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LIE SUPERALGEBRAS BASED ON $\mathfrak{sl}(2,\mathbb{F})$

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Abstract. In this paper, we study a class of Lie superalgebras based on the Lie algebra $\mathfrak{sl}(2,\mathbb{F})$ over a field of characteristic not equal to 2. Applying matrix techniques and methods, we determine their automorphisms group and local automorphisms, and characterize their superderivations and local superderivations.

1. Introduction and basics

The even part of a Lie superalgebra is a Lie algebra and the odd part is a module of the Lie algebra by means of the adjoint representation. Thus, one can construct Lie superalgebras from a Lie algebra and its modules. This point of view of constructing Lie superalgebras is quite useful for studying Lie superalgebras [1].

Let $\mathfrak{g}_{\bar{0}}$ be a Lie algebra with multiplication $\langle\ ,\ \rangle$, $\mathfrak{g}_{\bar{1}}$ an $\mathfrak{g}_{\bar{0}}$-module with module action "$\cdot$", and $P : \mathfrak{g}_{\bar{1}} \times \mathfrak{g}_{\bar{1}} \longrightarrow \mathfrak{g}_{\bar{0}}$ a symmetric bilinear mapping. We construct a super vector space $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$, where $\mathfrak{g}_{\bar{0}}$ is even part and $\mathfrak{g}_{\bar{1}}$ is odd part. Define a multiplication $[\ ,\ ]$ on $\mathfrak{g}$ by

$$[x,y] = \langle x,y\rangle,\ \ [x,u] = -[u,x] = x\cdot u,\ \ [u,v] = P(u,v),\ \ x,y \in \mathfrak{g}_0,\ \ u,v \in \mathfrak{g}_{\bar{1}}.$$

Then $\mathfrak{g}$ is a Lie superalgebra if and only if the mapping P satisfies that

$$P(u\cdot v,w) + P(v,u\cdot w) = [u,P(v,w)], \quad u \in \mathfrak{g}_{\bar{0}},\ \ v,w \in \mathfrak{g}_{\bar{1}}; \tag{1.1}$$

$$P(u,v)\cdot w + P(v,w)\cdot u + P(w,u)\cdot v = 0, \quad u,v,w \in \mathfrak{g}_{\bar{1}}. \tag{1.2}$$

A Lie superalgebra $\mathfrak{g}$ constructed in such way is called a Lie superalgebra based on the Lie algebra $\mathfrak{g}_{\bar{0}}$ and $\mathfrak{g}_{\bar{0}}$-module $\mathfrak{g}_{\bar{1}}$.

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Throughout the paper, $\mathbb{F}$ is a field of characteristic not 2, any additional assumption will be mentioned explicitly. $\mathbb{F}^*$ refers the multiplicative group of $\mathbb{F}$. Let V be a 2-dimensional linear space over $\mathbb{F}$ and ψ a non-degenerate skew-symmetric bilinear form on V. Then there exists a basis $\{\omega_1, \omega_{-1}\}$ of V such that $\psi(\omega_1, \omega_{-1}) = 1$. Let $\mathfrak{g}_{\bar{0}}$ be the symplectic Lie algebra $\mathfrak{sp}(\psi)$ and $\mathfrak{g}_{\bar{1}} = V$. Suppose that the bilinear mapping

$$p : V \times V \to \mathfrak{sp}(\psi)$$

satisfies

$$p(u,v)\omega = \psi(v,\omega)u - \psi(\omega,u)v, \;\; u,v,\omega \in V. \tag{1.3}$$

Obviously, p is symmetric and satisfies (1.1) and (1.2). Then $\mathfrak{g} = \mathfrak{sp}(\psi) \oplus V$ is a Lie superalgebra. Since $\mathfrak{sp}(\psi)$ is isomorphic to $\mathfrak{sl}(2,\mathbb{F})$, we call $\mathfrak{g}$ a Lie superalgebra based on Lie algebra $\mathfrak{sl}(2,\mathbb{F})$ and its module V.

From [1, Page 17], $\mathfrak{g} = \mathfrak{sp}(\psi) \oplus V$ is a Lie superalgebra if and only if there exists $d \in \mathbb{F}$ such that $[u,v] = dp(u,v)$, $u,v \in V$. Denote $\mathfrak{g} = \Gamma(d)$. Write $\Pi = \{\Gamma(d) | d \in \mathbb{F}\}$ for all Lie superalgebras based on Lie algebra $\mathfrak{sl}(2,\mathbb{F})$ and its module V.

In this paper, we will give the isomorphic classification of Π, determine their automorphisms, local automorphisms, superderivations and local superderivations.

For a Lie superalgebra $\mathfrak{g}$, denote by $\mathrm{Aut}(\mathfrak{g})$ and $\mathrm{LAut}(\mathfrak{g})$ the automorphism group and the local automorphism group of the Lie superalgebra $\mathfrak{g}$, respectively. Denote by $\mathrm{Der}(\mathfrak{g})$ and $\mathrm{ad}(\mathfrak{g})$ the superderivation algebra and inner superderivation algebra, and $\mathrm{LDer}(\mathfrak{g})$ the set of all local superderivations, respectively. We denote by $A \oplus B$ the block matrix $\begin{pmatrix} A & O \\ O & B \end{pmatrix}$, and by $A \overline{\oplus} B$ the block matrix $\begin{pmatrix} O & A \\ B & O \end{pmatrix}$, respectively.

The concepts of local automorphism and local derivation first appeared in references [2] and [3]. Here the notion of local superderivation are from [4]. In view of the difference of algebra structure of Lie superalgebra and Lie algebra, it is slightly different from local derivation in [2] and [3]. Next, we introduce the definitions of local automorphism and local superderivation of a Lie superalgebra.

DEFINITION 1.1. Let φ be a linear transformation of a Lie superalgebra $\mathfrak{g}$. We call φ a local automorphism of $\mathfrak{g}$, if for any $x \in \mathfrak{g}$ there exists an automorphism ϕ_x of $\mathfrak{g}$ such that $\varphi(x) = \phi_x(x)$.

DEFINITION 1.2. Suppose that $\mathfrak{g}$ is a Lie superalgebra, $\varphi : \mathfrak{g} \to \mathfrak{g}$ is a linear homogeneous mapping of degree α, $\alpha \in \{\bar{0}, \bar{1}\}$. If for any $x \in \mathfrak{g}$ there exists a superderivation ϕ_x of $\mathfrak{g}$ such that $\varphi(x) = \phi_x(x)$, then we call φ a local homogeneous superderivation of degree α. Let $\mathrm{LDer}_\alpha(\mathfrak{g})$ be the set of all local homogeneous superderivations of degree α, $\mathrm{LDer}(\mathfrak{g}) = \mathrm{LDer}_{\bar{0}}(\mathfrak{g}) \oplus \mathrm{LDer}_{\bar{1}}(\mathfrak{g})$. The element of $\mathrm{LDer}(\mathfrak{g})$ is called a local superderivation of $\mathfrak{g}$.

REMARK 1.3. It is easy to see that, by Definition 1.2, if φ is a local automorphism, then φ is invertible, and φ^{-1} is also a local automorphism.

2. Isomorphism classification of Π

The matrix of $p(u,v)$ with respect to the basis $\{\omega_1,\omega_{-1}\}$ is

$$p(\omega_1,\omega_{-1})=-h,\quad p(\omega_1,\omega_1)=2e,\quad p(\omega_{-1},\omega_{-1})=-2f,$$

where

$$h=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad e=\begin{pmatrix}0&1\\0&0\end{pmatrix},\quad f=\begin{pmatrix}0&0\\1&0\end{pmatrix}.$$

For any $x\in\mathfrak{sp}(\psi)$, we also denote by x its matrix with respect to the basis $\{\omega_1,\omega_{-1}\}$.

In the following, if we refer to the matrix of a linear transformation of $\Gamma(d)$, then it means the matrix with respect to the fixed basis $\{h,e,f,\omega_1,\omega_{-1}\}$.

Similar to [5, Lemma 2.5], we have the following lemma.

LEMMA 2.1. *Suppose that φ is an invertible linear mapping on $\mathfrak{sl}(2,\mathbb{F})$ whose matrix with respect to the basis $\{h,e,f\}$ is A. Then φ is an automorphism of Lie algebras if and only*

$$P^{-1}A^TP=A^*, \tag{2.1}$$

where $P=E_{11}+\frac{1}{2}E_{23}+\frac{1}{2}E_{32}$, A^T and A^* are the transpose and adjugate matrix of A, respectively.

For any $d_1,d_2\in\mathbb{F}$, let $\varphi:\Gamma(d_1)\to\Gamma(d_2)$ be a linear mapping such that

$$\varphi(h,e,f,\omega_1,\omega_{-1})=(h,e,f,\omega_1,\omega_{-1})\begin{pmatrix}A&O\\O&B\end{pmatrix}, \tag{2.2}$$

where $A\in M_3(\mathbb{F})$. Denote $A=(a_{ij}),B=(b_{ij})$. Then,

$$\mathrm{ad}(\varphi(h))(\omega_1,\omega_{-1})=(\omega_1,\omega_{-1})A_h, \tag{2.3}$$

$$\mathrm{ad}(\varphi(e))(\omega_1,\omega_{-1})=(\omega_1,\omega_{-1})A_e, \tag{2.4}$$

$$\mathrm{ad}(\varphi(f))(\omega_1,\omega_{-1})=(\omega_1,\omega_{-1})A_f, \tag{2.5}$$

where

$$A_h=\begin{pmatrix}a_{11}&a_{21}\\a_{31}&-a_{11}\end{pmatrix},\quad A_e=\begin{pmatrix}a_{12}&a_{22}\\a_{32}&-a_{12}\end{pmatrix},\quad A_f=\begin{pmatrix}a_{13}&a_{23}\\a_{33}&-a_{13}\end{pmatrix}. \tag{2.6}$$

Using these symbols, we characterize the conditions under which φ becomes an isomorphic mapping.

THEOREM 2.2. *Suppose that φ is described as above. If φ is invertible, then φ is a Lie superalgebra isomorphism of $\Gamma(d_1)$ to $\Gamma(d_2)$ if and only if $A_xB=Bx$, for $x=h,e$ and f, and one of the following conditions holds.*

(1) $d_1=d_2=0$;

(2) $d_1d_2\neq 0$ and $\det(B)=\dfrac{d_1}{d_2}$.

Proof. By definition of isomorphism we have

$$\varphi([x,y]) = [\varphi(x),\varphi(y)],\ x,y \in \Gamma(d_1). \tag{2.7}$$

Since $[h,\omega_1]=\omega_1$ and $[h,\omega_{-1}]=-\omega_{-1}$, we have $\phi(\omega_1)=[\phi(h),\phi(\omega_1)]$ and $-\phi(\omega_{-1}) = [\phi(h),\phi(\omega_{-1})]$. By (2.2) and (2.6), we have

$$\begin{aligned} b_{11}\omega_1 + b_{21}\omega_{-1} &= [a_{11}h + a_{21}e + a_{31}f, b_{11}\omega_1 + b_{21}\omega_{-1}] \\ &= (a_{11}b_{11} + a_{21}b_{21})\omega_1 + (-a_{11}b_{21} + a_{31}b_{11})\omega_{-1}, \\ -(b_{12}\omega_1 + b_{22}\omega_{-1}) &= [a_{11}h + a_{21}e + a_{31}f, b_{12}\omega_1 + b_{22}\omega_{-1}] \\ &= (a_{11}b_{12} + a_{21}b_{22})\omega_1 + (-a_{11}b_{22} + a_{31}b_{12})\omega_{-1}. \end{aligned}$$

Then

$$b_{11} = a_{11}b_{11} + a_{21}b_{21},\ \ b_{12} = -a_{11}b_{12} - a_{21}b_{22},$$

$$b_{21} = a_{31}b_{11} - a_{11}b_{21},\ \ b_{22} = a_{11}b_{22} - a_{31}b_{12},$$

i.e.,

$$\begin{pmatrix} a_{11} & a_{21} \\ a_{31} & -a_{11} \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} b_{11} & -b_{12} \\ b_{21} & -b_{22} \end{pmatrix} = B\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Thus, $A_hB = Bh$. Similarly, we have $A_eB = Be$ and $A_fB = Bf$. That is

$$A_xB = Bx, \quad x \in \{h,e,f\}. \tag{2.8}$$

Replacing x and y by ω_1 in (2.7) yields

$$\begin{cases} d_1a_{12} = -d_2b_{11}b_{21}, \\ d_1a_{22} = d_2b_{11}^2, \\ d_1a_{32} = -d_2b_{21}^2. \end{cases} \tag{2.9}$$

Then, both d_1 and d_2 are 0 or neither is 0.

If $d_1d_2 \neq 0$, by (2.8) and (2.9) we have

$$\begin{cases} \det(B)a_{12} = -b_{11}b_{21}, \\ \det(B)a_{22} = b_{11}^2, \\ \det(B)a_{32} = -b_{21}^2. \end{cases}$$

Comparing the above equations with (2.9), it can be concluded that

$$\left(\det(B) - \frac{d_1}{d_2}\right)a_{k2} = 0,\ \ k = 1,2,3.$$

Thus, $\det(B) = \dfrac{d_1}{d_2}$.

Conversely, if $d_1d_2 \neq 0$, $\det(B) = \dfrac{d_1}{d_2}$ and $A_xB = Bx$, for $x = h$, e and f, from the proof of the necessity part we know (2.7) holds for any $x \in \Gamma(d_1)_{\bar{0}}, y \in \Gamma(d_1)_{\bar{1}}$. By direct verification we have (2.7) holds for any $x,y \in \Gamma(d_1)_{\bar{1}}$. Moreover, (2.1) can be deduced by (2.8). By Lemma 2.1, (2.7) holds for any $x,y \in \Gamma(d_1)_{\bar{0}}$. Therefore, φ is a Lie superalgebra isomorphism of $\Gamma(d_1)$ into $\Gamma(d_2)$. Else if $d_1 = d_2 = 0$ and $A_x B = Bx$, for $x = h, e$ and f, we can prove that φ is an automorphism of $\Gamma(0)$ similarly. □

By Theorem 2.2 and its proof, we have the following conclusions.

COROLLARY 2.3. *Linear transformation of $\Gamma(d)$ is a Lie superalgebra automorphism if and only if its matrix is of the form*

$$b^{-1}\begin{pmatrix} b_{11}b_{22}+b_{12}b_{21} & -b_{11}b_{21} & b_{12}b_{22} \\ -2b_{11}b_{12} & b_{11}^2 & -b_{12}^2 \\ 2b_{21}b_{22} & -b_{21}^2 & b_{22}^2 \end{pmatrix} \oplus \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix},$$

where $b = \det(b_{ij}) \neq 0$, and if $d \neq 0$ then $b = 1$.

COROLLARY 2.4. $\mathrm{Aut}(\Gamma(0))$ *is isomorphic to $GL(2,\mathbb{F})$ (the general linear group), and* $\mathrm{Aut}(\Gamma(\mathrm{d}))$ *is isomorphic to $SL(2,\mathbb{F})$ (the special linear group), where $d \neq 0$.*

THEOREM 2.5. *Up to the Lie superalgebra isomorphism, there are only two classes in Π: $\Gamma(0)$ and $\Gamma(1)$.*

Proof. By Theorem 2.2, the only one that can be isomorphic to $\Gamma(0)$ is $\Gamma(0)$. If $0 \neq d \in \mathbb{F}$, we can choose a 2×2 matrix B over $\mathbb{F}$ such that $\det(B) = d$, then the matrix A is determined by (2.8). Thus, the proof of Theorem 2.2 shows that $\Gamma(d)$ is isomorphic to $\Gamma(1)$. □

3. Local automorphisms of $\Gamma(1)$ and $\Gamma(0)$

LEMMA 3.1. *Suppose that $\mathfrak{g} = \Gamma(0)$ or $\Gamma(1)$. If $\phi \in \mathrm{LAut}(\mathfrak{g})$, then the matrix of ϕ is of the form $A \oplus B$, where*

$$A = b^{-1}\begin{pmatrix} b_{11}b_{22}+b_{12}b_{21} & -\rho_2 b_{11}b_{21} & \rho_3 b_{12}b_{22} \\ -2b_{11}b_{12} & \rho_2 b_{11}^2 & -\rho_3 b_{12}^2 \\ 2b_{21}b_{22} & -\rho_2 b_{21}^2 & \rho_3 b_{22}^2 \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & \rho_1 b_{12} \\ b_{21} & \rho_1 b_{22} \end{pmatrix},$$

$\det(b_{ij}) = b \neq 0$ *and $\rho_i \in \mathbb{F}^*$, $i = 1,2,3$.*

Proof. By definition of local automorphism, we have

$$\phi(x) = \phi_x(x), \forall x \in \mathfrak{g}. \tag{3.1}$$

where ϕ_x is an automorphism of $\mathfrak{g}$. Using Corollary 2.3, we can write $A^x \oplus B^x$ for the matrix of ϕ_x, where $A^x = (A_1^x, A_2^x, A_3^x), B^x = (B_1^x, B_2^x)$. Therefore, by (3.1) we can obtain easily that the matrix of ϕ is of the form $A \oplus B$, where $A = (A_1, A_2, A_3)$ and $B = (B_1, B_2)$.

In a similar way to (2.3)–(2.5), we denote the matrix of $\mathrm{ad}(\phi(y))|_V$ and $\mathrm{ad}(\phi_x(y))|_V$ with respect to the fixed basis $\{\omega_1, \omega_{-1}\}$ by A_y and A_y^x, respectively, where $x \in \mathfrak{g}, y \in \{h, e, f\}$.

For any $i \in \{1, -1\}$ and $y \in \{h, e, f\}$, substituting $x = y + \omega_i$ into (3.1), then we have

$$B_1^{y+\omega_1} = B_1, \quad B_2^{y+\omega_{-1}} = B_2, \tag{3.2}$$

$$A_1^{h+\omega_i} = A_1, \quad A_2^{e+\omega_i} = A_2, \quad A_3^{f+\omega_i} = A_3. \tag{3.3}$$

Thus,

$$A_h^{h+\omega_i} = A_h, \quad A_e^{e+\omega_i} = A_e, \quad A_f^{f+\omega_i} = A_f, \quad i = 1, -1. \tag{3.4}$$

By Theorem 2.2 we have

$$A_y^x(B_1^x, B_2^x) = (B_1^x, B_2^x)y, \;\; y = h, e, f.$$

Then using (3.4) we conclude that

$$A_h(B_1^{h+\omega_1}, B_2^{h+\omega_1}) = (B_1^{h+\omega_1}, B_2^{h+\omega_1}) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3.5}$$

$$A_h(B_1^{h+\omega_{-1}}, B_2^{h+\omega_{-1}}) = (B_1^{h+\omega_{-1}}, B_2^{h+\omega_{-1}}) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$A_e(B_1^{e+\omega_1}, B_2^{e+\omega_1}) = (B_1^{e+\omega_1}, B_2^{e+\omega_1}) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{3.6}$$

$$A_e(B_1^{e+\omega_{-1}}, B_2^{e+\omega_{-1}}) = (B_1^{e+\omega_{-1}}, B_2^{e+\omega_{-1}}) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{3.7}$$

$$A_f(B_1^{f+\omega_1}, B_2^{f+\omega_1}) = (B_1^{f+\omega_1}, B_2^{f+\omega_1}) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{3.8}$$

$$A_f(B_1^{f+\omega_{-1}}, B_2^{f+\omega_{-1}}) = (B_1^{f+\omega_{-1}}, B_2^{f+\omega_{-1}}) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{3.9}$$

It is easy to see that there exist $\rho_i \in \mathbb{F}^*$, $i = 1, 2, 3$ such that

$$B_2^{h+\omega_{-1}} = \rho_1 B_2^{h+\omega_1}, \quad B_1^{e+\omega_{-1}} = \rho_2 B_1^{e+\omega_1}, \quad B_2^{f+\omega_{-1}} = \rho_3 B_2^{f+\omega_1}. \tag{3.10}$$

Then,

$$
\begin{aligned}
A_h(B_1^{h+\omega_1},B_2^{h+\omega_1}) &\overset{(3.5)}{=} (B_1^{h+\omega_1},B_2^{h+\omega_1})h,\\
A_e(B_1^{h+\omega_1},B_2^{h+\omega_1}) &\underset{(3.10)}{\overset{(3.2)}{=}} A_e(B_1^{e+\omega_1},\rho_1^{-1}B_2^{h+\omega_{-1}}) \overset{(3.2)}{=} A_e(B_1^{e+\omega_1},\rho_1^{-1}B_2^{e+\omega_{-1}})\\
&= (A_eB_1^{e+\omega_1},\rho_1^{-1}A_eB_2^{e+\omega_{-1}}) \underset{(3.7)}{\overset{(3.6)}{=}} (0,\rho_1^{-1}B_1^{e+\omega_{-1}})\\
&\overset{(3.10)}{=} (0,\rho_1^{-1}\rho_2B_1^{e+\omega_1}) \overset{(3.2)}{=} (0,\rho_1^{-1}\rho_2B_1^{h+\omega_1})\\
&= \rho_1^{-1}\rho_2(B_1^{h+\omega_1},B_2^{h+\omega_1})e,\\
A_f(B_1^{h+\omega_1},B_2^{h+\omega_1}) &\underset{(3.10)}{\overset{(3.2)}{=}} A_f(B_1^{f+\omega_1},\rho_1^{-1}B_2^{h+\omega_{-1}}) \overset{(3.2)}{=} A_f(B_1^{f+\omega_1},\rho_1^{-1}B_2^{f+\omega_{-1}})\\
&= (A_fB_1^{f+\omega_1},\rho_1^{-1}A_fB_2^{f+\omega_{-1}}) \underset{(3.9)}{\overset{(3.8)}{=}} (B_2^{f+\omega_1},0) \overset{(3.10)}{=} (\rho_3^{-1}B_2^{f+\omega_{-1}},0)\\
&\overset{(3.2)}{=} (\rho_3^{-1}B_2^{h+\omega_{-1}},0) \overset{(3.10)}{=} (\rho_3^{-1}\rho_1B_2^{h+\omega_1},0)\\
&= \rho_3^{-1}\rho_1(B_1^{h+\omega_1},B_2^{h+\omega_1})f.
\end{aligned}
$$

Therefore, $A_h = A_h^{h+\omega_1}$, $A_e = \rho_1^{-1}\rho_2A_e^{h+\omega_1}$, $A_f = \rho_3^{-1}\rho_1A_f^{h+\omega_1}$. Thus, using (3.3), (3.2) and (3.10) we have $A = (A_1^{h+\omega_1},\ \rho_1^{-1}\rho_2A_2^{h+\omega_1},\ \rho_3^{-1}\rho_1A_3^{h+\omega_1})$ and $B = (B_1^{h+\omega_1},\rho_1B_2^{h+\omega_1})$. Denote $B^{h+\omega_1} = (b_{ij})_{2\times 2}$ and $\det(B^{h+\omega_1}) = b$, then by Corollary 2.3 we know

$$
A = b^{-1}\begin{pmatrix} b_{11}b_{22}+b_{12}b_{21} & -\rho_1^{-1}\rho_2b_{11}b_{21} & \rho_3^{-1}\rho_1b_{12}b_{22}\\ -2b_{11}b_{12} & \rho_1^{-1}\rho_2b_{11}^2 & -\rho_3^{-1}\rho_1b_{12}^2\\ 2b_{21}b_{22} & -\rho_1^{-1}\rho_2b_{21}^2 & \rho_3^{-1}\rho_1b_{22}^2\end{pmatrix}. \qquad \square
$$

THEOREM 3.2. $\mathrm{LAut}(\Gamma(0)) = \mathrm{Aut}(\Gamma(0))$.

Proof. Suppose that $\phi \in \mathrm{LAut}(\Gamma(0))$. By Lemma 3.1 we can assume the matrix of ϕ is $A\oplus B$, where

$$
A = b^{-1}\begin{pmatrix} b_{11}b_{22}+b_{12}b_{21} & -\rho_2b_{11}b_{21} & \rho_3b_{12}b_{22}\\ -2b_{11}b_{12} & \rho_2b_{11}^2 & -\rho_3b_{12}^2\\ 2b_{21}b_{22} & -\rho_2b_{21}^2 & \rho_3b_{22}^2\end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & \rho_1b_{12}\\ b_{21} & \rho_1b_{22}\end{pmatrix}
$$

$\rho_1,\rho_2,\rho_3 \in \mathbb{F}^*$ and $b = \det(b_{ij}) \neq 0$. Then

$$
A^{-1} = b^{-1}\begin{pmatrix} b_{11}b_{22}+b_{12}b_{21} & b_{21}b_{22} & -b_{12}b_{11}\\ 2\rho_2^{-1}b_{22}b_{12} & \rho_2^{-1}b_{22}^2 & -\rho_2^{-1}b_{12}^2\\ -2\rho_3^{-1}b_{21}b_{11} & -\rho_3^{-1}b_{21}^2 & \rho_3^{-1}b_{11}^2\end{pmatrix},
$$

$$B^{-1}=\begin{pmatrix} b^{-1}b_{22} & -b^{-1}b_{12} \\ -\rho_1^{-1}b^{-1}b_{21} & \rho_1^{-1}b^{-1}b_{11} \end{pmatrix}.$$

But, ϕ^{-1} is also a local automorphism of $\Gamma(0)$. By Lemma 3.1, we can assume that the matrix of ϕ^{-1} is $G\oplus C$, where

$$G=c^{-1}\begin{pmatrix} c_{11}c_{22}+c_{12}c_{21} & -\varepsilon_2c_{11}c_{21} & \varepsilon_3c_{12}c_{22} \\ -2c_{11}c_{12} & \varepsilon_2c_{11}^2 & -\varepsilon_3c_{12}^2 \\ 2c_{21}c_{22} & -\varepsilon_2c_{21}^2 & \varepsilon_3c_{22}^2 \end{pmatrix},\quad C=\begin{pmatrix} c_{11} & \varepsilon_1c_{12} \\ c_{21} & \varepsilon_1c_{22} \end{pmatrix}, \tag{3.11}$$

$c=\det(c_{ij})_{2\times 2}\neq 0$, $\varepsilon_1\varepsilon_2\varepsilon_3\neq 0$. Then $G=A^{-1}$ and $C=B^{-1}$. Therefore,

$$c_{11}=b^{-1}b_{22},\ \ \varepsilon_1c_{12}=-b^{-1}b_{12},\ \ c_{21}=-\rho_1^{-1}b^{-1}b_{21},\ \ \varepsilon_1c_{22}=\rho_1^{-1}b^{-1}b_{11}. \tag{3.12}$$

Case 1. If $b_{11}\neq 0$, then using (3.12) and by the (3,3)-entry of A^{-1} and G, we have $\rho_1^2\varepsilon_1^2bc=\rho_3\varepsilon_3$.

Subcase 1.1. Suppose that $b_{12}\neq 0$. Then using (3.12) and by the (1,3)-entry and (2,3)-entry of A^{-1} and G, we have $\rho_2^{-1}=\rho_3=\rho_1$. Thus, by Corollary 2.3 we know $\phi\in \mathrm{Aut}(\Gamma(0))$.

Subcase 1.2. Suppose that $b_{12}=0$ and $b_{21}\neq 0$. Then using (3.12) and by the (3,1)-entry and (3,2)-entry of A^{-1} and G, we have $\varepsilon_2^{-1}=\varepsilon_3=\varepsilon_1$. Thus, by Corollary 2.3 we know $\phi^{-1}\in \mathrm{Aut}(\Gamma(0))$ and therefore $\phi\in\mathrm{Aut}(\Gamma(0))$.

Subcase 1.3. Suppose that $b_{12}=b_{21}=0$. Then $b_{22}\neq 0$ and

$$A=\frac{1}{b_{11}b_{22}}\begin{pmatrix} b_{11}b_{22} & 0 & 0 \\ 0 & \rho_2b_{11}^2 & 0 \\ 0 & 0 & \rho_3b_{22}^2 \end{pmatrix},\quad B=\begin{pmatrix} b_{11} & 0 \\ 0 & \rho_1b_{22} \end{pmatrix}.$$

Since $\phi(h+e+f+\omega_1)=\phi_{h+e+f+\omega_1}(h+e+f+\omega_1)$, $\rho_2\rho_3=1$. Denote $b_{11}=\delta_1$, $\rho_1b_{22}=\delta_2$ and $\rho_2b_{11}b_{22}^{-1}=\delta_3$, then

$$A=\begin{pmatrix} 1 & 0 & 0 \\ 0 & \delta_3 & 0 \\ 0 & 0 & \delta_3^{-1} \end{pmatrix},\quad B=\begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{pmatrix}. \tag{3.13}$$

Finally, let us prove $\delta_3=\delta_1\delta_2^{-1}$, and therefore, by Corollary 2.3, we will obtain $\phi\in\mathrm{Aut}(\Gamma(0))$.

By definition of local automorphism, there exists an automorphism $\phi_{e+f+\omega_1+\omega_{-1}}$ such that

$$\phi(e+f+\omega_1+\omega_{-1})=\phi_{e+f+\omega_1+\omega_{-1}}(e+f+\omega_1+\omega_{-1}). \tag{3.14}$$

By Corollary 2.3, we assume that the matrix of $\phi_{h+e+f+\omega_1+\omega_{-1}}$ is

$$d^{-1}\begin{pmatrix} d_{11}d_{22}+d_{12}d_{21} & -d_{11}d_{21} & d_{12}d_{22} \\ -2d_{11}d_{12} & d_{11}^2 & -d_{12}^2 \\ 2d_{21}d_{22} & -d_{21}^2 & d_{22}^2 \end{pmatrix}\oplus\begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix},$$

where $d=\det(d_{ij})\neq 0$. Then, by (3.14) we have

$$-d_{11}d_{21}+d_{12}d_{22}=0, \tag{3.15}$$

$$d_{11}^2-d_{12}^2=d\delta_3, \tag{3.16}$$

$$d_{22}^2-d_{21}^2=d\delta_3^{-1}, \tag{3.17}$$

$$d_{11}+d_{12}=\delta_1, \tag{3.18}$$

$$d_{21}+d_{22}=\delta_2. \tag{3.19}$$

Subcase 1.3.1. Suppose that $d_{21}=0$. Then $d_{22}\neq 0$. By (3.15) we have $d_{12}=0$. Using (3.16),(3.18) and (3.19) we obtain $\delta_3=\delta_1\delta_2^{-1}$.

Subcase 1.3.2. Suppose that $d_{21}\neq 0$. Then by (3.16) and (3.18) we have $2d_{11}=\delta_1+d\delta_3\delta_1^{-1}$ and $2d_{12}=\delta_1-d\delta_3\delta_1^{-1}$. Similarly, by (3.17) and (3.19) we have $2d_{22}=\delta_2+d\delta_3^{-1}\delta_2^{-1}$ and $2d_{21}=\delta_2-d\delta_3^{-1}\delta_2^{-1}$. Then, by (3.15) we can obtain

$$\delta_3^2=\delta_1^2\delta_2^{-2} \tag{3.20}$$

and

$$4d=4d_{11}d_{22}+d_{12}d_{21}=(\delta_1+d\delta_3\delta_1^{-1})(\delta_2+d\delta_3^{-1}\delta_2^{-1})-(\delta_1-d\delta_3\delta_1^{-1})(\delta_2-d\delta_3^{-1}\delta_2^{-1}).$$

Thus,

$$\delta_1\delta_2^{-1}\delta_3^{-1}+\delta_2\delta_3\delta_1^{-1}=2. \tag{3.21}$$

Hence, by (3.20) and (3.21) we obtain $\delta_3=\delta_1\delta_2^{-1}$.

Case 2. If $b_{11}=0$ and $b_{22}\neq 0$, then $b_{21}\neq 0$. By the (1,2)-entry and (2,2)-entry of A^{-1} and G, we have $\rho_2^{-1}=\rho_3=\rho_1$. Thus, by Corollary 2.3 we know $\phi\in\mathrm{Aut}(\Gamma(0))$.

Case 3. If $b_{11}=b_{22}=0$, then we can deduce that

$$A=\begin{pmatrix}-1&0&0\\0&0&\tau_3\\0&\tau_3^{-1}&0\end{pmatrix},\quad B=\begin{pmatrix}0&\tau_1\\ \tau_2&0\end{pmatrix},\quad \tau_1,\tau_2,\tau_3\in\mathbb{F}^*.$$

In a similar way to Case 1, we obtain $\phi\in\mathrm{Aut}(\Gamma(0))$. □

THEOREM 3.3. $\mathrm{LAut}(\Gamma(1))=\mathrm{Aut}(\Gamma(1))$.

Proof. Suppose that $\phi\in\mathrm{LAut}(\Gamma(1))$. Then by Lemma 3.1 and the proof of Theorem 3.2, we can assume the matrix of ϕ is

$$A=b^{-1}\begin{pmatrix}b_{11}b_{22}+b_{12}b_{21}&-b_{11}b_{21}&b_{12}b_{22}\\-2b_{11}b_{12}&b_{11}^2&-b_{12}^2\\2b_{21}b_{22}&-b_{21}^2&b_{22}^2\end{pmatrix}\oplus\begin{pmatrix}b_{11}&b_{12}\\b_{21}&b_{22}\end{pmatrix}$$

where $b=\det(b_{ij})\neq 0$.

By definition of local automorphism, there exists an automorphism $\phi_{f+\omega_{-1}}$ such that

$$\phi(f+\omega_{-1}) = \phi_{f+\omega_{-1}}(f+\omega_{-1}). \tag{3.22}$$

By Corollary 2.3, we assume that the matrix of $\phi_{f+\omega_{-1}}$ is

$$\begin{pmatrix} c_{11}c_{22}+c_{12}c_{21} & -c_{11}c_{21} & c_{12}c_{22} \\ -2c_{11}c_{12} & c_{11}^2 & -c_{12}^2 \\ 2c_{21}c_{22} & -c_{21}^2 & c_{22}^2 \end{pmatrix} \oplus \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix},$$

where $\det(c_{ij}) = 1$. Then, by (3.22) we have

$$b^{-1}b_{12}^2 = c_{12}^2, \ \ b^{-1}b_{22}^2 = c_{22}^2, \ \ b_{12} = c_{12}, \ \ b_{22} = c_{22}.$$

Thus, $b = 1$. By Corollary 2.3, $\phi \in \mathrm{Aut}(\Gamma(1))$. $\square$

4. Superderivations of $\Gamma(0)$ and $\Gamma(1)$

In this section, $\mathbb{F}$ is a field of characteristic different from 2 and 3.

THEOREM 4.1. *A linear transformation of $\Gamma(0)$ is a superderivation if and only if its matrix is of the form*

$$\begin{pmatrix} 0 & -b & c & 0 & 0 \\ -2c & -a & 0 & 0 & 0 \\ 2b & 0 & a & 0 & 0 \\ \theta & d & 0 & \delta & c \\ -d & 0 & \theta & b & \delta+a \end{pmatrix}, \tag{4.1}$$

where $a,b,c,d,\delta,\theta \in \mathbb{F}$.

Proof. Regard $\mathfrak{g}$ as a $\mathfrak{g}$-module, by [6, Lemma 2.1], any superderivation of $\mathfrak{g}$ is the sum of a zero weight-derivation and an inner superderivation. It is easy to see that $\mathfrak{g}_0 = \langle h \rangle$ is the Cartan subalgebra of $\mathfrak{g}_{\bar{0}}$. Suppose that ε is the dual basis of $\{h\}$. Then

$$\mathfrak{g}_{-2\varepsilon} = \langle f \rangle, \ \mathfrak{g}_{-\varepsilon} = \langle \omega_{-1} \rangle, \ \mathfrak{g}_{\varepsilon} = \langle \omega_1 \rangle, \ \mathfrak{g}_{2\varepsilon} = \langle e \rangle,$$

and the weight space decomposition of $\mathfrak{g}$ is $\mathfrak{g} = \mathfrak{g}_{-2\varepsilon} \oplus \mathfrak{g}_{-\varepsilon} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{\varepsilon} \oplus \mathfrak{g}_{2\varepsilon}$. By direct calculation, the matrix of any zero weight-derivation is of the form $\mathrm{diag}(0,k,-k,l,l-k)$, and the matrix of any inner superderivation is of the form

$$\begin{pmatrix} 0 & -x_3 & x_2 & 0 & 0 \\ -2x_2 & 2x_1 & 0 & 0 & 0 \\ 2x_3 & 0 & -2x_1 & 0 & 0 \\ -x_4 & -x_5 & 0 & x_1 & x_2 \\ x_5 & 0 & -x_4 & x_3 & -x_1 \end{pmatrix},$$

where $k,l,x_i \in \mathbb{F}$, $i=1,2,\cdots,5$. Thus, we deduce that the matrix of any superderivation of $\mathfrak{g}$ is of the form (4.1), where $a=-2x_1-k$, $b=x_3$, $c=x_2$, $d=-x_5$, $\theta=-x_4$, $\delta=x_1+l$.

Conversely, if the matrix of linear transformation ϕ of $\Gamma(0)$ is of the form (4.1), then it is easy to verify that $\phi \in \mathrm{Der}(\Gamma(0))$ by direct calculation. $\square$

THEOREM 4.2. $\mathrm{LDer}(\Gamma(0)) = \mathrm{Der}(\Gamma(0))$.

Proof. Suppose that $\phi \in \mathrm{LDer}_{\bar{0}}(\Gamma(0))$. Then for any $x \in \Gamma(0)$, there exists $\phi_x \in \mathrm{Der}(\Gamma(0))$ such that

$$\phi(x) = \phi_x(x). \tag{4.2}$$

Suppose that the matrix of ϕ and ϕ_x are $A \oplus B$ and $\begin{pmatrix} A_x & C_x \\ D_x & B_x \end{pmatrix}$ respectively, where $A=(a_{ij})_{3\times 3}$, $B=(b_{ij})_{2\times 2}$ and

$$A_x = \begin{pmatrix} 0 & -b_x & c_x \\ -2c_x & -a_x & 0 \\ 2b_x & 0 & a_x \end{pmatrix}, \quad C_x = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad D_x = \begin{pmatrix} \theta_x & d_x & 0 \\ -d_x & 0 & \theta_x \end{pmatrix}, \quad B_x = \begin{pmatrix} \delta_x & c_x \\ b_x & \delta_x + a_x \end{pmatrix}.$$

Substituting x in (4.2) with h, we have

$$\begin{pmatrix} A & \\ & B \end{pmatrix} e_1 = \begin{pmatrix} A_h & C_h \\ D_h & B_h \end{pmatrix} e_1,$$

where e_1 is the unit vector with 1 in the 1-th entry and 0 elsewhere. Then

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & -b_h & c_h \\ -2c_h & -a_h & 0 \\ 2b_h & 0 & a_h \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Thus, $a_{11}=0$. Similarly, substituting x in (4.2) with f and e respectively, we have $a_{23}=a_{32}=0$. To make it easier to see the goal, we denote

$$A \oplus B = \begin{pmatrix} 0 & -b_1 & c_1 \\ -2c_2 & -a_1 & 0 \\ 2b_2 & 0 & a_2 \end{pmatrix} \oplus \begin{pmatrix} e & c_3 \\ b_3 & k \end{pmatrix}.$$

By Theorem 4.1, to prove $\phi \in \mathrm{Der}_{\bar{0}}(\Gamma(0))$, we only need to show that

$$a_1=a_2, \;\; b_1=b_2=b_3, \;\; c_1=c_2=c_3, \;\; k=e+a_1.$$

Substituting x in (4.2) with $e+f$, then

$$\begin{pmatrix} 0 & -b_1 & c_1 \\ -2c_2 & -a_1 & 0 \\ 2b_2 & 0 & a_2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & -b_{e+f} & c_{e+f} \\ -2c_{e+f} & -a_{e+f} & 0 \\ 2b_{e+f} & 0 & a_{e+f} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

Thus, $-a_1 = -a_{e+f}$, $a_2 = a_{e+f}$. Therefore, $a_1 = a_2$. Similarly, substituting x in (4.2) with the following vectors

$$h+e,\ \ h+f,\ \ f+\omega_{-1},\ \ e+\omega_1,$$

respectively, we have

$$b_1 = b_2,\ \ c_1 = c_2,\ \ c_1 = c_3,\ \ b_1 = b_3.$$

Finally, substituting x in (4.2) with $h-e+f+\omega_1+\omega_{-1}$, we obtain $k = e+a_1$.

Suppose that $\psi \in \mathrm{LDer}_{\bar{1}}(\Gamma(0))$. Then for any $x \in \Gamma(0)$, there exists $\phi_x \in \mathrm{Der}(\Gamma(0))$ such that

$$\psi(x) = \phi_x(x). \tag{4.3}$$

In a similar way as above, by Theorem 4.1 and (4.3), we can assume that the matrix of ψ is

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \overline{\oplus} \begin{pmatrix} \theta_1 & d_1 & 0 \\ -d_2 & 0 & \theta_2 \end{pmatrix}.$$

Substituting x in (4.3) with $h-e+f$ and $h+e-f$ respectively, we conclude that $d_1 = d_2$ and $\theta_1 = \theta_2$. By Theorem 4.1, $\psi \in \mathrm{Der}_{\bar{1}}(\Gamma(0))$. $\square$

Next, we consider the case of $d \neq 0$.

PROPOSITION 4.3. $\mathrm{ad}(\Gamma(d))$ *is isomorphic to* $\Gamma(d)$ *as a Lie superalgebra.*

Proof. It is obvious because of the injectivity of $\mathrm{ad}: \Gamma(d) \to \mathrm{Der}\Gamma(d)$. $\square$

By direct calculation we have the following conclusion.

LEMMA 4.4. *Suppose that* φ *is a linear transformation of* $\Gamma(1)$. *Then* $\varphi \in \mathrm{ad}(\Gamma(1))$ *if and only if its matrix is of the form*

$$\begin{pmatrix} 0 & -b & c & d & \theta \\ -2c & -2a & 0 & -2\theta & 0 \\ 2b & 0 & 2a & 0 & 2d \\ \theta & d & 0 & -a & c \\ -d & 0 & \theta & b & a \end{pmatrix},$$

where $a,b,c,d,\theta \in \mathbb{F}$.

PROPOSITION 4.5. $\mathrm{Der}(\Gamma(1)) = \mathrm{ad}(\Gamma(1))$.

Proof. By Lemma 4.4, it is easy to prove that the Killing form of $\Gamma(1)$ is non-degenerate, and therefore every superderivation of $\Gamma(1)$ is inner. $\square$

THEOREM 4.6. $\mathrm{LDer}(\Gamma(1)) = \mathrm{Der}(\Gamma(1))$.

Proof. Suppose that $\phi \in \mathrm{LDer}_{\bar{0}}(\Gamma(1))$. Then for any $x \in \Gamma(1)$, there exists $\varphi_x \in \mathrm{Der}(\Gamma(1))$ such that

$$\phi(x) = \varphi_x(x). \tag{4.4}$$

By Proposition 4.5, Lemma 4.4 and (4.4), we can assume that the matrix of ϕ is

$$\begin{pmatrix} 0 & -b_1 & c_1 \\ -2c_2 & -2a_1 & 0 \\ 2b_2 & 0 & 2a_2 \end{pmatrix} \oplus \begin{pmatrix} -a_3 & c_3 \\ b_3 & a_4 \end{pmatrix}.$$

Substituting x in (4.4) with the following vectors

$$e+f,\ h+e,\ h+f,$$

respectively, we have $a_1 = a_2$, $b_1 = b_2$ and $c_1 = c_2$. Similarly, substituting x in (4.4) with the following vectors

$$f+\omega_1,\ \ h+\omega_1,\ \ h+\omega_{-1},\ \ e+\omega_{-1},$$

respectively, we have

$$a_1 = a_3,\ \ b_1 = b_3,\ \ c_1 = c_3,\ \ a_1 = a_4.$$

By Proposition 4.5 and Lemma 4.4, $\phi \in \mathrm{Der}_{\bar{0}}(\Gamma(1))$.

Suppose that $\psi \in \mathrm{LDer}_{\bar{1}}(\Gamma(1))$. Then for any $x \in \Gamma(1)$, there exists $\varphi_x \in \mathrm{Der}(\Gamma(1))$ such that

$$\psi(x) = \varphi_x(x). \tag{4.5}$$

By Proposition 4.5, Lemma 4.4 and (4.5), we can assume that the matrix of ψ is

$$\begin{pmatrix} d_1 & \theta_1 \\ -2\theta_2 & 0 \\ 0 & 2d_2 \end{pmatrix} \overline{\oplus} \begin{pmatrix} \theta_3 & d_3 & 0 \\ -d_4 & 0 & \theta_4 \end{pmatrix}.$$

Substituting x in (4.3) with the following vectors

$$h+\omega_1,\ \ h+\omega_{-1},\ \ \omega_1+\omega_{-1},\ \ \omega_1+2\omega_{-1},\ \ e+f+\omega_1+\omega_{-1},\ \ 4e+f+4\omega_1+2\omega_{-1},$$

respectively, we obtain the following equations,

$$d_1 = d_4,\ \ \theta_1 = \theta_3,\ \ d_1+\theta_1 = d_2+\theta_2,\ \ d_1+2\theta_1 = d_2+2\theta_2,$$

$$d_1+\theta_4 = \theta_1+d_3,\ \ 2d_1-\theta_1 = 2d_3-\theta_4.$$

Thus, $d_1 = d_2 = d_3 = d_4$, $\theta_1 = \theta_2 = \theta_3 = \theta_4$. By Proposition 4.5 and Lemma 4.4, $\psi \in \mathrm{Der}_{\bar{1}}(\Gamma(1))$. □

REMARK 4.7. In this section, the condition that the characteristic of field $\mathbb{F}$ is not 3 is only used to prove the non-degeneracy of killing type of $\Gamma(1)$. So the conclusions about $\Gamma(0)$ in this section also hold when the characteristic of $\mathbb{F}$ is not 2.

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