac{(\vartheta^Q n^2 - \vartheta^Q n\alpha - n^2 t + tn\alpha - 2n\alpha + 2n)^p}{\alpha(2n-\alpha-1)^p (n-\alpha)^p + (n-\alpha)(\vartheta^Q n^2 - \vartheta^Q n\alpha - n^2 t + tn\alpha - 2n\alpha + 2n)^p} \right)^{\frac{1}{p}}.$$

The equality holds if and only if $G \cong K_n$.

Proof. Similar to the proof in Theorem 3.4, we have

$$\begin{aligned}
\vartheta^Q x_i &= Re(i)x_i + \sum_{k=1,k\neq i}^{\alpha} r_G(i,k)x_k + \sum_{k=\alpha+1}^{n} r_G(i,k)x_k \\
&\geqslant Re(i)x_i + \sum_{k=1,k\neq i}^{\alpha} r_G(i,k)x_i + \sum_{k=\alpha+1}^{n} r_G(i,k)x_j \\
&\geqslant Re(i)x_i + \sum_{k=1,k\neq i}^{\alpha} r_G(i,k)x_i + \frac{2n-\alpha-1}{n}x_j \\
&\geqslant Re(i)x_i + \frac{2(\alpha-1)}{n-\alpha}x_i + \frac{2n-\alpha-1}{n}x_j \\
&\geqslant tx_i + \frac{2(\alpha-1)}{n-\alpha}x_i + \frac{2n-\alpha-1}{n}x_j \\
\vartheta^Q x_j &= Re(j)x_j + \sum_{k=1}^{\alpha} r_G(j,k)x_k + \sum_{k=\alpha+1,k\neq j}^{n} r_G(j,k)x_k \\
&\geqslant Re(j)x_j + \sum_{k=1}^{\alpha} r_G(j,k)x_i + \sum_{k=\alpha+1,k\neq j}^{n} r_G(j,k)x_j \\
&\geqslant tx_j + \frac{\alpha(2n-\alpha-1)}{n(n-\alpha)}x_i + \frac{2(n-\alpha-1)}{n}x_j
\end{aligned}$$

thus

$$x_i \geqslant \frac{(2n-\alpha-1)(n-\alpha)}{\vartheta^Q n^2 - \vartheta^Q n\alpha - n^2 t + tn\alpha - 2n\alpha + 2n}x_j, \tag{3.14}$$

$$x_j \geqslant \frac{\alpha(2n-\alpha-1)}{(\vartheta^Q - t)n(n-\alpha) - 2(n-\alpha-1)(n-\alpha)}x_i. \tag{3.15}$$

By (3.5) and (3.14)–(3.15), we have $x^{\min_p} \leqslant \min\{\overline{w_1}, \overline{w_2}\}$. Then the first part of the proof is done.

If $G \cong K_n$, then $\alpha = 1$. Then $A = \{1\}$ and $B = \{2, \ldots, n\}$ and $\vartheta^Q = \frac{4(n-1)}{n}$, $x = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$. By direct calculation, we have $\overline{u_1} = \overline{u_2} = (\frac{1}{n})^{\frac{1}{p}}$. That is $x^{\min_p} = \min\{\overline{u_1}, \overline{u_2}\} = \overline{u_1} = \overline{u_2}$.

If $x^{\min_p} = \min\{\overline{u_1}, \overline{u_2}\}$, each of in (3.6) and (3.7) the equality holds. Obviously, it must have $Re(i) = Re(j) = t$, and $t = \frac{2(\alpha-1)}{n-\alpha} + \frac{2n-\alpha-1}{n} = \frac{2(n-\alpha-1)}{n} + \frac{\alpha(2n-\alpha-1)}{n(n-\alpha)}$, then $\alpha = 1$. So $G \cong K_n$. $\square$

Similar to the proof of Corollary 3.3, we have the following results.

COROLLARY 3.6. *Let $G \in \mathcal{G}(n,\alpha)$. Fix $p \geqslant 1$, let $x = (x_1, x_2, \ldots, x_n)$ be the p-normalized principal eigenvector corresponding to the spectral radius ϑ of R and $w_1, w_2, \overline{w_1}, \overline{w_2}$ be the numbers in theorems 3.1 and 3.2, respectively. Then*

(i) $t \geqslant \frac{2n^2-n\alpha-3n+\alpha^2+\alpha}{n^2-n\alpha}$, $x^{\min_p} \leqslant \min\{\overline{u_1}, \overline{u_2}\}$. *The equality holds if and only if* $G \cong K_n$.

(ii) $\frac{2n^2-2n\alpha-2n+\alpha^2+\alpha}{n^2-n\alpha} < t < \frac{2n^2-n\alpha-3n+\alpha^2+\alpha}{n^2-n\alpha}$, $x^{\min_p} \leqslant \min\{\overline{u_1}, u_2\}$.

(iii) $t \leqslant \frac{2n^2-2n\alpha-2n+\alpha^2+\alpha}{n^2-n\alpha}$, $x^{\min_p} \leqslant \min\{u_1, u_2\}$. *The equality holds if and only if* $G \cong \overline{K_\alpha} \vee K_{n-\alpha}$.

4. Upper bounds on the maximum entries of the p-normalized principal eigenvector

THEOREM 4.1. *Let G be a connected graph of order n and $x = (x_1, x_2, \ldots, x_n)$ be the $p(p \geqslant 1)$-normalized principal eigenvector corresponding to the spectral radius ϑ of $R(G)$ with $x_1 \geqslant x_2 \geqslant \ldots \geqslant x_n$. Then*

$$x_1 \leqslant \left(\frac{2\delta|E|DT^{p-1}\vartheta^p + 4n(|E|DT^{p-1})^2}{\delta^2\vartheta^{2p} + 4\delta|E|DT^{p-1}\vartheta^p - 4n^2(|E|DT^{p-1})^2 + 8n(|E|DT^{p-1})^2} \right)^{\frac{1}{p}},$$

where D, δ denote the diameter, the minimum degree, respectively.

Proof. Since $x = (x_1, x_2, \ldots, x_n)$ be the p-normalized principal eigenvector, we have $\sum_{k=1}^n x_k^p = 1$. From $R(G)x = \vartheta x$, we have $\vartheta x_i = \sum_{k=1,k\neq i}^n r_{i,k} x_k$. For $p \geqslant 1$, by weighted power mean inequality, we have $\left(\frac{\sum_{k=1,k\neq i}^n r_{i,k} x_k^p}{\sum_{k=1,k\neq i}^n r_{i,k}} \right)^{\frac{1}{p}} \geqslant \frac{\sum_{k=1,k\neq i}^n r_{i,k} x_k}{\sum_{k=1,k\neq i}^n r_{i,k}}$. Further by Lemma 1.6,

$$\begin{aligned}
\vartheta^p x_i^p &= \left(\sum_{k=1,k\neq i}^n r_{i,k} x_k \right)^p \leqslant \left(\sum_{k=1,k\neq i}^n r_{i,k} \right)^{p-1} \sum_{k=1,k\neq i}^n r_{i,k} x_k^p \\
&\leqslant Re(v_i)^{p-1} \sum_{k=1,k\neq i}^n 2|E|D \left(\frac{1}{d_i} + \frac{1}{d_k} \right) x_k^p \\
&= 2|E|DRe(v_i)^{p-1} \left[\frac{1}{d_i} \sum_{k=1,k\neq i}^n x_k^p + \sum_{k=1,k\neq i}^n \frac{1}{d_k} x_k^p \right] \\
&= 2|E|DRe(v_i)^{p-1} \left[\frac{1-x_i^p}{d_i} + \sum_{k=1,k\neq i}^n \frac{1}{d_k} x_k^p \right] \\
&\leqslant 2|E|DRe(v_i)^{p-1} \left[\frac{1-x_i^p}{\delta} + \sum_{k=1,k\neq i}^n \frac{1}{\delta} x_k^p \right]. \qquad (4.1)
\end{aligned}$$

Setting $i=1,2$ in (4.1), respectively, we have

$$\begin{aligned}
\vartheta^p x_1^p &\leqslant 2|E|DRe(v_1)^{p-1}\left[\frac{1-x_1^p}{\delta}+\sum_{k=2}^{n}\frac{1}{\delta}x_2^p\right]\\
&= 2|E|DRe(v_1)^{p-1}\left[\frac{1-x_1^p}{\delta}+\frac{n-1}{\delta}x_2^p\right]\\
&\leqslant \frac{2|E|DRe(v_1)^{p-1}}{\delta}[(1-x_1^p)+(n-1)x_2^p]\\
&\leqslant \frac{2|E|DT^{p-1}}{\delta}[(1-x_1^p)+(n-1)x_2^p]. \qquad (4.2)\\
\vartheta^p x_2^p &\leqslant 2|E|DRe(v_2)^{p-1}\left[\frac{1-x_2^p}{\delta}+\sum_{k=1,k\neq 2}^{n}\frac{1}{\delta}x_k^p\right]\\
&\leqslant 2|E|DRe(v_2)^{p-1}\left[\frac{1-x_2^p}{\delta}+\sum_{k=1,k\neq 2}^{n}\frac{1}{\delta}x_1^p\right]\\
&= \frac{2|E|DRe(v_2)^{p-1}}{\delta}[(1-x_2^p)+(n-1)x_1^p]\\
&\leqslant \frac{2|E|DT^{p-1}}{\delta}[(1-x_2^p)+(n-1)x_1^p]. \qquad (4.3)
\end{aligned}$$

From (4.2) and (4.3), we have

$$\begin{aligned}
x_2^p &\leqslant \frac{2|E|DT^{p-1}[1+(n-1)x_1^p]}{\delta\vartheta^p+2|E|DT^{p-1}},\\
x_1^p &\leqslant \frac{2\delta|E|DT^{p-1}\vartheta^p+4n(|E|DT^{p-1})^2}{\delta^2\vartheta^{2p}+4\delta|E|DT^{p-1}\vartheta^p-4n^2(|E|DT^{p-1})^2+8n(|E|DT^{p-1})^2}.
\end{aligned}$$

This completes the proof. □

THEOREM 4.2. *Let G be a connected graph of order n and $x=(x_1,x_2,\dots,x_n)$ be the p $(p\geqslant 1)$-normalized principal eigenvector corresponding to the spectral radius ϑ of $\mathcal{R}^Q(G)$ with $x_1\geqslant x_2\geqslant\dots\geqslant x_n$. Then*

$$x_1^p\leqslant\left(\frac{2\delta|E|DT^{p-1}(\vartheta^Q(G)-T)^p+4n(|E|DT^{p-1})^2}{\delta^2(\vartheta^Q(G)-T)^{2p}+4\delta|E|DT^{p-1}(\vartheta^Q(G)-T)^p-4n^2(|E|DT^{p-1})^2+8n(|E|DT^{p-1})^2}\right)^{\frac{1}{p}},$$

where D,δ denote the diameter, the minimum degree, respectively.

Proof. Since $x=(x_1,x_2,\dots,x_n)$ be the p-normalized principal eigenvector, we have $\sum_{k=1}^{n}x_k^p=1$. From $\mathcal{R}^Qx=\vartheta^Q(G)x$, we have $\vartheta^Q(G)x_i=Re(v_i)x_i+\sum_{k=1,k\neq i}^{n}r_{i,k}x_k$. Then

$$(\vartheta^Q(G)-T)x_i\leqslant(\vartheta^Q(G)-Re(v_i))x_i=\sum_{k=1,k\neq i}^{n}r_{i,k}x_k.$$

For $p \geqslant 1,$ by weighted power mean inequality, we have

$$\begin{aligned}(\vartheta^{Q}(G)-T)^{p}x_i^p &\leqslant \left(\sum_{k=1,k\neq i}^{n} r_{i,k}x_k\right)^{p} \leqslant \left(\sum_{k=1,k\neq i}^{n} r_{i,k}\right)^{p-1} \sum_{k=1,k\neq i}^{n} r_{i,k}x_k^p\\
&\leqslant Re(v_i)^{p-1}\sum_{k=1,k\neq i}^{n} 2|E|D\left(\frac{1}{d_i}+\frac{1}{d_k}\right)x_k^p\\
&= 2|E|DRe(v_i)^{p-1}\left[\frac{1}{d_i}\sum_{k=1,k\neq i}^{n} x_k^p+\sum_{k=1,k\neq i}^{n}\frac{1}{d_k}x_k^p\right]\\
&= 2|E|DRe(v_i)^{p-1}\left[\frac{1-x_i^p}{d_i}+\sum_{k=1,k\neq i}^{n}\frac{1}{d_k}x_k^p\right]\\
&\leqslant 2|E|DRe(v_i)^{p-1}\left[\frac{1-x_i^p}{\delta}+\sum_{k=1,k\neq i}^{n}\frac{1}{\delta}x_k^p\right]. \qquad (4.4)\end{aligned}$$

Setting $i=1,2$ in (4.4), respectively, we have

$$\begin{aligned}(\vartheta^{Q}(G)-T)^{p}x_1^p &\leqslant 2|E|DRe(v_1)^{p-1}\left[\frac{1-x_1^p}{\delta}+\sum_{k=2}^{n}\frac{1}{\delta}x_2^p\right]\\
&= 2|E|DRe(v_1)^{p-1}\left[\frac{1-x_1^p}{\delta}+\frac{n-1}{\delta}x_2^p\right]\\
&\leqslant \frac{2|E|DRe(v_1)^{p-1}}{\delta}[(1-x_1^p)+(n-1)x_2^p]\\
&\leqslant \frac{2|E|DT^{p-1}}{\delta}[(1-x_1^p)+(n-1)x_2^p]. \qquad (4.5)\\
(\vartheta^{Q}(G)-T)^{p}x_2^p &\leqslant 2|E|DRe(v_2)^{p-1}\left[\frac{1-x_2^p}{\delta}+\sum_{k=1,k\neq 2}^{n}\frac{1}{\delta}x_k^p\right]\\
&\leqslant 2|E|DRe(v_2)^{p-1}\left[\frac{1-x_2^p}{\delta}+\sum_{k=1,k\neq 2}^{n}\frac{1}{\delta}x_1^p\right]\\
&= \frac{2|E|DRe(v_2)^{p-1}}{\delta}[(1-x_2^p)+(n-1)x_1^p]\\
&\leqslant \frac{2|E|DT^{p-1}}{\delta}[(1-x_2^p)+(n-1)x_1^p] \qquad (4.6)\end{aligned}$$

From (4.5) and (4.6), we have

$$x_2^p \leqslant \frac{2|E|DT^{p-1}[1+(n-1)x_1^p]}{\delta(\vartheta^{Q}(G)-T)^{p}+2|E|DT^{p-1}},$$

$$x_1^p \leqslant \frac{2\delta|E|DT^{p-1}(\vartheta^{Q}(G)-T)^{p}+4n(|E|DT^{p-1})^2}{\delta^2(\vartheta^{Q}(G)-T)^{2p}+4\delta|E|DT^{p-1}(\vartheta^{Q}(G)-T)^{p}-4n^2(|E|DT^{p-1})^2+8n(|E|DT^{p-1})^2}.$$

This completes the proof. □

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