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HYPONORMALITY OF SPECIFIC UNBOUNDED PRODUCT OF DENSELY DEFINED COMPOSITION OPERATORS IN L^2 SPACES

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Abstract. Let $(X,\mathscr{A},\mu)$ be a σ-finite measure space. A transformation $\phi : X \to X$ is non-singular if $\mu\circ\phi^{-1}$ is absolutely continuous with respect with μ. For this non-singular transformation, the composition operator $C_\phi : \mathscr{D}(C_\phi) \to L^2(\mu)$ is defined by $C_\phi f = f\circ\phi$, $f \in \mathscr{D}(C_\phi)$.

For a fixed positive integer $n \geqslant 2$, basic properties of product $C_{\phi_n}\cdots C_{\phi_1}$ in $L^2(\mu)$ are conveyed in Section 3–5, including the dense definiteness, kernel, adjoint of (not necessarily bounded) $C_{\phi_n}\cdots C_{\phi_1}$. Under the assistance of these properties, when $C_{\phi_1}, C_{\phi_2}, \cdots, C_{\phi_n}$ are densely defined, hyponormality of specific (not necessarily bounded) $C_{\phi_n}\cdots C_{\phi_1}$ in $L^2(\mu)$ is characterized in Section 6.

1. Introduction

Theory of composition operators is an important branch of linear operator theory, which has a history of over six decades. Composition operators (also, weighted composition operators) on various spaces of analytic functions have been studied extensively during the past several decades. This theory of composition operators, originated by E. Nordgren in 1968 (see [11]), has been developing since 1987 (see [15]). The book [5] written by Cowen and MacCluer contains comprehensive treatments of these (weighted) composition operators.

There is still another context in which composition operators could be studied. Theory of composition operators in L^2 spaces over a σ-finite measure space, playing a significant part in ergodic theory (see, e.g. [9]), also becomes another seminal branch of linear operator theory. The bounded composition operators in L^2 spaces, initiated by Nordgren in 1978 (see [12]), are well-developed until now. And the boundedness, normality, subnormality, seminormality etc. of these bounded operators are extensively investigated (I wouldn't like to point out any references here since most of them have no closed relation with my study in this paper. Interested readers could figure them out in [2]).

Let $(X,\mathscr{A},\mu)$ be a σ-finite measure space. The Hilbert space of all square integrable complex functions is usually denoted by $L^2(X,\mathscr{A},\mu)$, sometimes abbreviated by $L^2(\mu)$. A mapping $\phi : X \to X$ is always called a transformation of X. We say

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$\phi : X \to X$ is an $\mathscr{A}$-measurable transformation if $\phi^{-1}(\mathscr{A}) \subset \mathscr{A}$, where $\phi^{-1}(\mathscr{A}) = \{\phi^{-1}(\Delta) : \Delta \in \mathscr{A}\}$. Let $\mu \circ \phi^{-1}$ denote the measure on the σ-algebra $\mathscr{A}$, which is given by $\mu \circ \phi^{-1}(\Delta) = \mu(\phi^{-1}(\Delta))$ for every $\Delta \in \mathscr{A}$. We say $\phi : X \to X$ is a non-singular transformation of X if $\mu \circ \phi^{-1}$ is absolutely continuous with respect to μ, which is denoted by $\mu \circ \phi^{-1} \ll \mu$.

For a non-singular transformation ϕ, the composition operator $C_\phi : \mathscr{D}(C_\phi) \to L^2(\mu)$ is defined by

$$C_\phi f = f \circ \phi, \quad f \in \mathscr{D}(C_\phi),$$

where $\mathscr{D}(C_\phi) = \{f \in L^2(\mu) : f \circ \phi \in L^2(\mu)\}$ stands for the domain of C_ϕ. It is noted that if the composition operator C_ϕ is well-defined, then the transformation ϕ is non-singular. This assertion is easily checked and a generalized proof for it could be found in [3] (see, Proposition 7 in this book).

For finite or σ-finite measure spaces, the Radon-Nikodym derivative is theoretically important, which is guaranteed by the well-known Radon-Nikodym Theorem (see, e.g. Theorem 4.2.4 in [4], Theorem 2.2.1 in [1] or Section 2 in this paper). The construction in the following is basically an essential tool to the study of composition operators in L^2 spaces.

Suppose that transformation ϕ is non-singular. By the Radon-Nikodym theorem, there exists an $\mathscr{A}$-measurable positive function (up to sets of measure zero) $h_\phi : X \to [0,\infty]$ satisfying

$$\mu \circ \phi^{-1}(\Delta) = \int_\Delta h_\phi d\mu, \ \ \Delta \in \mathscr{A}. \tag{1.1}$$

Therefore, by [[1], Theorem 1.6.21] and [[13], Theorem 1.29], for each $\mathscr{A}$-measurable function $f : X \to \overline{\mathbb{R}}_+$ (or $f : X \to \mathbb{C}$ satisfying $f \circ \phi \in L^1(\mu)$), we have

$$\int_X f \circ \phi d\mu = \int_X f h_\phi d\mu. \tag{1.2}$$

Obviously, $f \circ \phi \in L^1(\mu)$ if and only if $f h_\phi \in L^1(\mu)$. And the domain of composition operator C_ϕ is

$$\mathscr{D}(C_\phi) = L^2\big((1+h_\phi)d\mu\big).$$

For a fixed positive integer $n \in \mathbb{N}$ with $n \geqslant 2$, suppose that $\phi_1, \phi_2, \cdots, \phi_n$ are non-singular transformations of X. It is easily conducted that $\phi_1 \circ \phi_2 \circ \cdots \phi_n$ is non-singular if $\phi_1, \phi_2, \cdots, \phi_n$ are all non-singular. Moreover, since $h_{\phi_1}, h_{\phi_2}, \cdots, h_{\phi_n} < \infty$ a.e. $[\mu]$ can be guaranteed by the dense definiteness of $C_{\phi_1}, C_{\phi_2}, \cdots, C_{\phi_n}$ (see, Proposition 3.2 in [2]), we concentrate on the properties when $C_{\phi_1}, C_{\phi_2}, \cdots, C_{\phi_n}$ are all densely defined.

Therefore, to avoid the repetition, the assumption in the following is given, which is denoted by (AS) for abbreviation and used frequently in this paper.

- *Throughout this paper, $n \geqslant 2$ is always a fixed positive integer, $k \in \{1, 2, \cdots, n\}$ represents the positive integer depending on n and $j \in \{1, 2, \cdots, k\}$ represents the positive integer depending on k, n.*

- *The triple $(X,\mathscr{A},\mu)$ is a σ-finite measure space, $\phi_1,\phi_2,\cdots,\phi_n$ are $\mathscr{A}$-measurable non-singular transformation of X such that $C_{\phi_1},C_{\phi_2},\cdots,C_{\phi_n}$ are all densely defined. Moreover, for each $k\in\{1,2,\cdots,n\}$, the triple $(X,\Phi_k^{-1}(\mathscr{A}),\mu)$ is also a σ-finite measure space.*

1.1. Notations

- In this paper, $(X,\mathscr{A},\mu)$ is always a σ-finite measure space and $L^2(X,\mathscr{A},\mu)$ is always abbreviate by $L^2(\mu)$.

- $\mathbb{N}$, $\mathbb{Z}_+$ and $\mathbb{R}_+$ stand for the sets of positive integers, nonnegative integers and nonnegative real numbers, respectively. Set $\overline{\mathbb{R}}_+=\mathbb{R}_+\cup\{\infty\}$. Moreover, for each given $m\in\mathbb{N}$,
$$J_m=\{k\in\mathbb{N}:k\leqslant m\}.$$

- For given subsets Δ,Δ_m of X, $m\in\mathbb{N}$, "$\Delta_m\nearrow\Delta$ as $m\to\infty$" stands for $\Delta_m\subseteq\Delta_{m+1}$ for every $m\in\mathbb{N}$ and $\Delta=\bigcup_{m=1}^{\infty}\Delta_m$. Analogously, for given $f,f_m:X\to\overline{\mathbb{R}}_+$, "$f_m\nearrow f$ as $m\to\infty$" stands for $\{f_m(x)\}_m$ is monotonically increasing and converging to $f(x)$ for every $x\in X$.

- Let T be an operator on a Hilbert space $\mathscr{H}$. Denote by $\mathscr{D}(T),\mathscr{N}(T),\mathscr{R}(T)$, $\overline{T}$ and T^* the domain, the kernel, the range, the closure and the adjoint of T, respectively. Moreover, denote by $\langle\cdot,\cdot\rangle_T$ and $\|\cdot\|_T$ the graph inner product and the graph norm of T, which mean that, for $f,g\in\mathscr{D}(T)$,
$$\langle f,g\rangle_T=\langle f,g\rangle_{\mathscr{H}}+\langle Tf,Tg\rangle_{\mathscr{H}},\quad \|f\|_T^2=\langle f,f\rangle_T.$$

- Set $\Delta_1\vartriangle\Delta_2=(\Delta_1\setminus\Delta_2)\cup(\Delta_2\setminus\Delta_1)$ for any subsets Δ_1 and Δ_2 of X.

- For a given $n\in\mathbb{N}$, suppose that $\phi_1,\phi_2,\cdots,\phi_n$ are non-singular transformation of X. Denote by
$$\Phi_k\triangleq\phi_1\circ\phi_2\circ\cdots\circ\phi_k,\quad \tilde{\Phi}_k\triangleq\phi_k\circ\phi_{k+1}\circ\cdots\circ\phi_n.$$

1.2. Non-probabilistic conditional expectation

The non-probabilistic conditional expectation plays another crucial role in this paper (as well as the Radon-Nikodym derivative). The concept of conditional expectation was originated in the classical probability theory. Book [14] written by Rao is useful for interested readers to understand it. The conditional expectation was applied to the non-probabilistic setting and incorporated into theory of composition operators by Harrington and Whitley in [8]. For more information on the non-probability conditional expectation, one can refer to Appendix A in [3].

The following part is exactly about the non-probability conditional expectation (abbr. conditional expectation, in this setting) with respect to the sub-algebra $\Phi_n^{-1}(\mathscr{A})$.

Suppose that (AS) holds. For every $\Phi_n^{-1}(\mathscr{A})$-measurable function $f : X \to \overline{\mathbb{R}}_+$, the conditional expectation of f with respect to $\Phi^{-1}(\mathscr{A})$ and the measure μ, denoted by $E(f;\Phi_n^{-1}(\mathscr{A}),\mu)$ or $E_{\Phi_n}(f)$ for abbreviation, is satisfied with

$$\int_\Delta f d\mu = \int_\Delta E_{\Phi_n}(f) d\mu, \;\; \Delta \in \Phi_n^{-1}(\mathscr{A}).$$

Note that the existence of $E_{\Phi_n}(\cdot)$ (up to sets of measure zero) is guaranteed by the Radon-Nikodym theorem and Proposition 2.7.

The first two lemmas in the following have their roots in [3] (see, formula (2.8) in page 24 and formula (A.12), respectively).

LEMMA 1.1. *A function $\tilde{g} : X \to \overline{\mathbb{R}}_+$ (resp. $\tilde{g} : X \to \mathbb{R}_+$, $\tilde{g} : X \to \mathbb{C}$) is $\Phi_n^{-1}(\mathscr{A})$-measurable if and only if there exists an $\mathscr{A}$-measurable function $g : X \to \overline{\mathbb{R}}_+$ (resp. $\tilde{g} : X \to \mathbb{R}_+$, $\tilde{g} : X \to \mathbb{C}$) such that $\tilde{g} = g \circ \Phi_n$.*

LEMMA 1.2. *If $f, g : X \to \mathbb{C}$ are $\mathscr{A}$-measurable functions such that $f \in L^p(\mu)$ and $g \circ \Phi_n \in L^q(\mu)$, then*

$$\int_X g \circ \Phi_n f d\mu = \int_X g \circ \Phi_n E_{\Phi_n}(f) d\mu,$$

where $p, q \in [1, \infty]$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$.

The following lemma can be obtained by Proposition 14 in [3] and the remarks below it, which is also a consequence of Proposition 1.4, Lemma 1.5 and Lemma 1.6 in [3].

LEMMA 1.3. *Suppose that (AS) holds. For an $\mathscr{A}$-measurable function $f : X \to \overline{\mathbb{R}}_+$ (resp. $f : X \to \mathbb{C}$), there exists an $\mathscr{A}$-meausrable $\overline{\mathbb{R}}_+$-valued (resp. $\mathbb{C}$-valued) function g with $g = g \cdot \chi_{h_{\Phi_j} h_{\phi_{j+1}} \cdots h_{\phi_n} > 0}$ such that*

- $E_{\Phi_n}(f) = g \circ \Phi_n \cdot \chi_{h_{\Phi_j} h_{\phi_{j+1}} \cdots h_{\phi_n} > 0}$ *a.e.* $[\mu]$.
- $\left(E_{\Phi_n}(f) \circ \Phi_n^{-1}\right) \circ \Phi_n = E_{\Phi_n}(f)$ *a.e.* $[\mu]$.

1.3. Product of composition operators in $L^2(\mu)$

Suppose that (AS) holds and therefore $C_{\phi_1}, C_{\phi_2}, \cdots, C_{\phi_n}$ are all densely defined. Now it is natural to consider the product of composition operators $C_{\phi_1}, C_{\phi_2}, \cdots, C_{\phi_n}$, i.e. the product operator $C_{\phi_n} \cdots C_{\phi_1}$.

Note that for each $\mathscr{A}$-measurable function $f : X \to \overline{\mathbb{R}}_+$ (or $f : X \to \mathbb{C}$) satisfying $f \circ \phi \in L^1(\mu)$,

$$\int_X C_{\phi_n} \cdots C_{\phi_1} f d\mu = \int_X f \cdot h_{\phi_1 \circ \phi_2 \circ \cdots \circ \phi_n} d\mu$$
$$= \int_X f \cdot h_{\Phi_{n-1}} E_{\Phi_{n-1}} \left(h_{\phi_n} \circ \Phi_{n-1}^{-1}\right) d\mu$$

$$= \int_X f \cdot h_{\Phi_{n-2}} \cdot h_{\phi_n} \circ \Phi_{n-1}^{-1} \cdot E_{\Phi_{n-2}} \left(h_{\phi_{n-1}} \circ \Phi_{n-2}^{-1} \right) d\mu$$
$$= \cdots$$
$$= \int_X f \cdot h_{\Phi_j} \cdot \prod_{l=j+1}^{n-1} h_{\phi_{l+1}} \circ \Phi_l^{-1} \cdot E_{\Phi_j} \left(h_{\phi_{j+1}} \circ \Phi_j^{-1} \right) d\mu$$
$$\cdots$$
$$= \int_X f \cdot h_{\phi_1} \cdot (h_{\phi_3} \circ \Phi_2^{-1}) \cdot \cdots \cdot (h_{\phi_n} \circ \Phi_{n-1}^{-1}) \cdot E_{\phi_1} \left(h_{\phi_2} \circ \Phi_1^{-1} \right) d\mu.$$

And the domain of product $C_{\phi_n} \cdots C_{\phi_1}$ is

$$\mathscr{D}(C_{\phi_n} \cdots C_{\phi_1}) = L^2 \left[1 + \sum_{k=1}^{n} h_{\Phi_j} \cdot \left(\prod_{l=j+1}^{k-1} h_{\phi_{l+1}} \circ \Phi_l^{-1} \right) \cdot E_{\Phi_j} \left(h_{\phi_{j+1}} \circ \Phi_j^{-1} \right) \right] d\mu,$$

where j is an arbitrary number in J_k.

Moreover, $\mathscr{D}(C_{\phi_1 \circ \phi_2 \circ \cdots \circ \phi_n}) = L^2\left((1 + h_{\phi_1 \circ \phi_2 \circ \cdots \circ \phi_n}) d\mu\right)$, which implies that the composition operator $C_{\phi_1 \circ \phi_2 \cdots \circ \phi_n}$ is an extension (for the definition of the extension operator, see, e.g. Section 4.1 in [16]) of the product $C_{\phi_n} \cdots C_{\phi_1}$, which is always denoted by

$$C_{\phi_n} \cdots C_{\phi_1} \subseteq C_{\phi_1 \circ \phi_2 \cdots \circ \phi_n}.$$

For the product $C_{\phi_n} \cdots C_{\phi_1}$, the following statements are obtained by Proposition 3.2 and Proposition 4.1 in [2]:

- $C_{\phi_n} \cdots C_{\phi_1}$ is closable and $C_{\phi_1 \circ \phi_2 \circ \cdots \circ \phi_n}$ is closed.
- $C_{\phi_n} \cdots C_{\phi_1}$ is densely defined if and only if $C_{\phi_1 \circ \phi_2 \circ \cdots \phi_k}$ is densely defined for every $1 \leqslant k \leqslant n$.
- If $C_{\phi_{n-1}} \cdots C_{\phi_1}$ is densely defined, then $C_{\phi_1 \circ \phi_2 \circ \cdots \circ \phi_k} = \overline{C_{\phi_k} \cdots C_{\phi_1}}$ for every $k = 1, 2, \cdots, n$.
- $C_{\phi_n} \cdots C_{\phi_1} = C_{\phi_1 \circ \phi_2 \cdots \circ \phi_n}$ fails to be held in general. Example 5.4 in [2] is typically a counterexample, which showed that $C_\phi^n = C_{\phi^n}$ doesn't hold in general even if $\mathscr{D}^\infty(C_\phi)$ is dense in $L^2(\mu)$, where $\mathscr{D}(C_\phi)$ stands for the $\mathscr{C}^\infty$-vectors of C_ϕ.
- $C_{\phi_n} \cdots C_{\phi_1} = C_{\phi_1 \circ \phi_2 \cdots \circ \phi_n}$ is densely defined cannot generally implies $C_{\phi_1}, C_{\phi_2}, \cdots$, C_{ϕ_n} are all densely defined. Example 5.3 in [2] is typically a counterexample, which showed that $C_{\phi_1 \circ \phi_2}$ is densely defined but C_{ϕ_1} is not.

For more information of $C_{\phi_n} \cdots C_{\phi_1}$, interested readers can refer to Section 4 in [2].

During the past decades, much effort was put into the investigation of bounded (weighted) composition operators in L^2 spaces, including the selfadjointness, normality, quasinormality, hyponormality, cohyponormality. etc. And criteria for the subnormality and cosubnormality of bounded composition operators were invented.

However, before 2018, little was known about the properties of unbounded (weighted) composition operators in L^2 spaces since the unbounded case has much difference with the bounded case. Budzyński, Jabłoński, Jung and Stochel in their recent book [3] investigated the unbounded weighted composition operators in L^2 spaces, which is significantly a reference for investigation of the unbounded case.

Characterization in the hyponormality of weighted composition operator (not necessarily bounded) was exactly given by Campbell and Hornor in [6] under quite restrictive assumptions, which was promoted by Budzyński etc. in [3] (see, Section 5.1 in this book). However, the literature on product $C_{\phi_n}\cdots C_{\phi_1}$ in $L^2(\mu)$ is meager and the only study for $C_{\phi_n}\cdots C_{\phi_1}$ could be found in [2], which can basically distinguish between $C_{\phi_n}\cdots C_{\phi_1}$ and $C_{\phi_1\circ\phi_2\circ\cdots\circ\phi_n}$.

As far as I'm concerned, the question *"What is the difference between $C_{\phi_n}\cdots C_{\phi_1}$ and $C_{\phi_1\circ\phi_2\circ\cdots\circ\phi_n}$?"* is not completely answered. And the answer could be promoted by investigating the hyponormality of unbounded product $C_{\phi_n}\cdots C_{\phi_1}$.

In [12], Nordgren raised a good question of determining measure theoretic conditions of the transformation ϕ (see, [[12], Theorem 1]), which inspires me to investigate other questions in such way. Thus, hyponormality of unbounded product $C_{\phi_n}\cdots C_{\phi_1}$ in $L^2(\mu)$ is continued in this vein, which is characterized in this paper.

It is well-known that if $C_{\phi_n}\cdots C_{\phi_1}$ is densely defined, then so is $C_{\phi_k}\cdots C_{\phi_1}$ for each $k\in J_n$. Hence, by [[2], Proposition 4.1(iii)], we have that

$$C_{\phi_1\circ\phi_2\circ\cdots\circ\phi_k}=\overline{C_{\phi_k}\cdots C_{\phi_1}},\ \ k=1,2,\cdots,n.$$

If $C_{\phi_k}\cdots C_{\phi_1}$ is hyponormal, then by definition it is densely defined. Since a hyponormal operator is closable and its closure is hyponormal, $C_{\phi_1\circ\phi_2\circ\cdots\circ\phi_k}$ is hyponormal. Obviously, we can apply the well-known characterizations of hyponormality of composition operators, in this particular case with symbol $\phi_1\circ\phi_2\circ\cdots\circ\phi_k$, $k\in J_n$. *However, the characterizations in the present paper is different with the ones in* [3] *because the unconditional expectations are employed.*

Under the hypothesis that $C_{\phi_1},C_{\phi_2},\cdots,C_{\phi_n}$ are densely defined, this paper is arranged as follows:

A comprehensive introduction to some measure-theoretic tools is included in Section 2. Basic properties of $C_{\phi_n}\cdots C_{\phi_1}$ are seldom studies, which are essential for the study in this paper. Thus, they are conveyed in Section 3–Section 5, including the dense definiteness, kernel, adjoint of (not necessarily bounded) $C_{\phi_n}\cdots C_{\phi_1}$. This investigation also provides a useful introduction for the uninitiated readers. At last, hyponormality of specific unbounded product $C_{\phi_n}\cdots C_{\phi_1}$ in $L^2(\mu)$ is investigated in Section 6.

It is noted that this paper is investigated under strong assumption $C_{\phi_k}\cdots C_{\phi_1}$ is hyponormal for each $k\in J_{n-1}$, which is emphasized by "specific" throughout this paper.

2. Auxiliary

For the latest decades, there has been much success in characterizing in simple, direct measure-theoretic terms, exactly when composition operators may lie in several of these subclasses. Since these results are scattered through the literature and the approaches developed seem not to be widely known. Therefore, a comprehensive introduction to some measure-theoretic tools is included in this section.

The well-known Radon-Nikodym theorem is a useful characterization in measure theory.

LEMMA 2.1. [[4], Theorem 4.2.4 or [1], Theorem 2.2.1] *Let $(X,\mathscr{A})$ be a measurable space, let μ be a σ-finite positive measure on $(X,\mathscr{A})$, and let ν be a finite signed or complex measure on $(X,\mathscr{A})$. If ν is absolutely continuous with respect to μ, then there is a function g that belongs to $L^1(X,\mathscr{A},\mu,\mathbb{R})$ or to $L^1(X,\mathscr{A},\mu,\mathbb{C})$ and satisfies $\nu(\Delta)=\int_\Delta g d\mu$ for each $\Delta\in\mathscr{A}$. The function g is unique up to μ-almost everywhere equality.*

The $\mathscr{A}$-measurable function g is called a Radon-Nikodym derivative of ν with respect to μ, which is sometimes denoted by $\frac{d\nu}{d\mu}$.

The following two conclusions seem to be folklore and their proofs are included in [2] (see, Lemma 12.1 and Corollary 12.2).

CONCLUSION 2.2. Let $(X,\mathscr{A},\mu)$ be a measure space and let ρ_1,ρ_2 be $\mathscr{A}$-measurable scalar functions on X such that $0<\rho_m<\infty$ a.e. $[\mu]$ for $m=1,2$. Then $L^2(\rho_1 d\mu)\cap L^2(\rho_2 d\mu)$ is dense in $L^2(\rho_m d\mu)$ for $m=1,2$.

CONCLUSION 2.3. Let $(X,\mathscr{A},\mu_1)$ and $(X,\mathscr{A},\mu_2)$ be σ-finite measure spaces. If the measures μ_1 and μ_2 are mutually absolutely continuous, then $L^2(\mu_1)\cap L^2(\mu_2)$ is dense in $L^2(\mu_m)$ for $m=1,2$.

The following proposition is a consequence of [[10], Definition I-6-1] and [[1], Theorem 1.3.10], which was introduced in [3].

PROPOSITION 2.4. *Let $\mathscr{P}$ be a semi-algebra of subsets of a set X and ν_1,ν_2 be measures on $\sigma(\mathscr{P})$ such that $\nu_1(\Delta)=\nu_2(\Delta)$ for all $\Delta\in\mathscr{P}$. Suppose there exists a sequence $\{\Delta_m\}_{m=1}^\infty\subseteq\mathscr{P}$ such that $\Delta_m\nearrow X$ as $m\to\infty$ and $\nu_j(\Delta_m)<\infty$ for every $m\in\mathbb{N}$. Then $\nu_1=\nu_2$.*

The following three propositions were implicit in most of the definitions and used explicitly in many of the processes.

PROPOSITION 2.5. [[3], Lemma 2] *If $(X,\mathscr{A},\nu)$ is a σ-finite measure space and f,g are $\mathscr{A}$-measurable complex functions on X such that $\int_\Delta|f|d\nu<\infty$, $\int_\Delta|g|d\nu<\infty$ and $\int_\Delta|f|d\nu=\int_\Delta|g|d\nu$ for every $\Delta\in\mathscr{A}$ such that $\nu(\Delta)<\infty$, then $f=g$ a.e. $[\nu]$.*

PROPOSITION 2.6. [[3], Lemma 4] *Let A and B be positive self-adjoint operators in a Hilbert space $\mathscr{H}$ satisfying $\mathscr{D}(A)=\mathscr{D}(B)$ and $\|Af\|=\|Bf\|$ for every $f\in\mathscr{D}(A)$. Then $A=B$.*

PROPOSITION 2.7. [[13], Theorem 1.29] *Suppose that $f: X\to\overline{\mathbb{R}}_+$ is $\mathscr{A}$-measurable and $g(E)=\int_E f d\mu$, then g is a measure on the σ-algebra $\mathscr{A}$ and*

$$\int_X h dg = \int_X h f d\mu$$

for every $\mathscr{A}$-measurable function g on X with range in $\overline{\mathbb{R}}_+$.

3. Basic properties of $C_{\phi_n}\cdots C_{\phi_1}$

PROPOSITION 3.1. *Suppose that (AS) holds. Then the following assertions are valid:*

(i) $\mathscr{D}(C_{\phi_1}\cdots C_{\phi_n}) = L^2\left(\left(1+\sum_{k=1}^{n} h_{\Phi_j}\cdot\left(\prod_{l=j+1}^{k-1} h_{\phi_{l+1}}\circ\Phi_l^{-1}\right)\cdot E_{\Phi_j}\left(h_{\phi_{j+1}}\circ\Phi_j^{-1}\right)\right)d\mu\right)$ *and* $\overline{\mathscr{D}(C_{\phi_n}\cdots C_{\phi_1})} = \chi_{\{\sum_{k=1}^{n} h_{\Phi_j}\cdot\left(\prod_{l=j+1}^{k-1} h_{\phi_{l+1}}\circ\Phi_l^{-1}\right)\cdot E_{\Phi_j}\left(h_{\phi_{j+1}}\circ\Phi_j^{-1}\right)<\infty\}}\cdot L^2(\mu)$, *where j is an arbitrary number in J_k.*

(ii) For every $f\in\mathscr{D}(C_{\phi_n}\cdots C_{\phi_1})$, the graph norm of $C_{\phi_n}\cdots C_{\phi_1}$ is given by

$$\|f\|^2_{C_{\phi_n}\cdots C_{\phi_1}} = \int_X |f|^2\left(1+h_{\Phi_j}\cdot\left(\prod_{l=j+1}^{k-1} h_{\phi_{l+1}}\circ\Phi_l^{-1}\right)\cdot E_{\Phi_j}\left(h_{\phi_{j+1}}\circ\Phi_j^{-1}\right)\right)d\mu,$$

where j is an arbitrary number in J_k.

(iii) If $h_{\Phi_j}\cdot\left(\prod_{l=j+1}^{k-1} h_{\phi_{l+1}}\circ\Phi_l^{-1}\right)\cdot E_{\Phi_j}\left(h_{\phi_{j+1}}\circ\Phi_j^{-1}\right)<\infty$ a.e. $[\mu]$, then $\left(h_{\Phi_j}\cdot\left(\prod_{l=j+1}^{k-1} h_{\phi_{l+1}}\circ\Phi_l^{-1}\right)\cdot E_{\Phi_j}\left(h_{\phi_{j+1}}\circ\Phi_j^{-1}\right)\right)\circ\Phi_k<\infty$ a.e. $[\mu]$ and for every $\mathscr{A}$-measurable function $f: X\to\overline{R}_+$, we have that

$$\int_X \frac{f\circ\Phi_k}{\left(h_{\Phi_j}\cdot\left(\prod_{l=j+1}^{k-1} h_{\phi_{l+1}}\circ\Phi_l^{-1}\right)\cdot E_{\Phi_j}\left(h_{\phi_{j+1}}\circ\Phi_j^{-1}\right)\right)\circ\Phi_k} d\mu = \int_{\{h_{\Phi_j}\cdot\left(\prod_{l=j+1}^{k-1} h_{\phi_{l+1}}\circ\Phi_l^{-1}\right)\cdot E_{\Phi_j}\left(h_{\phi_{j+1}}\circ\Phi_j^{-1}\right)>0\}} f d\mu,$$

where j is an arbitrary number in J_k.

Proof. (i) and (ii) are obviously hold. Since

$$\mu\left\{h_{\Phi_j}\cdot\left(\prod_{l=j+1}^{k-1}h_{\phi_{l+1}}\circ\Phi_l^{-1}\right)\cdot E_{\Phi_j}\left(h_{\phi_{j+1}}\circ\Phi_j^{-1}\right)=\infty\right\}=0,$$

it follows that

$$\begin{aligned}&\mu\left\{\left(h_{\Phi_j}\cdot\left(\prod_{l=j+1}^{k-1}h_{\phi_{l+1}}\circ\Phi_l^{-1}\right)\cdot E_{\Phi_j}\left(h_{\phi_{j+1}}\circ\Phi_j^{-1}\right)\right)\circ\Phi_k=\infty\right\}\\&=\int_X\chi_{\left\{h_{\Phi_j}\cdot\left(\prod_{l=j+1}^{k-1}h_{\phi_{l+1}}\circ\Phi_l^{-1}\right)\cdot E_{\Phi_j}\left(h_{\phi_{j+1}}\circ\Phi_j^{-1}\right)=\infty\right\}}\circ\Phi_k d\mu=0,\end{aligned}$$

which implies that $\left(h_{\Phi_j}\cdot\left(\prod_{l=j+1}^{k-1}h_{\phi_{l+1}}\circ\Phi_l^{-1}\right)\cdot E_{\Phi_j}\left(h_{\phi_{j+1}}\circ\Phi_j^{-1}\right)\right)\circ\Phi_k<\infty$ a.e. $[\mu]$, where j is an arbitrary number in J_k.

Moreover, combining (i) and (ii), for each $k\in J_n$, we have that

$$\begin{aligned}&\int_X\frac{f\circ\Phi_k}{\left(h_{\Phi_j}\cdot\left(\prod_{l=j+1}^{k-1}h_{\phi_{l+1}}\circ\Phi_l^{-1}\right)\cdot E_{\Phi_j}\left(h_{\phi_{j+1}}\circ\Phi_j^{-1}\right)\right)\circ\Phi_k}d\mu\\&=\int_X\left(\frac{f\cdot\chi_{\left\{0<h_{\Phi_j}\cdot\left(\prod_{l=j+1}^{k-1}h_{\phi_{l+1}}\circ\Phi_l^{-1}\right)\cdot E_{\Phi_j}\left(h_{\phi_{j+1}}\circ\Phi_j^{-1}\right)<\infty\right\}}}{h_{\Phi_j}\cdot\left(\prod_{l=j+1}^{k-1}h_{\phi_{l+1}}\circ\Phi_l^{-1}\right)\cdot E_{\Phi_j}\left(h_{\phi_{j+1}}\circ\Phi_j^{-1}\right)}\right)\circ\Phi_k d\mu\\&=\int_{\left\{h_{\Phi_j}\cdot\left(\prod_{l=j+1}^{k-1}h_{\phi_{l+1}}\circ\Phi_l^{-1}\right)\cdot E_{\Phi_j}\left(h_{\phi_{j+1}}\circ\Phi_j^{-1}\right)>0\right\}}f d\mu,\end{aligned}$$

where j is an arbitrary number in J_k. This implies (iii) and therefore completes the proof. □

The following proposition describes the dense definiteness of $C_{\phi_n}\cdots C_{\phi_1}$.

PROPOSITION 3.2. *Suppose that (AS) holds. Then the following statements are equivalent:*

(i) $C_{\phi_n}\cdots C_{\phi_1}$ *is densely defined.*

(ii) $\sum_{k=1}^n h_{\Phi_j}\cdot\left(\prod_{l=j+1}^{k-1}h_{\phi_{l+1}}\circ\Phi_l^{-1}\right)\cdot E_{\Phi_j}\left(h_{\phi_{j+1}}\circ\Phi_j^{-1}\right)<\infty$, *where* j *is an arbitrary number in* J_k.

(iii) $\mu\circ\Phi_k^{-1}$ *is* σ*-finite.*

(iv) $\mu|_{\Phi_k^{-1}(\mathscr{A})}$ *is* σ*-finite.*

Proof. (iii)$\Rightarrow$(iv) is trivial and (i)$\Leftrightarrow$(ii) holds by Proposition 3.1.

(ii)$\Rightarrow$(iii): Let $\{X_m\}_{m=1}^{\infty} \subseteq \mathscr{A}$ be any sequence of sets such that $\mu(X_m) < \infty$, $X_m \nearrow X$ as $m \to \infty$ and $h_{\Phi_j} \cdot \left(\prod_{l=j+1}^{k-1} h_{\phi_{l+1}} \circ \Phi_l^{-1}\right) \cdot E_{\Phi_j}\left(h_{\phi_{j+1}} \circ \Phi_j^{-1}\right) \leqslant m$ a.e. $[\mu]$ on X_m, $m \in \mathbb{N}$, where j is an arbitrary number in J_k. Then $\mu \circ \Phi_k(X_m) < \infty$, which yields to (iii).

(iv)$\Rightarrow$(ii): Let $\{Y_m\}_{m=1}^{\infty} \subseteq \mathscr{A}$ be a sequence of sets such that $\mu(\Phi_k^{-1}(Y_m)) < \infty$ and $\Phi_k^{-1}(Y_m) \nearrow X$ as $m \to \infty$. Without loss of generality, we can assume that $Y_m \nearrow Y_\infty \triangleq \bigcup_{m=1}^{\infty} Y_m$ as $m \to \infty$. We have $h_{\Phi_j} \cdot \left(\prod_{l=j+1}^{k-1} h_{\phi_{l+1}} \circ \Phi_l^{-1}\right) \cdot E_{\Phi_j}\left(h_{\phi_{j+1}} \circ \Phi_j^{-1}\right) < \infty$ a.e. $[\mu]$ on Y_m and consequently on Y_∞, where j is an arbitrary number in J_k. Observe that $\Phi_k^{-1}(Y_m) \nearrow \Phi_k^{-1}(Y_\infty)$ and $\Phi_k^{-1}(Y_m) \nearrow X$ as $m \to \infty$. It follows that $\mu(\Phi_k^{-1}(X \setminus Y_\infty)) = 0$ and therefore $h_{\Phi_j} \cdot \left(\prod_{l=j+1}^{k-1} h_{\phi_{l+1}} \circ \Phi_l^{-1}\right) \cdot E_{\Phi_j}\left(h_{\phi_{j+1}} \circ \Phi_j^{-1}\right) = 0$ a.e. $[\mu]$ on $X \setminus Y_\infty$, where j is an arbitrary number in J_k. Combining what we have been observed above, we prove (ii). This completes the proof. $\square$

Set

$$N^{k,j} \triangleq \left\{x \in X : h_{\Phi_j} \cdot \left(\prod_{l=j+1}^{k-1} h_{\phi_{l+1}} \circ \Phi_l^{-1}\right) \cdot E_{\Phi_j}\left(h_{\phi_{j+1}} \circ \Phi_j^{-1}\right)(x) = 0\right\},$$

where j is an arbitrary number in J_k.

Then the following propositions characterize the injectivity of $C_{\phi_n} \cdots C_{\phi_1}$.

PROPOSITION 3.3. *Suppose that (AS) holds. Then*

$$\mathscr{N}(C_{\phi_n} \cdots C_{\phi_1}) = \prod_{k=1}^{n} \chi_{N^{k,j}} \cdot L^2(\mu),$$

where j *is an arbitrary number in* J_n.

Proof. Only observe that

$$\int_X C_{\phi_k} \cdots C_{\phi_1} f d\mu = \int_X f h_{\Phi_j} \cdot \left(\prod_{l=j+1}^{k-1} h_{\phi_{l+1}} \circ \Phi_l^{-1}\right) \cdot E_{\Phi_j}\left(h_{\phi_{j+1}} \circ \Phi_j^{-1}\right) d\mu,$$

where j is an arbitrary number in J_k. $\square$

PROPOSITION 3.4. *Suppose that (AS) holds. Then the following statements are equivalent:*

(i) $\mathscr{N}(C_{\phi_n}\cdots C_{\phi_1})=\{0\}$.

(ii) $\sum_{k=1}^{n}\mu(N^{k,j})=0$.

(iii) $\chi_{N^{k,j}}\circ\Phi_k=\chi_{N^{k,j}}$ *a.e.* $[\mu]$ *for each* $k\in J_n$.

Moreover, if $C_{\phi_1},C_{\phi_2},\cdots,C_{\phi_n}$ *are all densely defined, then (i)–(iii) are equivalent with*

(iv) $\mathscr{N}(C_{\phi_k}\cdots C_{\phi_1})\subseteq\mathscr{N}\big((C_{\phi_k}\cdots C_{\phi_1})^*\big)$,

where j *is an arbitrary number in* J_k *with* $k\in J_n$.

Proof. We always assume that $k\in J_n$ in this proof.

(ii)$\Leftrightarrow$(iii): Suppose that (ii) holds. By the non-singularity of Φ_k, we have $\mu(\Phi_k^{-1}(N^{k,j}))=0$. Then $\mu\big(N^{k,j}\triangle\Phi_k^{-1}(N^{k,j})\big)=0$ and thus (iii) holds.

Suppose that (iii) holds.

$$\begin{aligned}\mu(N^{k,j})&=\int_X\chi_{N^{k,j}}\circ\Phi_k d\mu\\&=\int_X\chi_{N^{k,j}}h_{\Phi_j}\cdot\left(\prod_{l=j+1}^{k-1}h_{\phi_{l+1}}\circ\Phi_l^{-1}\right)\cdot E_{\Phi_j}\left(h_{\phi_{j+1}}\circ\Phi_j^{-1}\right)d\mu=0,\end{aligned}$$

where j is an arbitrary number in J_k. This implies (ii).

(iv)$\Rightarrow$(ii): Let $\{X_m\}_{m=1}^{\infty}$ be a sequence in $\mathscr{A}$ such that $X_m\nearrow X$ as $n\to\infty$, $\mu(X_m)<\infty$ for each $m\in\mathbb{N}$ and $\sum_{j=1}^{m}h_{\Phi_j}\leqslant m$ for μ-a.e. $x\in X_m$ and $m\in\mathbb{N}$. (Observe the existence of $\{X_m\}_{m=1}^{\infty}$ is guaranteed by the densely definiteness of $C_{\phi_1},C_{\phi_2},\cdots,C_{\phi_k}$.)

Therefore, $\chi_{X_m},\chi_{N^{k,j}\cap X_m}\in\mathscr{D}(C_{\phi_k}\cdots C_{\phi_1})$ for each $m\in\mathbb{N}$. Since $\chi_{N^{k,j}\cap X_m}\in\mathscr{N}(C_{\phi_k}\cdots C_{\phi_1})$, by (iv), for each $m\in\mathbb{N}$,

$$0=\langle(C_{\phi_k}\cdots C_{\phi_1})^*\chi_{N^{k,j}\cap X_m},\chi_{X_m}\rangle=\mu(N^{k,j}\cap X_m\cap\Phi_k^{-1}(X_m)).$$

Note that $N^{k,j}\cap X_m\cap\Phi_k^{-1}(X_m)\nearrow N^{k,j}$ as $m\to\infty$, where j is an arbitrary number in J_k. Then the continuity of μ implies (ii).

Moreover, the equivalent between (i) and (ii) can be obtained by Proposition 3.3 and (1)$\Rightarrow$(iv) is obvious. This completes the proof. □

Recall that if an operator T is hyponormal, then $\mathscr{N}(T)\subseteq\mathscr{N}(T^*)$, which implies the following corollary.

COROLLARY 3.5. *Suppose that (AS) holds. Then the following assertions are valid:*

(i) If $C_{\phi_n}\cdots C_{\phi_1}$ *is hyponormal, then* $\mathscr{N}(C_{\phi_n}\cdots C_{\phi_1})=\{0\}$.

(ii) If $C_{\phi_n}\cdots C_{\phi_1}$ is cohyponormal, then $\mathscr{N}\left((C_{\phi_n}\cdots C_{\phi_1})^*\right)=\{0\}$.

(iii) If $C_{\phi_n}\cdots C_{\phi_1}$ if formally normal, then

$$\mathscr{D}(C_{\phi_n}\cdots C_{\phi_1})\cap\mathscr{D}\left((C_{\phi_n}\cdots C_{\phi_1})^*\right)=\{0\}.$$

(iv) If $C_{\phi_n}\cdots C_{\phi_1}$ is normal, then

$$\mathscr{N}(C_{\phi_n}\cdots C_{\phi_1})=\mathscr{N}\left((C_{\phi_n}\cdots C_{\phi_1})^*\right)=\{0\}.$$

PROPOSITION 3.6. *Suppose that (AS) holds. Then*

$$\left(h_{\Phi_j}\cdot\left(\prod_{l=j+1}^{k-1}h_{\phi_{l+1}}\circ\Phi_l^{-1}\right)\cdot E_{\Phi_j}\left(h_{\phi_{j+1}}\circ\Phi_j^{-1}\right)\right)\circ\Phi_k>0$$

a.e. $[\mu]$. Moreover, if

$$\begin{aligned}&\left(h_{\Phi_j}\cdot\left(\prod_{l=j+1}^{k-1}h_{\phi_{l+1}}\circ\Phi_l^{-1}\right)\cdot E_{\Phi_j}\left(h_{\phi_{j+1}}\circ\Phi_j^{-1}\right)\right)\circ\Phi_k\\&=h_{\Phi_j}\cdot\left(\prod_{l=j+1}^{k-1}h_{\phi_{l+1}}\circ\Phi_l^{-1}\right)\cdot E_{\Phi_j}\left(h_{\phi_{j+1}}\circ\Phi_j^{-1}\right)\end{aligned}$$

a.e. $[\mu]$, then $\mathscr{N}(C_{\phi_n}\cdots C_{\phi_1})=\{0\}$, where j is an arbitrary number in J_k.

Proof. The first statement is obtained by $\mu(\Phi_k^{-1}(N^{k,j}))=0$ and the "moreover" part is obtained by Proposition 3.4. □

4. Boundedness of $C_{\phi_n}\cdots C_{\phi_1}$

One can find the original proof of the boundedness in [12].

THEOREM 4.1. *Suppose that (AS) holds. Then necessary and sufficient conditions for $C_{\phi_n}\cdots C_{\phi_1}$ to be bounded are*

$$h_{\Phi_j}\cdot\left(\prod_{l=j+1}^{k-1}h_{\phi_{l+1}}\circ\Phi_l^{-1}\right)\cdot E_{\Phi_j}\left(h_{\phi_{j+1}}\circ\Phi_j^{-1}\right)\in L^\infty(\mu), \tag{4.1}$$

where j is an arbitrary number in J_k.

Proof. Note that $C_{\phi_n}\cdots C_{\phi_1}$ is bounded if and only if $C_{\phi_k}\cdots C_{\phi_1}$ is bounded.

Necessity: (4.1) is obviously the necessary condition for $C_{\phi_n}\cdots C_{\phi_1}$ to be bounded.

Sufficiency: Suppose that $C_{\phi_n}\cdots C_{\phi_1}$ is bounded. Thus, $C_{\phi_k}\cdots C_{\phi_1}$ is bounded. Since

$$\int_X|f|^2\frac{d\mu\circ\Phi_k^{-1}}{d\mu}d\mu=\|C_{\phi_k}\cdots C_{\phi_1}\|^2_{L^2(\mu)}\leqslant\|C_{\phi_k}\cdots C_{\phi_1}\|^2\int_X|f|^2d\mu,$$

which implies $\frac{d\mu\circ\Phi_k^{-1}}{d\mu} \leqslant \|C_{\phi_k}\cdots C_{\phi_1}\|^2$. Then the proof is completed by the fact $h_{\Phi_j}\cdot\left(\prod_{l=j+1}^{k-1} h_{\phi_{l+1}}\circ\Phi_l^{-1}\right)\cdot E_{\Phi_j}\left(h_{\phi_{j+1}}\circ\Phi_j^{-1}\right) = \frac{d\mu\circ\Phi_k^{-1}}{d\mu}$ a.e. $[\mu]$, where j is an arbitrary number in J_k. □

REMARK 4.2. Suppose that (AS) holds. By Theorem 4.1, the boundedness of $C_{\phi_n}\cdots C_{\phi_1}$ implies the boundedness of $C_{\phi_1\circ\phi_2\circ\cdots\circ\phi_n}$. However, the converse assertion fails to be held in general.

5. Adjoint of $C_{\phi_n}\cdots C_{\phi_1}$

THEOREM 5.1. *Suppose that (AS) holds and $C_{\phi_n}\cdots C_{\phi_1}$ is densely defined. Then the following assertions are valid:*

(i) $\mathscr{D}\big((C_{\phi_n}\cdots C_{\phi_1})^*\big) =$

$$\left\{f\in L^2(\mu): \sum_{k=1}^{n} h_{\Phi_j}\cdot\left(\prod_{l=j+1}^{k-1} h_{\phi_{l+1}}\circ\Phi_l^{-1}\right)\cdot E_{\Phi_j}\left(h_{\phi_{j+1}}\circ\Phi_j^{-1}\right) E_{\Phi_k}(f)\circ\Phi_k^{-1}\in L^2(\mu)\right\},$$

where j is an arbitrary number in J_k.

(ii) For each $f\in\mathscr{D}\big((C_{\phi_n}\cdots C_{\phi_1})^*\big)$, we have

$$(C_{\phi_n}\cdots C_{\phi_1})^*(f) = h_{\Phi_j}\cdot\left(\prod_{l=j+1}^{k-1} h_{\phi_{l+1}}\circ\Phi_l^{-1}\right)\cdot E_{\Phi_j}\left(h_{\phi_{j+1}}\circ\Phi_j^{-1}\right) E_{\Phi_n}(f)\circ\Phi_n^{-1}$$

and

$$\|(C_{\phi_n}\cdots C_{\phi_1})^*(f)\|^2_{L^2(\mu)} = \int_X h^2_{\Phi_j} E^2_{\Phi_j}\left(\prod_{l=j}^{k-1} h_{\phi_{l+1}}\circ\Phi_l^{-1}\right)|E_{\Phi_n}(f)\circ\Phi_n^{-1}|^2 d\mu,$$

where j is an arbitrary number in J_k.

(iii) $\mathscr{N}\left((C_{\phi_n}\cdots C_{\phi_1})^*\right) = \left\{f\in L^2(\mu): E_{\Phi_n}(f) = 0 \text{ a.e. } [\mu]\right\}$, *where j is an arbitrary number in J_k.*

Proof. Note that $\mathscr{D}\big((C_{\phi_n}\cdots C_{\phi_1})^*\big) = \bigcap_{k=1}^{n}\mathscr{D}\big((C_{\phi_k}\cdots C_{\phi_1})^*\big)$.

For $f\in L^2(\mu), g\in\mathscr{D}\big((C_{\phi_k}\cdots C_{\phi_1})^*\big)$, obviously,

$$E_{\Phi_k}(f),\ C_{\phi_k}\cdots C_{\phi_1}(g)\in L^2(\mu).$$

By Lemma 1.2 and $(A.11)$ in [3], we have

$$\begin{aligned}\langle g,(C_{\phi_k}\cdots C_{\phi_1})^*f\rangle_{L^2(\mu)} &= \int_X g\circ\Phi_k\cdot E_{\Phi_k}(\overline{f})d\mu = \int_X g\circ\Phi_k\cdot\overline{E_{\Phi_k}(f)}d\mu \\ &= \int_X g\cdot\left(\overline{E_{\Phi_k}(f)\circ\Phi_k^{-1}h_{\Phi_j}\cdot\left(\prod_{l=j+1}^{k-1}h_{\phi_{l+1}}\circ\Phi_l^{-1}\right)\cdot E_{\Phi_j}\left(h_{\phi_{j+1}}\circ\Phi_j^{-1}\right)}\right)d\mu \\ &= \left\langle g,h_{\Phi_j}\cdot\left(\prod_{l=j+1}^{k-1}h_{\phi_{l+1}}\circ\Phi_l^{-1}\right)\cdot E_{\Phi_j}\left(h_{\phi_{j+1}}\circ\Phi_j^{-1}\right)E_{\Phi_k}(f)\circ\Phi_k^{-1}\right\rangle_{L^2(\mu)}, \qquad (5.1)\end{aligned}$$

where j is an arbitrary number in J_k.

Let

$$\mathscr{F}_k = \left\{f\in L^2(\mu): h_{\Phi_j}\cdot\left(\prod_{l=j+1}^{k-1}h_{\phi_{l+1}}\circ\Phi_l^{-1}\right)\cdot E_{\Phi_j}\left(h_{\phi_{j+1}}\circ\Phi_j^{-1}\right)E_{\Phi_k}(f)\circ\Phi_k^{-1}\in L^2(\mu)\right\}.$$

For any $f\in\mathscr{F}_k$, by (5.1), we have $f\in\mathscr{D}(C_{\phi_k}\cdots C_{\phi_1})^*$ and

$$(C_{\phi_k}\cdots C_{\phi_1})^*(f) = h_{\Phi_j}\cdot\left(\prod_{l=j+1}^{k-1}h_{\phi_{l+1}}\circ\Phi_l^{-1}\right)\cdot E_{\Phi_j}\left(h_{\phi_{j+1}}\circ\Phi_j^{-1}\right)E_{\Phi_k}\circ\Phi_k^{-1},$$

where j is an arbitrary number in J_k. This proves (ii).

In the sequence, we only need to show that for any $f\in\mathscr{D}(C_{\phi_k}\cdots C_{\phi_1})^*$,

$$h_{\Phi_j}\cdot\left(\prod_{l=j+1}^{k-1}h_{\phi_{l+1}}\circ\Phi_l^{-1}\right)\cdot E_{\Phi_j}\left(h_{\phi_{j+1}}\circ\Phi_j^{-1}\right)\overline{E_{\Phi_k}(f)\circ\Phi_k^{-1}}\in L^2(\mu).$$

By (5.1),

$$g\cdot h_{\Phi_j}\cdot\left(\prod_{l=j+1}^{k-1}h_{\phi_{l+1}}\circ\Phi_l^{-1}\right)\cdot E_{\Phi_j}\left(h_{\phi_{j+1}}\circ\Phi_j^{-1}\right)\overline{E_{\Phi_k}(f)\circ\Phi_k^{-1}}\in L^1(\mu)$$

and

$$\begin{aligned}&\int_X g\cdot h_{\Phi_j}\cdot\left(\prod_{l=j+1}^{k-1}h_{\phi_{l+1}}\circ\Phi_l^{-1}\right)\cdot E_{\Phi_j}\left(h_{\phi_{j+1}}\circ\Phi_j^{-1}\right)\overline{E_{\Phi_k}(f)\circ\Phi_k^{-1}}d\mu \\ &= \int_X g\cdot\overline{(C_{\phi_k}\cdots C_{\phi_1})^*f}d\mu\end{aligned}$$

for any $g\in\mathscr{D}(C_{\phi_k}\cdots C_{\phi_1})$, where j is an arbitrary number in J_k.

Let $\{X_m\}_{m=1}^{\infty}$ be a sequence of sets as in [[3], Lemma 9]. For any $\Delta\in\mathscr{A}$, $m\in\mathbb{N}$, let $g=\chi_{\Delta\cap X_m}\in\mathscr{D}(C_{\phi_k}\cdots C_{\phi_1})$. By Proposition 2.5, we have $g\cdot h_{\Phi_j}\cdot\left(\prod_{l=j+1}^{k-1}h_{\phi_{l+1}}\circ\Phi_l^{-1}\right)\cdot E_{\Phi_j}\left(h_{\phi_{j+1}}\circ\Phi_j^{-1}\right)\overline{E_{\Phi_k}(f)\circ\Phi_k^{-1}} = (C_{\phi_k}\cdots C_{\phi_1})^*f$ a.e. $[\mu]$ on X_m for $m\in\mathbb{N}$ and consequently on X, which proves (i).

Now we prove (iii). For $f \in \mathcal{N}\left((C_{\phi_n}\cdots C_{\phi_1})^*\right)$,

$$h_{\Phi_j}\cdot\left(\prod_{l=j+1}^{k-1} h_{\phi_{l+1}}\circ\Phi_l^{-1}\right)\cdot E_{\Phi_j}\left(h_{\phi_{j+1}}\circ\Phi_j^{-1}\right)E_{\Phi_n}(f)\circ\Phi_n^{-1}=0$$

a.e. $[\mu]$. By Lemma 1.3, $E_{\Phi_n}(f)\circ\Phi_n^{-1}=0$ a.e. $[\mu]$ on

$$\left\{h_{\Phi_j}\cdot\left(\prod_{l=j+1}^{k-1} h_{\phi_{l+1}}\circ\Phi_l^{-1}\right)\cdot E_{\Phi_j}\left(h_{\phi_{j+1}}\circ\Phi_j^{-1}\right)=0\right\}$$

and consequently on X. Then, by [[3], Lemma 5], $E_{\Phi_n}(f)=0$ a.e. $[\mu]$, where j is an arbitrary number in J_k.

Conversely, for $f\in\left\{f\in L^2(\mu): E_{\Phi_n}(f)=0 \text{ a.e. } [\mu]\right\}$, we have

$$\int_\Delta h_{\Phi_j}\cdot\left(\prod_{l=j+1}^{k-1} h_{\phi_{l+1}}\circ\Phi_l^{-1}\right)\cdot E_{\Phi_j}\left(h_{\phi_{j+1}}\circ\Phi_j^{-1}\right)(E_{\Phi_n}(f)\circ\Phi_n^{-1})d\mu=\mu(E_{\Phi_n}(f))=0,$$

where j is an arbitrary number in J_k. Hence, $E_{\Phi_n}(f)\circ\Phi_n^{-1}=0$ a.e. $[\mu]$. By (i), $f\in\mathcal{N}\left((C_{\phi_n}\cdots C_{\phi_1})^*\right)$, which proves (iii). This completes the proof. $\square$

REMARK 5.2. By (iii) of Theorem 5.1, the kernel of $(C_{\phi_n}\cdots C_{\phi_1})^*$ coincides with the one of E_{Φ_n}, where $E_{\Phi_n}(\cdot)$ is seen as the operator $E(\cdot;\Phi_n^{-1}(\mathscr{A}),\mu): L^2_+(\mu)\to L^2(\mu)$ (See Appendix A in [3]).

6. Hyponormality of unbounded $C_{\phi_n}\cdots C_{\phi_1}$

The hyponormality of specific unbounded product $C_{\phi_n}\cdots C_{\phi_1}$ is investigated in this section, which is the main result in this paper.

Conveniently, the following lemma is proved before the main result.

LEMMA 6.1.

$$\begin{aligned}&\int_X\left(h_{\Phi_j}\cdot\left(\prod_{l=j+1}^{k-1} h_{\phi_{l+1}}\circ\Phi_l^{-1}\right)\cdot E_{\Phi_j}\left(h_{\phi_{j+1}}\circ\Phi_j^{-1}\right)\right)\circ\Phi_k\cdot|E_{\Phi_k}(f)|^2d\mu\\&\leqslant\int_X h_{\Phi_j}\cdot\left(\prod_{l=j+1}^{k-1} h_{\phi_{l+1}}\circ\Phi_l^{-1}\right)\cdot E_{\Phi_j}\left(h_{\phi_{j+1}}\circ\Phi_j^{-1}\right)|f|^2d\mu\end{aligned}\tag{6.1}$$

holds for every function $f\in L^2\left(\left(1+\sum_{l=1}^{k} h_{\Phi_j}E_{\Phi_j}\left(\prod_{t=j}^{l-1} h_{\phi_{t+1}}\circ\Phi_t^{-1}\right)\right)d\mu\right)$ *if and only if*

$$\begin{aligned}&\int_X\left(h_{\Phi_j}\cdot\left(\prod_{l=j+1}^{k-1} h_{\phi_{l+1}}\circ\Phi_l^{-1}\right)\cdot E_{\Phi_j}\left(h_{\phi_{j+1}}\circ\Phi_j^{-1}\right)\right)\circ\Phi_k\cdot E_{\Phi_k}(f)^2d\mu\\&\leqslant\int_X h_{\Phi_j}\cdot\left(\prod_{l=j+1}^{k-1} h_{\phi_{l+1}}\circ\Phi_l^{-1}\right)\cdot E_{\Phi_j}\left(h_{\phi_{j+1}}\circ\Phi_j^{-1}\right)f^2d\mu\end{aligned}\tag{6.2}$$

holds for every $\mathscr{A}$-measurable function $f : X \to \mathbb{R}_+$, where j is an arbitrary number in J_k.

Proof. (6.2)$\Rightarrow$(6.1) could be conducted by [[3], Proposition A.3].

(6.1)$\Rightarrow$(6.2): By [[3], Theorem A.4], for any $f \in L^2_+\left(\left(1+\sum_{l=1}^{k} h_{\Phi_j} E_{\Phi_j}\left(\prod_{t=j}^{l-1} h_{\phi_{t+1}} \circ \Phi_t^{-1}\right)\right) d\mu\right)$, we have that

$$\int_X \left(h_{\Phi_j} \cdot \left(\prod_{l=j+1}^{k-1} h_{\phi_{l+1}} \circ \Phi_l^{-1}\right) \cdot E_{\Phi_j}\left(h_{\phi_{j+1}} \circ \Phi_j^{-1}\right)\right) \circ \Phi_k \cdot E_{\Phi_k}(f)^2 d\mu$$
$$\leqslant \int_X h_{\Phi_j} \cdot \left(\prod_{l=j+1}^{k-1} h_{\phi_{l+1}} \circ \Phi_l^{-1}\right) \cdot E_{\Phi_j}\left(h_{\phi_{j+1}} \circ \Phi_j^{-1}\right) f^2 d\mu, \tag{6.3}$$

in which $E_{\Phi_k}(f) \in L^2_+\left(\left(1+\sum_{l=1}^{k} h_{\Phi_j} E_{\Phi_j}\left(\prod_{t=j}^{l-1} h_{\phi_{t+1}} \circ \Phi_t^{-1}\right)\right) d\mu\right)$, and j is an arbitrary number in J_k.

Define a measure $\nu_k : \mathscr{A} \to \overline{\mathbb{R}}_+$ by

$$\nu_k(\Delta) = \int_\Delta \left(1+\sum_{l=1}^{k} h_{\Phi_j} E_{\Phi_j}\left(\prod_{t=j}^{l-1} h_{\phi_{t+1}} \circ \Phi_t^{-1}\right)\right) d\mu, \Delta \in \mathscr{A},$$

where j is an arbitrary number in J_k. By the dense definiteness of $C_{\phi_n} \cdots C_{\phi_1}$, ν_k is σ-finite. Choose $\{\Omega_m\}_{m=1}^{\infty} \subseteq \mathscr{A}$ satisfying $\nu_k(\Omega_m) < \infty$, $m \in \mathbb{N}$ and $\Omega_m \nearrow X$ as $m \to \infty$. For an $\mathscr{A}$-measurable function $f : X \to \mathbb{R}_+$, by [[13], Theorem 1.17], there exists a sequence $\{s_\alpha\}_{\alpha=1}^{\infty}$ of $\mathscr{A}$-measurable simple functions such that

$$0 \leqslant s_\alpha \nearrow \chi_{\Omega_m} f \quad \text{as} \quad \alpha \to \infty.$$

Observe that $\{s_\alpha\}_{\alpha=1}^{\infty} \subseteq L^2_+(\nu_k)$. In the view of formula (A.6) in [3],

$$0 \leqslant E_{\Phi_k}(s_\alpha) \nearrow E_{\Phi_k}(\Omega_k f) \quad \text{a.e.} \quad [\mu] \quad \text{as} \quad \alpha \to \infty.$$

Hence, by Lebesgue monotone convergence theorem and (6.3),

$$\int_X \left(h_{\Phi_j} \cdot \left(\prod_{l=j+1}^{k-1} h_{\phi_{l+1}} \circ \Phi_l^{-1}\right) \cdot E_{\Phi_j}\left(h_{\phi_{j+1}} \circ \Phi_j^{-1}\right)\right) \circ \Phi_k \cdot E_{\Phi_k}(\chi_{\Omega_m} f)^2 d\mu$$
$$= \lim_{\alpha\to\infty} \int_X \left(h_{\Phi_j} \cdot \left(\prod_{l=j+1}^{k-1} h_{\phi_{l+1}} \circ \Phi_l^{-1}\right) \cdot E_{\Phi_j}\left(h_{\phi_{j+1}} \circ \Phi_j^{-1}\right)\right) \circ \Phi_k \cdot E_{\Phi_k}(s_\alpha)^2 d\mu$$
$$\leqslant \lim_{\alpha\to\infty} \int_X h_{\Phi_j} \cdot \left(\prod_{l=j+1}^{k-1} h_{\phi_{l+1}} \circ \Phi_l^{-1}\right) \cdot E_{\Phi_j}\left(h_{\phi_{j+1}} \circ \Phi_j^{-1}\right) s_\alpha^2 d\mu \tag{6.4}$$
$$= \int_X h_{\Phi_j} \cdot \left(\prod_{l=j+1}^{k-1} h_{\phi_{l+1}} \circ \Phi_l^{-1}\right) \cdot E_{\Phi_j}\left(h_{\phi_{j+1}} \circ \Phi_j^{-1}\right) (\chi_{\Omega_m} f)^2 d\mu, \tag{6.5}$$

where $m \in \mathbb{N}$ and j is an arbitrary number in J_k. Repeating the same process above, we obtain (6.2). This completes the proof. $\square$

THEOREM 6.2. *Suppose that (AS) holds and $C_{\phi_n}\cdots C_{\phi_1}$ is densely defined. Then the following statements are equivalent:*

(i) $C_{\phi_k}\cdots C_{\phi_1}$ *is hyponormal for each* $k \in J_n$.

(ii) $\sum_{l=1}^{k} h_{\Phi_j} E_{\Phi_j}\left(\prod_{t=j}^{l-1} h_{\phi_{t+1}} \circ \Phi_t^{-1}\right) > 0$ *a.e.* $[\mu]$ *and for each* $\mathscr{A}$*-measurable function* $f : X \to \mathbb{R}_+$,

$$E_{\Phi_k}\left(\sqrt{\frac{\left(h_{\Phi_j}\cdot\left(\prod_{l=j+1}^{k-1} h_{\phi_{l+1}}\circ\Phi_l^{-1}\right)\cdot E_{\Phi_j}\left(h_{\phi_{j+1}}\circ\Phi_j^{-1}\right)\right)\circ\Phi_k}{h_{\Phi_j}\cdot\left(\prod_{l=j+1}^{k-1} h_{\phi_{l+1}}\circ\Phi_l^{-1}\right)\cdot E_{\Phi_j}\left(h_{\phi_{j+1}}\circ\Phi_j^{-1}\right)}}\cdot f\right)^2 \leqslant E_{\Phi_k}(f^2)$$

a.e. $[\mu]$, *where* j *is an arbitrary number in* J_k *and* $k \in J_n$.

(iii) $\sum_{l=1}^{k} h_{\Phi_j} E_{\Phi_j}\left(\prod_{t=j}^{l-1} h_{\phi_{t+1}} \circ \Phi_t^{-1}\right) > 0$ *a.e.* $[\mu]$ *and*

$$E_{\Phi_k}\left(\frac{\left(h_{\Phi_j}\cdot\left(\prod_{l=j+1}^{k-1} h_{\phi_{l+1}}\circ\Phi_l^{-1}\right)\cdot E_{\Phi_j}\left(h_{\phi_{j+1}}\circ\Phi_j^{-1}\right)\right)\circ\Phi_k}{h_{\Phi_j}\cdot\left(\prod_{l=j+1}^{k-1} h_{\phi_{l+1}}\circ\Phi_l^{-1}\right)\cdot E_{\Phi_j}\left(h_{\phi_{j+1}}\circ\Phi_j^{-1}\right)}\right) \leqslant 1$$

a.e. $[\mu]$, *where* j *is an arbitrary number in* J_k *and* $k \in J_n$.

(iv) $\sum_{l=1}^{k} h_{\Phi_j} E_{\Phi_j}\left(\prod_{t=j}^{l-1} h_{\phi_{t+1}} \circ \Phi_t^{-1}\right) > 0$ *a.e.* $[\mu]$ *and*

$$E_{\Phi_k}\left(\frac{1}{h_{\Phi_j}\cdot\left(\prod_{l=j+1}^{k-1} h_{\phi_{l+1}}\circ\Phi_l^{-1}\right)\cdot E_{\Phi_j}\left(h_{\phi_{j+1}}\circ\Phi_j^{-1}\right)}\right)$$
$$\leqslant \frac{1}{\left(h_{\Phi_j}\cdot\left(\prod_{l=j+1}^{k-1} h_{\phi_{l+1}}\circ\Phi_l^{-1}\right)\cdot E_{\Phi_j}\left(h_{\phi_{j+1}}\circ\Phi_j^{-1}\right)\right)\circ\Phi_k}$$

a.e. $[\mu]$, *where* j *is an arbitrary number in* J_k *and* $k \in J_n$.

Proof. (i)$\Leftrightarrow$(ii): By the definition, the hyponormality of $C_{\phi_k}\cdots C_{\phi_1}$ is equivalent with

- $L^2\left(\left(1+\sum_{l=1}^{k} h_{\Phi_j}E_{\Phi_j}\left(\prod_{t=j}^{l-1} h_{\phi_{t+1}}\circ\Phi_t^{-1}\right)\right)d\mu\right)$
$\subseteq\left\{f\in L^2(\mu):\sum_{l=1}^{k} h_{\Phi_j}E_{\Phi_j}\left(\prod_{t=j}^{l-1} h_{\phi_{t+1}}\circ\Phi_t^{-1}\right)\cdot E_{\Phi_l}(f)\circ\Phi_l^{-1}\in L^2(\mu)\right\}.$
-

$$\int_X h_{\Phi_j}\cdot\left(\prod_{l=j+1}^{k-1} h_{\phi_{l+1}}\circ\Phi_l^{-1}\right)\cdot E_{\Phi_j}\left(h_{\phi_{j+1}}\circ\Phi_j^{-1}\right)^2\cdot|E_{\Phi_k}(f)\circ\Phi_k^{-1}|^2d\mu$$
$$\leqslant\int_X h_{\Phi_j}\cdot\left(\prod_{l=j+1}^{k-1} h_{\phi_{l+1}}\circ\Phi_l^{-1}\right)\cdot E_{\Phi_j}\left(h_{\phi_{j+1}}\circ\Phi_j^{-1}\right)|f|^2d\mu, \tag{6.6}$$

where $f\in L^2\left(\left(1+\sum_{l=1}^{k} h_{\Phi_j}E_{\Phi_j}\left(\prod_{t=j}^{l-1} h_{\phi_{t+1}}\circ\Phi_t^{-1}\right)\right)d\mu\right)$,

where j is an arbitrary number in J_k and $k\in J_n$. By (2.10) and (A.7) in [3], (6.6) is equivalent with

$$\int_X\left|E_{\Phi_k}\left(\sqrt{\left(h_{\Phi_j}\cdot\left(\prod_{l=j+1}^{k-1} h_{\phi_{l+1}}\circ\Phi_l^{-1}\right)\cdot E_{\Phi_j}\left(h_{\phi_{j+1}}\circ\Phi_j^{-1}\right)\right)\circ\Phi_k f}\right)\right|^2d\mu$$
$$\leqslant\int_X h_{\Phi_j}\cdot\left(\prod_{l=j+1}^{k-1} h_{\phi_{l+1}}\circ\Phi_l^{-1}\right)\cdot E_{\Phi_j}\left(h_{\phi_{j+1}}\circ\Phi_j^{-1}\right)|f|^2d\mu,$$

where $f\in L^2\left(\left(1+\sum_{l=1}^{k} h_{\Phi_j}E_{\Phi_j}\left(\prod_{t=j}^{l-1} h_{\phi_{t+1}}\circ\Phi_t^{-1}\right)\right)d\mu\right)$. By Lemma 6.1, the above formula is equivalent with

$$\int_X\left[E_{\Phi_k}\left(\sqrt{\left(h_{\Phi_j}\cdot\left(\prod_{l=j+1}^{k-1} h_{\phi_{l+1}}\circ\Phi_l^{-1}\right)\cdot E_{\Phi_j}\left(h_{\phi_{j+1}}\circ\Phi_j^{-1}\right)\right)\circ\Phi_k f}\right)\right]^2d\mu$$
$$\leqslant\int_X h_{\Phi_j}\cdot\left(\prod_{l=j+1}^{k-1} h_{\phi_{l+1}}\circ\Phi_l^{-1}\right)\cdot E_{\Phi_j}\left(h_{\phi_{j+1}}\circ\Phi_j^{-1}\right)f^2d\mu, \tag{6.7}$$

where $f:X\to\mathbb{R}_+$ is an $\mathscr{A}$-measurable function. Set

$$\theta_{k,j}=\sqrt{\frac{\left(h_{\Phi_j}\cdot\left(\prod_{l=j+1}^{k-1} h_{\phi_{l+1}}\circ\Phi_l^{-1}\right)\cdot E_{\Phi_j}\left(h_{\phi_{j+1}}\circ\Phi_j^{-1}\right)\right)\circ\Phi_k}{h_{\Phi_j}\cdot\left(\prod_{l=j+1}^{k-1} h_{\phi_{l+1}}\circ\Phi_l^{-1}\right)\cdot E_{\Phi_j}\left(h_{\phi_{j+1}}\circ\Phi_j^{-1}\right)}}\quad a.e.\quad[\mu],$$

where j is an arbitrary number in J_k and $k \in J_n$. By making the substitution $f \rightsquigarrow \sqrt{h_{\Phi_j} \cdot \left(\prod_{l=j+1}^{k-1} h_{\phi_{l+1}} \circ \Phi_l^{-1}\right) \cdot E_{\Phi_j}\left(h_{\phi_{j+1}} \circ \Phi_j^{-1}\right)} f$ in (6.7), for an $\mathscr{A}$-measurable function $f : X \to \mathbb{R}_+$, we have that (6.7) is equivalent with

$$\int_X E_{\Phi_k}(\theta_{k,j} f)^2 d\mu \leqslant \int_X h_{\Phi_j} \cdot \left(\prod_{l=j+1}^{k-1} h_{\phi_{l+1}} \circ \Phi_l^{-1}\right) \cdot E_{\Phi_j}\left(h_{\phi_{j+1}} \circ \Phi_j^{-1}\right) f^2 d\mu, \quad (6.8)$$

where j is an arbitrary number in J_k and $k \in J_n$. For any $\Delta \in \mathscr{A}$, by substituting $\chi_{\Phi_k^{-1}(\Delta)} f$ in place of f and formulae (A.7), (A.1) in [3], (6.8) is equivalent with

$$\int_{\Phi_k^{-1}(\Delta)} E_{\Phi_k}(\theta_{k,j} f)^2 d\mu \leqslant \int_{\Phi_k^{-1}(\Delta)} E_{\Phi_k}(f^2) d\mu, \quad (6.9)$$

where j is an arbitrary number in J_k and $k \in J_n$. Then Proposition 3.2 and [[1], Theorem 1.6.11] complete the proof of this part.

(ii)$\Rightarrow$(iii): By (iii), $E_{\Phi_k}(\theta_{k,j}^2)^2 \leqslant E_{\Phi_k}(\theta_{k,j}^2)$, which implies that $E_{\Phi_k}(\theta_{k,j}^2) \leqslant 1$ a.e. $[\mu]$, where j is an arbitrary number in J_k and $k \in J_n$. This implies (iii).

(iii)$\Rightarrow$(ii): By [[3], Lemma A.1], for every $\mathscr{A}$-measurable function $f : X \to \mathbb{R}_+$,

$$E_{\Phi_k}(\theta_{k,j} f)^2 \leqslant E_{\Phi_k}(\theta_{k,j}^2) \cdot E_{\Phi_k}(f^2) \leqslant E_{\Phi_k}(f^2),$$

where j is an arbitrary number in J_k and $k \in J_n$. This implies (ii).

(iii)$\Leftrightarrow$(iv): Observe that

$$\left(h_{\Phi_j} \cdot \left(\prod_{l=j+1}^{k-1} h_{\phi_{l+1}} \circ \Phi_l^{-1}\right) \cdot E_{\Phi_j}\left(h_{\phi_{j+1}} \circ \Phi_j^{-1}\right)\right) \circ \Phi_k$$

is $\Phi_k^{-1}(\mathscr{A})$-measurable, where j is an arbitrary number in J_k and $k \in J_n$. Then (iv) is obtained by formula (A.7) in [3].

This completes the proof. $\square$

The following corollary is trivially obtained by Theorem 6.2.

COROLLARY 6.3. *Suppose that (AS) holds and $C_{\phi_n} \cdots C_{\phi_1}$ is hyponormal. Then the assertions in the following are valid:*

(i) For every $\mathscr{A}$-measurable function $f : X \to \mathbb{R}_+$, we have that

$$E_{\Phi_k}(\theta_{k,j}^{n+1})^2 \leqslant E_{\Phi_k}(\theta_{k,j}^{2n}) \quad a.e. \quad [\mu].$$

(ii) $E_{\Phi_k}(\theta_{k,j})^2 \leqslant 1$ *a.e.* $[\mu]$.

(iii) $E_{\Phi_k}\left(\frac{1}{\theta_{k,j}^2}\right) \geqslant 1$ *a.e.* $[\mu]$,

$$\theta_{k,j} = \sqrt{\frac{\left(h_{\Phi_j} \cdot \left(\prod_{l=j+1}^{k-1} h_{\phi_{l+1}} \circ \Phi_l^{-1}\right) \cdot E_{\Phi_j}\left(h_{\phi_{j+1}} \circ \Phi_j^{-1}\right)\right) \circ \Phi_k}{h_{\Phi_j} \cdot \left(\prod_{l=j+1}^{k-1} h_{\phi_{l+1}} \circ \Phi_l^{-1}\right) \cdot E_{\Phi_j}\left(h_{\phi_{j+1}} \circ \Phi_j^{-1}\right)}} \quad a.e. \quad [\mu],$$

where j is an arbitrary number in J_k and $k \in J_n$.

7. Counter-examples

In this section, several counter-examples are given to show that Theorem 6.2 can be valid for the unbounded case.

EXAMPLE 7.1. For a fixed $N \in \mathbb{N}$, let $X = J_N$, $\mathscr{A} = 2^X$, μ be a finite measure on $\mathscr{A}$ such that for $m \in X$,

$$\mu(m) = \begin{cases} m & m \neq 0, \\ 1 & m = 0. \end{cases}$$

Define a transformation $\phi : X \to X$ such that for $m \in X$,

$$\phi(m) = \begin{cases} k_m & \exists k_m \in X, \quad s.t. \quad m = k_m^2, \\ 0 & else. \end{cases}$$

Set $\Lambda = \{m \in X : m \neq 0, m \neq k^2, \forall k \in X\}$. Then we have

$$h(m) = \begin{cases} m & m \neq 0, \\ \mu(\Lambda) & m = 0. \end{cases}$$

Hence, $h < \infty$ a.e. $[\mu]$ but $h \notin L^\infty$. Furthermore,

$$h \circ \phi(m) = \begin{cases} k_m & \exists k_m \in X, \quad s.t. \quad m = k_m^2, \\ \mu(\Lambda) & else. \end{cases}$$

It follows that for any $m \in \Lambda^c$, $h \circ \phi(m) > h(m)$.

EXAMPLE 7.2. Let $X, \mathscr{A}, \mu$ be defined as in Example 7.1. On the measure space $(\Lambda^c, 2^{\Lambda^c}, \mu)$, we have $h < \infty$ a.e. $[\mu]$ but $h \notin L^\infty$ and $h \circ \phi \leqslant h$.

The following example shows that Theorem 6.2 can be satisfied when $C_{\phi_n} \cdots C_{\phi_1}$ is unbounded.

EXAMPLE 7.3. As constructed in Example 7.2, we still consider the measure space $(\Lambda^c, 2^{\Lambda^c}, \mu)$. Let $\psi_1 = \phi$ defined in Example 7.1 and $\psi_k = id$, $k \neq 1$, where *id* stands for the identity transformation. Then

$$E_{\Phi_k}\left(\frac{\left(h_{\Phi_j} \cdot \left(\prod_{l=j+1}^{k-1} h_{\phi_{l+1}} \circ \Phi_l^{-1}\right) \cdot E_{\Phi_j}\left(h_{\phi_{j+1}} \circ \Phi_j^{-1}\right)\right) \circ \Phi_k}{h_{\Phi_j} \cdot \left(\prod_{l=j+1}^{k-1} h_{\phi_{l+1}} \circ \Phi_l^{-1}\right) \cdot E_{\Phi_j}\left(h_{\phi_{j+1}} \circ \Phi_j^{-1}\right)}\right) \leqslant 1$$

a.e. $[\mu]$, where j is an arbitrary number in J_k and $k \in J_n$. However, $C_{\phi_n} \cdots C_{\phi_1}$ is unbounded.

8. Conclusion

In this paper, basic properties of product $C_{\phi_n} \cdots C_{\phi_1}$ in $L^2(\mu)$ are conveyed in Section 3–5, including the dense definiteness, kernel, adjoint of (not necessarily bounded) $C_{\phi_n} \cdots C_{\phi_1}$ in $L^2(\mu)$. The hyponormality of specific (not necessarily bounded) $C_{\phi_n} \cdots C_{\phi_1}$ is characterized in Section 6. This exactly shows the difference between (unbounded) $C_{\phi_n} \cdots C_{\phi_1}$ and $C_{\phi_1 \circ \phi_2 \circ \cdots \circ \phi_n}$ in another way.

To summarize this paper, the following open questions are raised.

OPEN QUESTION 8.1. Is there other new examples which are not trivial to show that Theorem 6.2 hold when $C_{\phi_n} \cdots C_{\phi_1}$ is unbounded?

It is noted that this paper is investigated under strong assumption. Thus we ask:

OPEN QUESTION 8.2. What about the hyponormality of $C_{\phi_n} \cdots C_{\phi_1}$ when $C_{\phi_1}, C_{\phi_2}, \cdots, C_{\phi_n}$ are all densely defined if we drop the assumption $C_{\phi_k} \cdots C_{\phi_1}$ is normal for each $k \in J_n$?

For the unbounded case, the polar decomposition isn't valid and the approaches in the study are quietly different. Since the hyponormality of $C_{\phi_n} \cdots C_{\phi_1}$ is investigated in this paper, it is natural to ask:

OPEN QUESTION 8.3. What about the cohypnormality and, even the normality of unbounded product $C_{\phi_n} \cdots C_{\phi_1}$ in $L^2(\mu)$?

As the investigation in [3], the quasinormality and subnormality of unbounded composition operators are of vital significance to the study of theory of composition operators in L^2 spaces over a σ-finite measure space. Thus we ask:

OPEN QUESTION 8.4. What about the quasinormality and subnormality of unbounded product $C_{\phi_n} \cdots C_{\phi_1}$ in $L^2(\mu)$?

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