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REMARKS ON THE PRODUCT OF TWO PROJECTIONS

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Abstract. In this paper we investigate complex projections A and B so that AB is a diagonalizable matrix. Particularly, we provide necessary and/or sufficient conditions so that AB is a diagonalizable matrix with its eigenvalues belonging to the real segment $[0,1]$. Moreover, we investigate on eigenspaces and eigenvalues of the product of two projections.

1. Introduction

Throughout this paper, the matrices used are complex of order n. The symbols A^*, $\mathrm{Tr}(A)$, $\sigma(A)$, $\mathrm{Im}(A)$, $\mathrm{Ker}(A)$, α_A, δ_A and σ_A denote the conjugate transpose, the trace, the spectrum, the range, the null space, the algebraic multiplicity of zero as eigenvalue, the number of eigenvalues from $\mathbb{C}\setminus\{0,1\}$ and the number of singular values from $\mathbb{R}\setminus\{0,1\}$, respectively, of some matrix A. A matrix A is called an EP matrix if $\mathrm{Im}(A)=\mathrm{Im}(A^*)$, or equivalently if $\mathrm{Im}(A)=(\mathrm{Ker}(A))^{\perp}$. More generally, a matrix A is called a core matrix, that is, a matrix of index one, if $\mathrm{Im}(A)\cap\mathrm{Ker}(A)=\{0\}$, or equivalently if $\mathrm{Im}(A)\oplus\mathrm{Ker}(A)=\mathbb{C}^{n\times 1}$. Particularly, a matrix A is called a projection if $A^2=A$. We denote $\mathbb{C}_P^{n\times n}$, $\mathbb{C}_{HP}^{n\times n}$, $\mathbb{C}_D^{n\times n}$, $\mathbb{C}_N^{n\times n}$, $\mathbb{C}_{EP}^{n\times n}$ and $\mathbb{C}_U^{n\times n}$ the sets of all the projections, of all the Hermitian projections, of all the diagonalizable matrices, of all the normal matrices, of all the EP matrices and of all the unitary matrices, respectively.

Clearly, if A and B are projections, then A and B are diagonalizable matrices, but in general, neither AB nor BA are diagonalizable matrices. Note that if A is a diagonalizable matrix, then A is a core matrix because $\mathbb{C}^{n\times 1}=\mathrm{Ker}(A)\oplus\mathrm{Ker}(A-\lambda_1 I)\oplus\ldots\oplus\mathrm{Ker}(A-\lambda_k I)$, with $\lambda_1,\ldots,\lambda_k\in\sigma(A)\setminus\{0\}$ and $\mathrm{Im}(A)=\mathrm{Ker}(A-\lambda_1 I)\oplus\ldots\oplus\mathrm{Ker}(A-\lambda_k I)$. We shall also use a definition of the polar decomposition of a complex matrix A: Any singular complex matrix A can be represented in the form $A=UP$, where P is a Hermitian nonnegative definite matrix $(P\geqslant 0)$ and U is a unitary matrix. If A is nonsingular such a representation is unique, and so P is a Hermitian positive definite matrix $(P>0)$. Moreover, we shall use some information concerning the Moore-Penrose inverse for some $A\in\mathbb{C}^{m\times k}$: Recall that the Moore-Penrose inverse $A^{\dagger}$ is the unique matrix which satisfies $AA^{\dagger}A=A$, $A^{\dagger}AA^{\dagger}=A^{\dagger}$, $(AA^{\dagger})^*=AA^{\dagger}$ and $(A^{\dagger}A)^*=A^{\dagger}A$.

In this paper, we continue the investigations carried out in [5, section 3] on the product of two projections A and B. Thus, in section 2, given $A,B\in\mathbb{C}_P^{n\times n}$, we

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carry out some investigation on the eigenspaces and eigenvalues of AB. We start section 2 with our first main result which establishes $\operatorname{Ker}(I-AB)=\operatorname{Im}(A)\cap(\operatorname{Im}(B)\oplus(\operatorname{Ker}(A)\cap\operatorname{Ker}(B)))$ whenever $AB\in\mathbb{C}_D^{n\times n}$. Taking into account that, by [5, Remark 3], $\delta_{AB}\leqslant\min\{\dim\operatorname{Ker}(A),\dim\operatorname{Ker}(B)\}$, we shall show, throughout section 2, some results refining this last result. Moreover, we shall show results that provide a necessary and/or sufficient condition so that $\delta_{AB}=0$ or $\delta_{AB}=\operatorname{Tr}(A)$.

In section 3, we take up, above all, with the following question: Once a projection A is fixed, we investigate projections B so that AB is a diagonalizable matrix with $\sigma(AB)\subset[0,1]$ or with arbitrary spectrum. Moreover, we shall show results that provide a necessary and/or sufficient condition so that AB is diagonalizable, where A and B are projections with some restrictions. Particularly, in [7, Theorem 1], for example, Groß and Trenkler provided a necessary and sufficient condition so that AB is a projection whenever A and B are projections, and in this case $\delta_{AB}=0$. In this section, our main result is the Theorem 3.1 that takes up with the following problem: Once fixed a Hermitian projection A and given a projection B, the normality of AB implies that AB is a Hermitian projection, and soon after, Remark 6 characterizes such projections B.

2. On eigenspaces and eigenvalues of the product of two projections

For any two projections A and B of same order, by [10, Corollary 9], we have that $\operatorname{Im}(AB)=\operatorname{Im}(A)\cap(\operatorname{Im}(B)+\operatorname{Ker}(A))$. Particularly, in our first main result, we shall prove that $\operatorname{Ker}(I-AB)=\operatorname{Im}(A)\cap(\operatorname{Im}(B)\oplus(\operatorname{Ker}(A)\cap\operatorname{Ker}(B)))$ whenever AB is diagonalizable, and for that we shall make use of the following lemma:

LEMMA 2.1. *If* $A,B\in\mathbb{C}_P^{n\times n}$, *then* $\dim\operatorname{Ker}(I-AB)=\dim(\operatorname{Im}(A)\cap\operatorname{Im}(B))+\dim((\operatorname{Im}(A)+\operatorname{Im}(B))\cap\operatorname{Ker}(A)\cap\operatorname{Ker}(B))$.

Proof. Let W and U be two subspaces such that $W\oplus\operatorname{Im}(A)\cap\operatorname{Im}(B)=\operatorname{Ker}(I-AB)$ and $U\oplus\operatorname{Im}(A)\cap\operatorname{Im}(B)=\operatorname{Ker}(I-BA)$. Consider $v=w+u\in(\operatorname{Im}(A)+\operatorname{Im}(B))\cap\operatorname{Ker}(A)\cap\operatorname{Ker}(B)$, where $w\in\operatorname{Im}(A)$ and $u\in\operatorname{Im}(B)$, and so $Aw=w$, $Bu=u$ and $Av=Bv=0$. Hence, $Av=w+Au=0$ and $Bv=Bw+u=0$, which implies $ABw=w$ and $BAu=u$. If $w,u\in\operatorname{Im}(A)\cap\operatorname{Im}(B)$, then clearly $v=0$. Thus, let $v=w+u=w-Bw=(I-B)w$, with $w\in W$ and $u\in U$. Since $\operatorname{Im}(B)\cap W=\operatorname{Ker}(I-B)\cap W=\{0\}$, it follows that $\dim((\operatorname{Im}(A)+\operatorname{Im}(B))\cap\operatorname{Ker}(A)\cap\operatorname{Ker}(B))\leqslant\dim W$.

Conversely, let $v=w+u$, where $ABw=w$ and $BAu=u$ for all $w\in W$ and $u\in U$, hence $Av=w+Au=ABw+ABu=ABv$, and so $A(I-B)v=0$, which implies $(I-B)v\in\operatorname{Ker}(A)\cap\operatorname{Ker}(B)$. Since $(I-B)v=w+u-Bw-u=(I-B)w\in\operatorname{Im}(A)+\operatorname{Im}(B)$, it follows that $(I-B)w\in(\operatorname{Im}(A)+\operatorname{Im}(B))\cap\operatorname{Ker}(A)\cap\operatorname{Ker}(B)$, which implies $\dim W\leqslant\dim((\operatorname{Im}(A)+\operatorname{Im}(B))\cap\operatorname{Ker}(A)\cap\operatorname{Ker}(B))$, and therefore $\dim W=\dim((\operatorname{Im}(A)+\operatorname{Im}(B))\cap\operatorname{Ker}(A)\cap\operatorname{Ker}(B))$. □

REMARK 1. According to Lemma 2.1 and keeping in mind that $(\operatorname{Im}(A)\cap\operatorname{Im}(B))\subset\operatorname{Ker}(I-AB)$, we may conclude that $\operatorname{Ker}(I-AB)=\operatorname{Im}(A)\cap\operatorname{Im}(B)$ if and only if $(\operatorname{Im}(A)+\operatorname{Im}(B))\cap\operatorname{Ker}(A)\cap\operatorname{Ker}(B)=\{0\}$.

THEOREM 2.2. *If* $A,B \in \mathbb{C}_P^{n\times n}$ *and* $AB \in \mathbb{C}_D^{n\times n}$, *then* $\mathrm{Ker}(I-AB) = \mathrm{Im}(A)\cap(\mathrm{Im}(B)\oplus(\mathrm{Ker}(A)\cap\mathrm{Ker}(B)))$.

Proof. Clearly, $(\mathrm{Im}(A)\cap\mathrm{Im}(B)) \subset \mathrm{Ker}(I-AB)$ and $(\mathrm{Im}(A)\cap\mathrm{Im}(B)) \subset \mathrm{Im}(A)\cap(\mathrm{Im}(B)\oplus(\mathrm{Ker}(A)\cap\mathrm{Ker}(B)))$. Now, note that $\mathrm{Im}(A)\cap(\mathrm{Im}(B)\oplus(\mathrm{Ker}(A)\cap\mathrm{Ker}(B))) \subset \mathrm{Im}(A)\cap(\mathrm{Im}(B)+\mathrm{Ker}(A)) = \mathrm{Im}(AB) = \mathrm{Ker}(I-AB)\oplus\mathrm{Ker}(\lambda_1 I-AB)\oplus\ldots\oplus\mathrm{Ker}(\lambda_k I-AB)$ since AB is diagonalizable, where $\lambda_1,\ldots,\lambda_k \in \sigma(AB)\cap\mathbb{C}\setminus\{0,1\}$. Thus, consider $v \in \mathbb{C}^{n\times 1}$ and $\lambda \in \mathbb{C}\setminus\{0\}$ so that $ABv = \lambda v$ and $v = w+u$, where $v \in \mathrm{Im}(A)$, $w \in \mathrm{Im}(B)$ and $u \in \mathrm{Ker}(A)\cap\mathrm{Ker}(B)$. Hence, $Av = v = Aw+Au = Aw$. Moreover, $ABv = \lambda v = ABw+ABu = Aw = v$, which implies $\lambda = 1$, and so we conclude that $\mathrm{Im}(A)\cap(\mathrm{Im}(B)\oplus(\mathrm{Ker}(A)\cap\mathrm{Ker}(B))) \subset \mathrm{Ker}(I-AB)$.

In order to conclude that $\mathrm{Ker}(I-AB) = \mathrm{Im}(A)\cap(\mathrm{Im}(B)\oplus(\mathrm{Ker}(A)\cap\mathrm{Ker}(B)))$, it suffices to prove that, taking into account Lemma 2.1, $\dim\mathrm{Ker}(I-AB) = \dim(\mathrm{Im}(A)\cap\mathrm{Im}(B)) + \dim((\mathrm{Im}(A)+\mathrm{Im}(B))\cap\mathrm{Ker}(A)\cap\mathrm{Ker}(B)) = \dim(\mathrm{Im}(A)\cap(\mathrm{Im}(B)\oplus(\mathrm{Ker}(A)\cap\mathrm{Ker}(B))))$. Indeed, $\dim(\mathrm{Im}(A)+\mathrm{Im}(B)+(\mathrm{Ker}(A)\cap\mathrm{Ker}(B))) = \dim(\mathrm{Im}(A)+\mathrm{Im}(B)) + \dim(\mathrm{Ker}(A)\cap\mathrm{Ker}(B)) - \dim((\mathrm{Im}(A)+\mathrm{Im}(B))\cap\mathrm{Ker}(A)\cap\mathrm{Ker}(B)) = \dim\mathrm{Im}(A) + \dim\mathrm{Im}(B) - \dim(\mathrm{Im}(A)\cap\mathrm{Im}(B)) + \dim(\mathrm{Ker}(A)\cap\mathrm{Ker}(B)) - \dim((\mathrm{Im}(A)+\mathrm{Im}(B))\cap\mathrm{Ker}(A)\cap\mathrm{Ker}(B))$, which implies $\dim(\mathrm{Im}(A)\cap\mathrm{Im}(B)) + \dim((\mathrm{Im}(A)+\mathrm{Im}(B))\cap\mathrm{Ker}(A)\cap\mathrm{Ker}(B)) = \dim\mathrm{Ker}(I-AB) = \dim\mathrm{Im}(A) + \dim\mathrm{Im}(B) + \dim(\mathrm{Ker}(A)\cap\mathrm{Ker}(B)) - \dim(\mathrm{Im}(A)+\mathrm{Im}(B)+(\mathrm{Ker}(A)\cap\mathrm{Ker}(B)))$. On the other hand, we have that $\dim(\mathrm{Im}(A)+\mathrm{Im}(B)+(\mathrm{Ker}(A)\cap\mathrm{Ker}(B))) = \dim\mathrm{Im}(A) + \dim(\mathrm{Im}(B)+(\mathrm{Ker}(A)\cap\mathrm{Ker}(B))) - \dim(\mathrm{Im}(A)\cap(\mathrm{Im}(B)\oplus(\mathrm{Ker}(A)\cap\mathrm{Ker}(B)))) = \dim\mathrm{Im}(A) + \dim(\mathrm{Im}(B)) + \dim(\mathrm{Ker}(A)\cap\mathrm{Ker}(B)) - \dim(\mathrm{Im}(B)\cap\mathrm{Ker}(A)\cap\mathrm{Ker}(B)) - \dim(\mathrm{Im}(A)\cap(\mathrm{Im}(B)\oplus(\mathrm{Ker}(A)\cap\mathrm{Ker}(B)))) = \dim\mathrm{Im}(A) + \dim(\mathrm{Im}(B) + \dim(\mathrm{Ker}(A)\cap\mathrm{Ker}(B)) - \dim(\mathrm{Im}(A)\cap(\mathrm{Im}(B)\oplus(\mathrm{Ker}(A)\cap\mathrm{Ker}(B))))$, and therefore we may conclude that

$$\dim\mathrm{Ker}(I-AB) = \dim(\mathrm{Im}(A)\cap(\mathrm{Im}(B)\oplus(\mathrm{Ker}(A)\cap\mathrm{Ker}(B)))). \quad \square$$

Let $c_{AB}(x) = x^{m_o}(x-1)^{m_1}(x-\lambda_2)^{m_2}\ldots(x-\lambda_{k+1})^{m_{k+1}}$ and $m_{AB}(x) = x^{n_o}(x-1)^{n_1}(x-\lambda_2)^{n_2}\ldots(x-\lambda_{k+1})^{n_{k+1}}$ be the characteristic and minimal polynomial, respectively, of AB, where A and B are projections of order n and $\lambda_2,\ldots,\lambda_{k+1} \in \mathbb{C}\setminus\{0,1\}$.

PROPOSITION 2.3. *Let* $A,B \in \mathbb{C}_P^{n\times n}$. *Thus, if* $c_{AB}(x) = x^{m_o}(x-1)^{m_1}(x-\lambda_2)^{m_2}\ldots(x-\lambda_{k+1})^{m_{k+1}}$ *and* $m_{AB}(x) = x^{m_o}(x-1)^{n_1}(x-\lambda_2)^{n_2}\ldots(x-\lambda_{k+1})^{n_{k+1}}$ *are the characteristic and minimal polynomial, respectively, of* AB, *then* $\delta_{AB} \leqslant 1$, *which implies* $k=1$ *and* $m_2 \leqslant 1$.

Proof. By hypothesis, clearly $\dim\mathrm{Ker}(AB) = 1$, and as $\dim\mathrm{Ker}(AB) = \dim(\mathrm{Ker}(A)\cap\mathrm{Im}(B)) + \dim\mathrm{Ker}(B)$ and $\dim\mathrm{Ker}(B) \geqslant 1$, it follows that $\dim\mathrm{Ker}(B) = 1$, and as $\delta_{AB} \leqslant \min\{\dim\mathrm{Ker}(A), \dim\mathrm{Ker}(B)\}$, we conclude that $\delta_{AB} \leqslant 1$, which implies $k=1$ and $m_2 \leqslant 1$, that is, AB has at most three distinct eigenvalues. $\square$

Now, take into account the following information: Let W_1, W_2, W_3 and W_4 be subspaces from $\mathbb{C}^{n\times 1}$ so that $W_1\oplus(\mathrm{Im}(A)\cap\mathrm{Im}(B))\oplus(\mathrm{Im}(A)\cap\mathrm{Ker}(B)) = \mathrm{Im}(A)$, $W_2\oplus(\mathrm{Im}(A)\cap\mathrm{Im}(B))\oplus(\mathrm{Im}(B)\cap\mathrm{Ker}(A)) = \mathrm{Im}(B)$, $W_3\oplus(\mathrm{Im}(B)\cap\mathrm{Ker}(A))\oplus(\mathrm{Ker}(A)\cap\mathrm{Ker}(B)) = \mathrm{Ker}(A)$ and $W_4\oplus(\mathrm{Im}(A)\cap\mathrm{Ker}(B))\oplus(\mathrm{Ker}(A)\cap\mathrm{Ker}(B)) = \mathrm{Ker}(B)$, where A and B are matrices of index one of order n.

LEMMA 2.4. *Let* $A,B \in \mathbb{C}_P^{n\times n}$. *Thus,* $\operatorname{rank}(AB) = \operatorname{rank}(BA)$ *and* $\operatorname{rank}(A(I-B)) = \operatorname{rank}((I-B)A)$ *if and and only if* $\dim W_1 = \dim W_2 = \dim W_3 = \dim W_4$.

Proof. According to [10, Corollary 9], $\operatorname{Ker}(AB) = \operatorname{Ker}(B) \oplus (\operatorname{Im}(B) \cap \operatorname{Ker}(A))$, which implies $n - \operatorname{rank}(AB) = \dim \operatorname{Ker}(AB) = \dim \operatorname{Ker}(B) + \dim(\operatorname{Im}(B) \cap \operatorname{Ker}(A))$, hence $n - \operatorname{rank}(AB) + \dim(\operatorname{Im}(A) \cap \operatorname{Im}(B)) + \dim W_2 = \dim \operatorname{Ker}(B) + \dim(\operatorname{Im}(B) \cap \operatorname{Ker}(A)) + \dim(\operatorname{Im}(A) \cap \operatorname{Im}(B)) + \dim W_2 = n$, and so $\operatorname{rank}(AB) = \dim(\operatorname{Im}(A) \cap \operatorname{Im}(B)) + \dim W_2$. Similarly, we have that $\operatorname{rank}(BA) = \dim(\operatorname{Im}(A) \cap \operatorname{Im}(B)) + \dim W_1$, $\operatorname{rank}(A(I-B)) = \dim(\operatorname{Im}(A) \cap \operatorname{Ker}(B)) + \dim W_4$ and $\operatorname{rank}((I-B)A) = \dim(\operatorname{Im}(A) \cap \operatorname{Ker}(B)) + \dim W_1$.

This implies that if $\operatorname{rank}(AB) = \operatorname{rank}(BA)$ and $\operatorname{rank}(A(I-B)) = \operatorname{rank}((I-B)A)$, then $\dim(\operatorname{Im}(A) \cap \operatorname{Im}(B)) + \dim W_2 = \dim(\operatorname{Im}(A) \cap \operatorname{Im}(B)) + \dim W_1$ and $\dim(\operatorname{Im}(A) \cap \operatorname{Ker}(B)) + \dim W_4 = \dim(\operatorname{Im}(A) \cap \operatorname{Ker}(B)) + \dim W_1$, which implies $\dim W_1 = \dim W_2 = \dim W_4$, and as $\dim W_1 + \dim W_3 = \dim W_2 + \dim W_4$, see [5, Lemma 3.1], we have that $\dim W_3 = \dim W_1$.

Conversely, if $\dim W_1 = \dim W_2 = \dim W_3 = \dim W_4$, then, clearly, $\operatorname{rank}(AB) = \operatorname{rank}(BA)$ and $\operatorname{rank}(A(I-B)) = \operatorname{rank}((I-B)A)$. $\square$

Let $A,B \in \mathbb{C}_P^{n\times n}$. Thus, we have that $\delta_{AB} \leqslant \min\{\dim \operatorname{Ker}(A), \dim \operatorname{Ker}(B)\}$, hence $\delta_{AB} \leqslant n/2$ (take, for example, $A,B \in \mathbb{C}_{HP}^{n\times n}$ with $\lambda_1, \dots, \lambda_{n/2} \in (0,1)$ eigenvalues of AB and $\operatorname{Im}(A) \cap \operatorname{Im}(B) = \{0\}$, $\operatorname{Im}(A) \cap \operatorname{Ker}(B) = \{0\}$, $\operatorname{Im}(B) \cap \operatorname{Ker}(A) = \{0\}$, $\operatorname{Ker}(A) \cap \operatorname{Ker}(B) = \{0\}$ and $\dim \operatorname{Ker}(A) = \dim \operatorname{Ker}(B) = n/2$, and so, in this case, $\delta_{AB} = n/2$). Moreover, it is easy to see that $\delta_{AB} \leqslant \dim W_1$ and $\delta_{BA} \leqslant \dim W_2$, and as $\delta_{AB} = \delta_{BA}$, we have that $\delta_{AB} \leqslant \dim W_2$, and by proof of [5, Theorem 3.3], $\dim W_2 \leqslant \dim W_3 + \dim(\operatorname{Ker}(A) \cap \operatorname{Ker}(B))$, hence $\delta_{AB} \leqslant \dim W_3 + \dim(\operatorname{Ker}(A) \cap \operatorname{Ker}(B))$.

Particularly, by Lemma 2.4, if $\operatorname{rank}(AB) = \operatorname{rank}(BA)$ and $\operatorname{rank}(A(I-B)) = \operatorname{rank}((I-B)A)$, then $\delta_{AB} \leqslant \dim W_1 = \dim W_3$. In this way, the following result provides another sufficient condition so that $\delta_{AB} \leqslant \dim W_3$.

PROPOSITION 2.5. *If* $A,B \in \mathbb{C}_P^{n\times n}$ *and* $(\operatorname{Ker}(A) \cap \operatorname{Ker}(B)) \subset (\operatorname{Im}(A) + \operatorname{Im}(B))$, *then* $\delta_{AB} \leqslant \dim W_3$.

Proof. According to Lemma 2.1, $\dim \operatorname{Ker}(I-AB) = \dim(\operatorname{Im}(A) \cap \operatorname{Im}(B)) + \dim((\operatorname{Im}(A) + \operatorname{Im}(B)) \cap \operatorname{Ker}(A) \cap \operatorname{Ker}(B))$, and as $(\operatorname{Ker}(A) \cap \operatorname{Ker}(B)) \subset (\operatorname{Im}(A) + \operatorname{Im}(B))$, we have that $\dim \operatorname{Ker}(I-AB) = \dim(\operatorname{Im}(A) \cap \operatorname{Im}(B)) + \dim(\operatorname{Ker}(A) \cap \operatorname{Ker}(B))$. Since $\operatorname{Ker}(I-AB) \oplus \operatorname{Ker}(\lambda_2 I - AB)^{n_2} \oplus \dots \oplus \operatorname{Ker}(\lambda_{k+1} I - AB)^{n_{k+1}} \subset \operatorname{Im}(AB)$. It follows that $\dim \operatorname{Ker}(I-AB) + \delta_{AB} = \dim(\operatorname{Im}(A) \cap \operatorname{Im}(B)) + \dim(\operatorname{Ker}(A) \cap \operatorname{Ker}(B)) + \delta_{AB} \leqslant \operatorname{rank}(AB) = \dim(\operatorname{Im}(A) \cap \operatorname{Im}(B)) + \dim W_2$ since $\delta_{AB} = \dim(\operatorname{Ker}(\lambda_2 I - AB)^{n_2} \oplus \dots \oplus \operatorname{Ker}(\lambda_{k+1} I - AB)^{n_{k+1}})$, which implies $\dim(\operatorname{Ker}(A) \cap \operatorname{Ker}(B)) + \delta_{AB} \leqslant \dim W_2$, and so, keeping in mind that $\dim W_2 \leqslant \dim W_3 + \dim(\operatorname{Ker}(A) \cap \operatorname{Ker}(B))$. we may conclude that $\delta_{AB} \leqslant \dim W_3$. $\square$

REMARK 2. Let $A,B \in \mathbb{C}_{HP}^{n\times n}$. Then $\operatorname{Im}(A) + \operatorname{Im}(B) = (\operatorname{Ker}(A^*) \cap \operatorname{Ker}(B^*))^{\perp} = (\operatorname{Ker}(A) \cap \operatorname{Ker}(B))^{\perp}$, which implies $(\operatorname{Im}(A) + \operatorname{Im}(B)) \cap \operatorname{Ker}(A) \cap \operatorname{Ker}(B) = \{0\}$, and by Lemma 2.1, $\operatorname{Ker}(I-AB) = \operatorname{Im}(A) \cap \operatorname{Im}(B)$. Moreover, since $AB, BA, A(I-B), (I-B)A \in \mathbb{C}_D^{n\times n}$, it follows that $\operatorname{rank}(AB) = \operatorname{rank}(BA)$ and $\operatorname{rank}(A(I-B)) = \operatorname{rank}((I-B)A)$, see [5, Theorem 3.7], and by Lemma 2.4, it follows that $\delta_{AB} = \dim W_1 = \dim W_3$.

Let $A \in \mathbb{C}_P^{n\times n}$. Thus, it is well known that $\sigma_A \geqslant 1$ if and only if $A \in \mathbb{C}_P^{n\times n} \setminus \mathbb{C}_{HP}^{n\times n}$. Moreover, if $\sigma_1, \dots, \sigma_k \in \mathbb{R} \setminus \{0,1\}$ are singular values of $A \in \mathbb{C}_P^{n\times n} \setminus \mathbb{C}_{HP}^{n\times n}$, then $\sigma_i > 1$ for each $i \in \{1, \dots, k\}$.

Now, consider $A \in \mathbb{C}_P^{n\times n} \setminus \mathbb{C}_{HP}^{n\times n}$. Then, by [8, Corollary 3.4.3.3], there is $U \in \mathbb{C}_U^{n\times n}$ so that $U^*AU = \operatorname{diag}(A_1, \dots, A_k, 1, \dots, 1, 0, \dots, 0)$, where $A_i = \begin{pmatrix} 1 & (\sigma_i - 1)^{\frac{1}{2}} \\ 0 & 0 \end{pmatrix}$ and $\sigma_i > 1$ is a singular value of A for each $i \in \{1, \dots, k\}$.

Consider, also, $B \in \mathbb{C}_{HP}^{n\times n}$ so that $U^*BU = \operatorname{diag}(P_1, \dots, P_s, 1, \dots, 1, 0, \dots, 0)$, where $P_i = \begin{pmatrix} a_i & \bar{b_i} \\ b_i & c_i \end{pmatrix} \in \mathbb{C}_{HP}^{2\times 2}$, $i = 1, \dots, s$ and $s \geqslant k$.

Thus, the following two results establish a relation between δ_{AB} and σ_A.

PROPOSITION 2.6. *Let $A \in \mathbb{C}_P^{n\times n} \setminus \mathbb{C}_{HP}^{n\times n}$, $B \in \mathbb{C}_{HP}^{n\times n}$ and $U \in \mathbb{C}_U^{n\times n}$ as defined above. Thus, if $b_i \in \mathbb{C} \setminus \mathbb{R}$ for each $i \in \{1, \dots, k\}$, then $\delta_{AB} = \sigma_A$.*

Proof. According to the notations above, we have that $U^*ABU = \operatorname{diag}(E_1, \dots, E_k, P_{k+1}, \dots, P_s, 1, \dots, 1, 0, \dots, 0)$, where

$$E_i = \begin{pmatrix} 1 & (\sigma_i - 1)^{\frac{1}{2}} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a_i & \bar{b_i} \\ b_i & c_i \end{pmatrix} = \begin{pmatrix} a_i + (\sigma_i - 1)^{\frac{1}{2}} b_i & \bar{b_i} + (\sigma_i - 1)^{\frac{1}{2}} c_i \\ 0 & 0 \end{pmatrix}.$$

Since $P_i \in \mathbb{C}_{HP}^{2\times 2}$, it follows that $a_i, c_i \in \mathbb{R}$. Suppose that $a_i + (\sigma_i - 1)^{\frac{1}{2}} b_i = 1$. Then $1 - a_i = (\sigma_i - 1)^{\frac{1}{2}} b_i$, but this represents a contradiction because $1 - a_i, (\sigma_i - 1)^{\frac{1}{2}} \in \mathbb{R}$ and, by hypothesis, $b_i \notin \mathbb{R}$.

Similarly, suppose that $a_i + (\sigma_i - 1)^{\frac{1}{2}} b_i = 0$. Then $a_i = -(\sigma_i - 1)^{\frac{1}{2}} b_i$, which implies a contradiction too. □

On the other hand, consider $b_i \in \mathbb{R}$ for each $i \in \{1, \dots, k\}$. Clearly, if $b_1 = b_2 = \dots = b_k = 0$, then we conclude that $\delta_{AB} = 0$. Thus, in our next result we shall take into account that $b_1, \dots, b_k \in \mathbb{R} \setminus \{0\}$.

PROPOSITION 2.7. *Let $A \in \mathbb{C}_P^{n\times n} \setminus \mathbb{C}_{HP}^{n\times n}$, $B \in \mathbb{C}_{HP}^{n\times n}$ and $U \in \mathbb{C}_U^{n\times n}$ as defined above. Thus, if $b_i \in \mathbb{R}$, $\sigma_i a_i \neq 1$ and $\sigma_i c_i \neq 1$ for each $i \in \{1, \dots, k\}$, then $\delta_{AB} = \sigma_A$.*

Proof. We have that $U^*ABU = \operatorname{diag}(E_1, \dots, E_k, P_{k+1}, \dots, P_s, 1, \dots, 1, 0, \dots, 0)$, where $E_i = \begin{pmatrix} a_i + (\sigma_i - 1)^{\frac{1}{2}} b_i & \bar{b_i} + (\sigma_i - 1)^{\frac{1}{2}} c_i \\ 0 & 0 \end{pmatrix}$.

Since $P_i \in \mathbb{C}_{HP}^{2\times 2}$, it follows that $a_i, c_i \in \mathbb{R}$, $a_i + c_i = 1$ and $a_i c_i = b_i^2$.

Suppose that $a_i + (\sigma_i - 1)^{\frac{1}{2}} b_i = 1 = a_i + c_i$. Then $c_i = (\sigma_i - 1)^{\frac{1}{2}} b_i$, which implies $a_i(\sigma_i - 1)^{\frac{1}{2}} b_i = b_i^2$, that is, $b_i = a_i(\sigma_i - 1)^{\frac{1}{2}}$. Hence, $c_i = (\sigma_i - 1)^{\frac{1}{2}} a_i (\sigma_i - 1)^{\frac{1}{2}} = (\sigma_i - 1) a_i$, and so $a_i + (\sigma_i - 1) a_i = \sigma_i a_i = 1$, but this represents a contradiction for each $i \in \{1, \dots, k\}$.

Similarly, suppose that $a_i + (\sigma_i - 1)^{\frac{1}{2}} b_i = 0$. Then $a_i = -(\sigma_i - 1)^{\frac{1}{2}} b_i$, which implies $-(\sigma_i - 1)^{\frac{1}{2}} b_i c_i = b_i^2$, that is, $b_i = -(\sigma_i - 1)^{\frac{1}{2}} c_i$. Hence, $a_i = -(\sigma_i - 1)^{\frac{1}{2}} (-(\sigma_i -$

$1)^{\frac{1}{2}}c_i) = (\sigma_i - 1)c_i$, and so $(\sigma_i - 1)c_i + c_i = \sigma_i c_i = 1$, but this represents a contradiction for each $i \in \{1,\dots,k\}$ too. □

Consider the decompositions given below for the projections A, B and C:

$$Y_A A Y_A^{-1} = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}, \; Y_B B Y_B^{-1} = \begin{pmatrix} I_s & 0 \\ 0 & 0 \end{pmatrix} \text{ and } Y_A B Y_A^{-1} = C = Y_C^{-1} \begin{pmatrix} I_s & 0 \\ 0 & 0 \end{pmatrix} Y_C = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix}, \tag{1}$$

where Y_A, Y_B and Y_C are nonsingular matrices, $\operatorname{rank}(A) = r$, $\operatorname{rank}(B) = s$ and $C_1 \in \mathbb{C}^{r\times r}$.

Moreover, consider that there is a simultaneous triangularization between two projections A and B, and so, clearly, $\delta_{AB} = 0$. Particularly, if $\operatorname{rank}(AB - BA) \leqslant 1$, then $\delta_{AB} = 0$ too, see [11, Theorem 40.5]. However, in our next result we shall characterize the projections A and B so that $\delta_{AB} = 0$.

PROPOSITION 2.8. *Let $A,B,C \in \mathbb{C}_P^{n\times n}$ be with representation in (1). Then $\delta_{AB} = 0$ if and only if $AB - BA$ is nilpotent.*

Proof. Taking into account that $Y_A AB Y_A^{-1} = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix} = \begin{pmatrix} C_1 & C_2 \\ 0 & 0 \end{pmatrix}$, we have that $\delta_{AB} = 0$ implies that $\delta_{C_1} = 0$, and so if λ is an eigenvalue of C_1, then $\lambda = 0$ or $\lambda = 1$. Since $C \in \mathbb{C}_P^{n\times n}$, it follows that $C_2C_3 = C_1 - C_1^2$. Hence, C_2C_3 is nilpotent.

Conversely, if C_2C_3 is nilpotent, then any eigenvalue of C_2C_3 is equal to zero. Thus, if λ is an eigenvalue of C_1, then $\lambda - \lambda^2$ is eigenvalue of $C_1 - C_1^2 = C_2C_3$, which implies $\lambda - \lambda^2 = 0$, hence $\lambda = 0$ or $\lambda = 1$, that is, $\delta_{C_1} = 0$, and so $\delta_{AB} = 0$.

Now, we have that $Y_A BA Y_A^{-1} = \begin{pmatrix} C_1 & 0 \\ C_3 & 0 \end{pmatrix}$, which implies $Y_A AB Y_A^{-1} - Y_A BA Y_A^{-1} = Y_A(AB - BA)Y_A^{-1} = \begin{pmatrix} C_1 & C_2 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} C_1 & 0 \\ C_3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & C_2 \\ -C_3 & 0 \end{pmatrix}$, hence $Y_A(AB - BA)^2 Y_A^{-1} = \begin{pmatrix} -C_2C_3 & 0 \\ 0 & -C_3C_2 \end{pmatrix}$.

Since C_2C_3 is nilpotent $\Leftrightarrow C_3C_2$ is nilpotent, it follows that $(AB - BA)^2$ is nilpotent $\Leftrightarrow C_2C_3$ is nilpotent, and as $(AB - BA)^2$ is nilpotent $\Leftrightarrow AB - BA$ is nilpotent, we may already conclude that $\delta_{AB} = 0 \Leftrightarrow AB - BA$ is nilpotent. □

Particularly, the following result provides a necessary and sufficient condition so that $AB \in \mathbb{C}_P^{n\times n}$ whenever $A,B \in \mathbb{C}_P^{n\times n}$.

COROLLARY 2.9. *Let $A,B,C \in \mathbb{C}_P^{n\times n}$ be with representation in (1). Then $AB \in \mathbb{C}_P^{n\times n}$ if and only if $C_2C_3 = 0$ and $\operatorname{Im}(C_2) \subset \operatorname{Im}(C_1)$.*

Proof. Consider that $Y_A AB Y_A^{-1} = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix} = \begin{pmatrix} C_1 & C_2 \\ 0 & 0 \end{pmatrix}$. If $AB \in \mathbb{C}_P^{n\times n}$, then, by [5, Theorem 2.11], $C_1 \in \mathbb{C}_P^{r\times r}$ and $\operatorname{Im}(C_2) \subset \operatorname{Im}(C_1)$. Since $C_2C_3 = C_1 - C_1^2$, it follows that $C_2C_3 = 0$.

Conversely, if $C_2C_3 = 0$, then $C_1 = C_1^2$, and as $\operatorname{Im}(C_2) \subset \operatorname{Im}(C_1)$, again by [5, Theorem 2.11], we have that $AB \in \mathbb{C}_P^{n\times n}$. □

On the other hand, the next result provides a sufficient condition so that δ_{AB} reaches its maximum value, that is, $\delta_{AB} = \dim W_1$.

PROPOSITION 2.10. *If* $A,B \in \mathbb{C}_P^{n\times n}$ *and* $AB-BA$ *is nonsingular, then* $\delta_{AB} = \operatorname{Tr}(A)$.

Proof. Let $C = AB-BA$ nonsingular Then, by [12, Corollary 2.10], $\operatorname{Im}(A)\oplus\operatorname{Im}(B) = \operatorname{Im}(A^*)\oplus\operatorname{Im}(B^*) = \mathbb{C}^{n\times 1}$ and $\operatorname{rank}(AB) = \operatorname{rank}(BA) = \operatorname{rank}(A) = \operatorname{rank}(B)$. Hence, $\operatorname{Im}(A)\cap\operatorname{Im}(B) = \{0\}$ and $(\operatorname{Im}(A^*)\oplus\operatorname{Im}(B^*))^\perp = (\operatorname{Im}(A^*))^\perp\cap(\operatorname{Im}(B^*))^\perp = \operatorname{Ker}(A)\cap\operatorname{Ker}(B) = (\mathbb{C}^{n\times 1})^\perp = \{0\}$. Since, by Lemma 2.1, $\dim\operatorname{Ker}(I-AB) = \dim(\operatorname{Im}(A)\cap\operatorname{Im}(B)) + \dim((\operatorname{Im}(A)+\operatorname{Im}(B))\cap\operatorname{Ker}(A)\cap\operatorname{Ker}(B))$, it follows that $1\notin\sigma(AB)$. Moreover, taking into account that $\dim(\operatorname{Im}(A)\cap\operatorname{Ker}(B)) = \operatorname{rank}(A)-\operatorname{rank}(BA) = \dim(\operatorname{Im}(B)\cap\operatorname{Ker}(A)) = \operatorname{rank}(B)-\operatorname{rank}(AB) = 0$, we have that $\operatorname{Im}(A)\cap\operatorname{Ker}(B) = \operatorname{Im}(B)\cap\operatorname{Ker}(A) = \{0\}$, which implies $\operatorname{Ker}(AB) = W_4$, and keeping in mind that $\operatorname{Im}(A)\cap\operatorname{Im}(B) = \{0\}$, we conclude that $W_1 = \operatorname{Im}(AB) = \operatorname{Im}(A)$, and therefore $\delta_{AB} = \operatorname{rank}(AB) = \operatorname{rank}(A) = \operatorname{Tr}(A)$. □

REMARK 3. From Corollary 3.5, we see that the requiring $AB-BA$ to be nonsingular is not necessary for the conclusion: $\delta_{AB} = \operatorname{Tr}(A)$ in Proposition 2.10.

REMARK 4. Let $A,B,C \in \mathbb{C}_P^{n\times n}$ be with representation in (1). Thus, we shall show a sufficient and necessary condition so that $\delta_{AB} = \operatorname{Tr}(A)$ whenever $\operatorname{rank}(A) = \operatorname{Tr}(A) \leqslant n/2$. Before, however, note that $\delta_{AB} \leqslant \dim(W_1) \leqslant \operatorname{rank}(A)$, and if $\delta_{AB} = \operatorname{Tr}(A) = \operatorname{rank}(A)$, then $\delta_{AB} = \dim(W_1)$ and $1\notin\sigma(AB)$. Moreover, $\operatorname{rank}(AB) \leqslant \operatorname{rank}(A) = \delta_{AB} \leqslant \operatorname{rank}(AB)$, which implies $\delta_{AB} = \operatorname{rank}(AB) = \operatorname{rank}(A)$, hence we may conclude that $AB \in \mathbb{C}_D^{n\times n}$. Thus, taking into account this information and using Proposition 3.4, in section 3, we have that $\delta_{AB} = \operatorname{Tr}(A)$ if and only if $C_2C_3 \in \mathbb{C}_D^{r\times r}$ and nonsingular whenever $\operatorname{Tr}(A) \leqslant n/2$.

We have already showed that, given A and B projections of order n, $\delta_{AB} \leqslant \dim(W_3\oplus(\operatorname{Ker}(A)\cap\operatorname{Ker}(B)))$. Now, in our last main result of this section, we shall show a more refined result, where $\delta_{AB} = \dim\operatorname{Ker}(\lambda_1 I - AB) + \ldots + \dim\operatorname{Ker}(\lambda_k I - AB) \leqslant \dim((W_3\oplus(\operatorname{Ker}(A)\cap\operatorname{Ker}(B)))\cap(W_1\oplus W_2\oplus(\operatorname{Im}(B)\cap\operatorname{Ker}(A))))$ whenever AB and BA are diagonalizable and $\lambda_1,\ldots,\lambda_k \in \sigma(AB)\setminus\{0,1\}$ distinct. However, first we shall show a preliminary result and relevant to Theorem 2.12.

PROPOSITION 2.11. *Let* $A,B \in \mathbb{C}_P^{n\times n}$. *Thus,* $(W_3\oplus(\operatorname{Ker}(A)\cap\operatorname{Ker}(B)))\cap(A\operatorname{Im}(B)+\operatorname{Im}(B)) = \{0\}$ *if and only if* $\operatorname{Im}(B) = (\operatorname{Im}(A)\cap\operatorname{Im}(B))\oplus(\operatorname{Ker}(A)\cap\operatorname{Im}(B))$.

Proof. Consider $v \in \operatorname{Im}(A)\cap(\operatorname{Im}(B)\oplus W_3\oplus(\operatorname{Ker}(A)\cap\operatorname{Ker}(B))) = \operatorname{Im}(AB)$. Hence $v = u+w$, where $v\in\operatorname{Im}(A)$, $u\in\operatorname{Im}(B)$ and $w\in W_3\oplus(\operatorname{Ker}(A)\cap\operatorname{Ker}(B))$. This implies that $Av = v = Au + Aw = Au = u+w$, hence $w = Au-u$, and so $w\in(W_3\oplus(\operatorname{Ker}(A)\cap\operatorname{Ker}(B)))\cap(A\operatorname{Im}(B)+\operatorname{Im}(B))$. Thus, if $\operatorname{Im}(B) = (\operatorname{Im}(A)\cap\operatorname{Im}(B))\oplus(\operatorname{Ker}(A)\cap\operatorname{Im}(B))$, then $u = u_1+u_2$, where $u_1\in\operatorname{Im}(A)\cap\operatorname{Im}(B)$ and $u_2\in\operatorname{Ker}(A)\cap\operatorname{Im}(B)$, which implies $w = A(u_1+u_2)-(u_1+u_2) = u_1-u_1-u_2 = -u_2$, and as $(\operatorname{Ker}(A)\cap\operatorname{Im}(B))\cap(W_3\oplus(\operatorname{Ker}(A)\cap\operatorname{Ker}(B))) = \{0\}$, we have that $w=0$.

Conversely, consider $W_2 \neq \{0\}$, $w_2\in W_2$ and $w_2\neq 0$. Hence, $u = u_1+u_2+w_2$, which implies $w = A(u_1+u_2+w_2)-(u_1+u_2+w_2) = u_1+Aw_2-u_1-u_2-w_2 =$

$Aw_2 - (u_2 + w_2)$. Since $Aw_2 \notin \operatorname{Im}(B)$ and $-(u_2+w_2) \in \operatorname{Im}(B)$, it follows that $w \neq 0$, see [5, Lemma 3.2].

Therefore, we may conclude that if $(W_3 \oplus (\operatorname{Ker}(A) \cap \operatorname{Ker}(B))) \cap (A\operatorname{Im}(B) + \operatorname{Im}(B)) = \{0\}$, then $W_2 = \{0\}$, that is, $\operatorname{Im}(B) = (\operatorname{Im}(A) \cap \operatorname{Im}(B)) \oplus (\operatorname{Ker}(A) \cap \operatorname{Im}(B))$. □

THEOREM 2.12. *If* $A, B \in \mathbb{C}_P^{n\times n}$ *and* $AB, BA \in \mathbb{C}_D^{n\times n}$, *then* $\delta_{AB} \leqslant \dim((W_3 \oplus (\operatorname{Ker}(A) \cap \operatorname{Ker}(B))) \cap (W_1 \oplus W_2 \oplus (\operatorname{Im}(B) \cap \operatorname{Ker}(A))))$.

Proof. Since $AB, BA \in \mathbb{C}_D^{n\times n}$, it follows that AB and BA are core matrices, which implies, by [5, Lemma 2.5], $\operatorname{rank}(AB) = \operatorname{rank}(BA)$, and according to proof of Lemma 2.4, $\operatorname{rank}(AB) = \dim(\operatorname{Im}(A) \cap \operatorname{Im}(B)) + \dim W_2 = \dim(\operatorname{Im}(A) \cap \operatorname{Im}(B)) + \dim W_1 = \operatorname{rank}(BA)$, hence $\dim W_1 = \dim W_2$ and $AW_2 = W_1$. Taking into account that $AB \in \mathbb{C}_D^{n\times n}$, we have that $\operatorname{Im}(AB) = (\operatorname{Im}(A) \cap \operatorname{Im}(B)) \oplus W_1 = \operatorname{Im}(A) \cap (\operatorname{Im}(B) + \operatorname{Ker}(A)) = \operatorname{Im}(A) \cap ((\operatorname{Im}(A) \cap \operatorname{Im}(B)) \oplus W_2 \oplus (\operatorname{Im}(B) \cap \operatorname{Ker}(A)) \oplus W_3 \oplus (\operatorname{Ker}(A) \cap \operatorname{Ker}(B)))$. Now, note that $\dim(\operatorname{Im}(A) \cap (W_2 \oplus (\operatorname{Im}(B) \cap \operatorname{Ker}(A)) \oplus W_3 \oplus (\operatorname{Ker}(A) \cap \operatorname{Ker}(B)))) = \operatorname{rank}(A) + \dim W_2 + \dim \operatorname{Ker}(A) - \dim(\operatorname{Im}(A) + (W_2 \oplus (\operatorname{Im}(B) \cap \operatorname{Ker}(A)) \oplus W_3 \oplus (\operatorname{Ker}(A) \cap \operatorname{Ker}(B)))) = \dim W_2 + n - n = \dim W_1$, and so we may conclude that $\operatorname{Im}(A) \cap (W_2 \oplus (\operatorname{Im}(B) \cap \operatorname{Ker}(A)) \oplus W_3 \oplus (\operatorname{Ker}(A) \cap \operatorname{Ker}(B))) = W_1$. Moreover, keeping in mind that $\dim W_1 = \dim W_2$, we have that $\dim W_1 \leqslant \dim W_2 + \dim(\operatorname{Im}(B) \cap \operatorname{Ker}(A)) = t$, and since $\operatorname{Ker}(\lambda_1 I - AB) \oplus \ldots \oplus \operatorname{Ker}(\lambda_k I - AB) \subset W_1$, it follows that $\delta_{AB} = \dim \operatorname{Ker}(\lambda_1 I - AB) + \ldots + \dim \operatorname{Ker}(\lambda_k I - AB) \leqslant \dim W_1 \leqslant t$.

On the other hand, consider $v \in \operatorname{Im}(A) \cap (W_2 \oplus (\operatorname{Im}(B) \cap \operatorname{Ker}(A)) \oplus W_3 \oplus (\operatorname{Ker}(A) \cap \operatorname{Ker}(B)))$, hence $v = u + w$, where $v \in \operatorname{Im}(A)$, $u \in W_2 \oplus (\operatorname{Im}(B) \cap \operatorname{Ker}(A))$ and $w \in W_3 \oplus (\operatorname{Ker}(A) \cap \operatorname{Ker}(B))$, which implies $Av = v = Au + Aw = Au = u + w$, that is, $w = Au - u$, and so $w \in (W_3 \oplus (\operatorname{Ker}(A) \cap \operatorname{Ker}(B))) \cap (W_1 \oplus W_2 \oplus (\operatorname{Im}(B) \cap \operatorname{Ker}(A)))$ since $A(W_2 \oplus (\operatorname{Im}(B) \cap \operatorname{Ker}(A)) = AW_2 = W_1$.

Let $\{u_1, \ldots, u_k, \ldots, u_t\}$ be a basis of $W_2 \oplus (\operatorname{Im}(B) \cap \operatorname{Ker}(A))$. Moreover, consider that $w_1 = Au_1 - u_1, \ldots, w_k = Au_k - u_k, \ldots, w_t = Au_t - u_t$ and $v_1 = u_1 + w_1, \ldots, v_k = u_k + w_k$, where $ABv_i = \lambda_i v_i$ and $BAu_i = \lambda_i u_i$ for each $i \in \{1, \ldots, k\}$ since $W_1 = \operatorname{Im}(A) \cap (W_2 \oplus (\operatorname{Im}(B) \cap \operatorname{Ker}(A)) \oplus W_3 \oplus (\operatorname{Ker}(A) \cap \operatorname{Ker}(B)))$.

Now, note that if $c_1 w_1 + \ldots + c_k w_k + \ldots + c_t w_t = 0$ with $c_1, \ldots, c_t \in \mathbb{C}$, then $c_1(Au_1 - u_1) + \ldots + c_k(Au_k - u_k) + \ldots + c_t(Au_t - u_t) = A(c_1 u_1 + \ldots + c_k u_k + \ldots + c_t u_t) - (c_1 u_1 + \ldots + c_k u_k + \ldots + c_t u_t) = 0$, which implies $c_1 u_1 + \ldots + c_k u_k + \ldots + c_t u_t \in \operatorname{Im}(A)$, but $c_1 u_1 + \ldots + c_k u_k + \ldots + c_t u_t \in W_2 \oplus (\operatorname{Im}(B) \cap \operatorname{Ker}(A))$, and as $\operatorname{Im}(A) \cap (W_2 \oplus (\operatorname{Im}(B) \cap \operatorname{Ker}(A))) = \{0\}$, we have that $c_1 u_1 + \ldots + c_k u_k + \ldots + c_t u_t = 0$, and so $c_1 = \ldots = c_k = \ldots = c_t = 0$. This implies that $\dim span(w_1, \ldots, w_t) = t$, hence we may conclude that $t \leqslant \dim((W_3 \oplus (\operatorname{Ker}(A) \cap \operatorname{Ker}(B))) \cap (W_1 \oplus W_2 \oplus (\operatorname{Im}(B) \cap \operatorname{Ker}(A)))$. □

REMARK 5. Regarding Theorem 2.12, keeping in mind that $AB, BA \in \mathbb{C}_D^{n\times n}$, we may easily conclude, by symmetry, that $\delta_{BA} \leqslant \dim((W_4 \oplus (\operatorname{Ker}(A) \cap \operatorname{Ker}(B))) \cap (W_1 \oplus W_2 \oplus (\operatorname{Im}(A) \cap \operatorname{Ker}(B))))$, and as $\delta_{AB} = \delta_{BA}$, we claim that $\delta_{AB} \leqslant \min\{\dim((W_3 \oplus (\operatorname{Ker}(A) \cap \operatorname{Ker}(B))) \cap (W_1 \oplus W_2 \oplus (\operatorname{Im}(B) \cap \operatorname{Ker}(A)))), \dim((W_4 \oplus (\operatorname{Ker}(A) \cap \operatorname{Ker}(B))) \cap (W_1 \oplus W_2 \oplus (\operatorname{Im}(A) \cap \operatorname{Ker}(B))))\}$.

3. When the product of two projections is a diagonalizable matrix

In [5, Corollary 3.9], we have proved that if $D \in \mathbb{C}_D^{n\times n}$ with $\operatorname{rank}(D) \leqslant n/2$ and with arbitrary spectrum, then there are $A,\ B \in \mathbb{C}_P^{n\times n}$ so that $AB = D$. Moreover, in [5, Theorem 3.12], we have proved that there are projections A and B so that $AB = D$, where D is diagonalizable, if and only if $\delta_D \leqslant \alpha_D$. Similarly, in [12, page 81], Ballantine proved that given a singular diagonalizable matrix D and A and B of same order, $D = AB$ if and only if $\operatorname{rank}(I-D) \leqslant 2\dim\operatorname{Ker}(D)$. Another relevant information is that, by [2, Theorem 3.2.11.1], we may conclude that given projections A and B, AB is diagonalizable if and only if BA is diagonalizable whenever AB and BA are core matrices. In this section, we shall show some necessary and/or sufficient conditions so that AB is diagonalizable with arbitrary spectrum or restricted to the real segment [0, 1].

In [5, Corollary 3.10], we have proved that if $N \in \mathbb{C}_N^{n\times n}$ with rank at most $n/2$ and arbitrary spectrum, then there are $A, B \in \mathbb{C}_P^{n\times n} \setminus \mathbb{C}_{HP}^{n\times n}$ so that $AB = N$. However, if $A \in \mathbb{C}_{HP}^{n\times n}$, $N \in \mathbb{C}_N^{n\times n} \setminus \mathbb{C}_{HP}^{n\times n}$, then $AB \neq N$ for any $B \in \mathbb{C}_P^{n\times n}$. Our next main result is able to demonstrate this.

THEOREM 3.1. *If $A \in \mathbb{C}_{HP}^{n\times n}$, $B \in \mathbb{C}_P^{n\times n}$ and $N \in \mathbb{C}_N^{n\times n}$ so that $AB = N$, then $N \in \mathbb{C}_{HP}^{n\times n}$.*

Proof. According to [1, p. 42], $AX = N$ if and only if $AA^\dagger N = N$ for some $X \in \mathbb{C}^{n\times n}$. Since $A \in \mathbb{C}_{HP}^{n\times n}$, it follows that $A = A^2 = A^* = A^\dagger$, so $AX = N$ is solvable if and only if $AN = N$. Suppose that $N \notin \mathbb{C}_{HP}^{n\times n}$. Thus, take $U \in \mathbb{C}_U^{n\times n}$ so that $U^*NU = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}$, where $D = \operatorname{diag}(\lambda_1,\ldots,\lambda_r)$, $\lambda_1,\ldots,\lambda_r \in \mathbb{C}\setminus\{0\}$ and at least one $\lambda_i \in \mathbb{C}\setminus\{0,1\}$ with $i \in \{1,\ldots,r\}$. Consider $U^*AU = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$ with $A_1 \in \mathbb{C}^{r\times r}$. Hence, $U^*ANU = U^*AUU^*NU = U^*NU$, which implies $\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}$, and so we have that $A_1 = I_r$ and $A_3 = 0$, and as $U^*AU \in \mathbb{C}_{HP}^{n\times n}$, we also have that $A_2 = 0$ and $A_4 \in \mathbb{C}_{HP}^{n-r\times n-r}$. Again by [1, p. 42], $X = A^\dagger N + (I - A^\dagger A)M = N + (I-A)M$ is the general solution of the equation $AX = N$ for any $M \in \mathbb{C}^{n\times n}$. This implies that $X = U\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}U^* + \left[\begin{pmatrix} I_r & 0 \\ 0 & I_{n-r} \end{pmatrix} - U\begin{pmatrix} I_r & 0 \\ 0 & A_4 \end{pmatrix}U^*\right]M$, that is,

$$U^*XU = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & I_{n-r} - A_4 \end{pmatrix}\begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} = \begin{pmatrix} D & 0 \\ (I_{n-r} - A_4)M_3 & (I_{n-r} - A_4)M_4 \end{pmatrix},$$

where $U^*MU = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}$ and $M_1 \in \mathbb{C}^{r\times r}$. Hence, taking into account that $\{\lambda_1,\ldots,\lambda_r\} = \sigma(D) \subset \sigma(X)$, we may conclude that $X \notin \mathbb{C}_P^{n\times n}$, but this contradicts our hypothesis that $X = B \in \mathbb{C}_P^{n\times n}$. $\square$

REMARK 6. Taking into account the proof of Theorem 3.1 and if $A, N \in \mathbb{C}_{HP}^{n\times n}$ and $AN = N$, then $D = I_r$ and $AN = N = N^* = NA$. Thus, take $U \in \mathbb{C}_U^{n\times n}$ so that

$U^*AUU^*NU = U^*NU = \begin{pmatrix} I_r & 0 \\ 0 & A_4 \end{pmatrix} \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$, where $A_4 = \begin{pmatrix} I_{t_1} & 0 \\ 0 & 0 \end{pmatrix}$ and $t_1 + t_2 = n - r$. Hence, $X = U \begin{pmatrix} I_r & 0 \\ V & W \end{pmatrix} U^*$ is the general solution of the equation $AX = N$, where

$$V = \left[\begin{pmatrix} I_{t_1} & 0 \\ 0 & I_{t_2} \end{pmatrix} - \begin{pmatrix} I_{t_1} & 0 \\ 0 & 0 \end{pmatrix} \right] M_3 = \begin{pmatrix} 0 & 0 \\ 0 & I_{t_2} \end{pmatrix} M_3$$

and

$$W = \left[\begin{pmatrix} I_{t_1} & 0 \\ 0 & I_{t_2} \end{pmatrix} - \begin{pmatrix} I_{t_1} & 0 \\ 0 & 0 \end{pmatrix} \right] M_4 = \begin{pmatrix} 0 & 0 \\ 0 & I_{t_2} \end{pmatrix} M_4$$

for any $M = U \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} U^* \in \mathbb{C}^{n\times n}$, and consequently M_1, M_2, M_3 and M_4 are arbitrary submatrices. Therefore, we may conclude that $X \in \mathbb{C}_P^{n\times n}$ if and only if $WV = 0$ and $W^2 = W$.

REMARK 7. Consider $E \in \mathbb{C}_{EP}^{n\times n}$. Thus, by [3, Lemma 2], $Q^*EQ = \begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix}$ for some $Q \in \mathbb{C}_U^{n\times n}$ and $T \in \mathbb{C}^{t\times t}$ nonsingular. Again, taking into account the proof of Theorem 3.1, we may similarly conclude that if $A \in \mathbb{C}_{HP}^{n\times n}$, $B \in \mathbb{C}_P^{n\times n}$ and $E \in \mathbb{C}_{EP}^{n\times n}$ so that $AB = E$, then $E \in \mathbb{C}_{HP}^{n\times n}$.

Now, it follows a result which provides a sufficient condition so that $AB \in \mathbb{C}_{EP}^{n\times n}$, where $A \in \mathbb{C}_P^{n\times n}$ and $B \in \mathbb{C}_P^{n\times n} \setminus \mathbb{C}_{HP}^{n\times n}$.

PROPOSITION 3.2. *Let $A \in \mathbb{C}_P^{n\times n}$ and $B \in \mathbb{C}_P^{n\times n} \setminus \mathbb{C}_{HP}^{n\times n}$. Let P_AU_A and U_BP_B be polar decompositions, respectively, of A and B, where $U_A, U_B \in \mathbb{C}_U^{n\times n}$, $P_A \geqslant 0$, $P_B \geqslant$ and $Q = U_AU_B$. Thus, if $P_AQ \in \mathbb{C}_{EP}^{n\times n}$, $\mathrm{Ker}(P_B) \subset \mathrm{Ker}(P_AQ)$ and $P_B(\mathrm{Ker}(P_AQ)) \subset \mathrm{Ker}(P_AQ)$, then $AB \in \mathbb{C}_{EP}^{n\times n}$.*

Proof. We have that $\mathrm{Ker}(P_B) \subset \mathrm{Ker}(P_AQ) \Rightarrow (\mathrm{Ker}(P_AQ))^\perp \subset (\mathrm{Ker}(P_B))^\perp \Rightarrow \mathrm{Im}((P_AQ)^*) \subset \mathrm{Im}(P_B^*) \Rightarrow \mathrm{Im}(P_AQ) \subset \mathrm{Im}(P_B)$ since $P_AQ \in \mathbb{C}_{EP}^{n\times n}$, hence $\mathrm{Im}(P_AQP_B) \subset \mathrm{Im}(P_B)$. Moreover, since $P_B(\mathrm{Ker}(P_AQ)) \subset \mathrm{Ker}(P_AQ)$, it follows that $\mathrm{Ker}(P_AQ) \subset \mathrm{Ker}(P_AQP_B)$. Keeping in mind that $P_AQ, P_B \in \mathbb{C}_{EP}^{n\times n}$ and by [9, Theorem 2], if $\mathrm{Im}(P_AQP_B) \subset \mathrm{Im}(P_B)$ and $\mathrm{Ker}(P_AQ) \subset \mathrm{Ker}(P_AQP_B)$, then $P_AQP_B = AB \in \mathbb{C}_{EP}^{n\times n}$. □

Now, consider the solvable matricial equation $AX = D$, where $A \in \mathbb{C}_{HP}^{n\times n}$ and $D \in \mathbb{C}_D^{n\times n} \setminus \mathbb{C}_N^{n\times n}$. Thus, if $\mathrm{rank}(A) = s \leqslant n - s$ and given $\alpha_1, \ldots, \alpha_r \in \mathbb{C} \setminus \{0\}$ with $\mathrm{rank}(D) = r \leqslant s$, then there are $D \in \mathbb{C}_D^{n\times n} \setminus \mathbb{C}_N^{n\times n}$ and $X \in \mathbb{C}_P^{n\times n}$ so that $\{\alpha_1, \ldots, \alpha_r\} \subset \sigma(D)$ and $AX = D$. Our next result provides sufficient conditions for the projection X to satisfy the equation $AX = D$, under the conditions above established.

Consider the following decompositions for the projections A and B of order n:

$U_A^*AU_A = \begin{pmatrix} I_s & 0 \\ 0 & 0 \end{pmatrix} = V_BBV_B^{-1}$, where $U_A \in \mathbb{C}_U^{n\times n}$, $V_B \in \mathbb{C}^{n\times n}$ and nonsingular, $B = \begin{pmatrix} I_s & A_2 \\ 0 & 0 \end{pmatrix}$, $A_2 \in \mathbb{C}^{s\times n-s}$ and $A_2 \neq 0$. Hence, $A = T \begin{pmatrix} I_s & A_2 \\ 0 & 0 \end{pmatrix} T^{-1} \in \mathbb{C}_{HP}^{n\times n}$ with $T = U_AV_B$. Consider, also, $D = T\mathrm{diag}(\lambda_1, \ldots, \lambda_r, 0, \ldots, 0)\ T^{-1} \in \mathbb{C}^{n\times n}$, $D_\lambda = \mathrm{diag}(\lambda_1 - 1, \ldots, \lambda_r - 1, -1, \ldots, -1) \in \mathbb{C}^{s\times s}$, $\lambda_1, \ldots, \lambda_r \in \mathbb{C} \setminus \{0\}$, $r \leqslant s$ and $M_3 = A_2^\dagger D_\lambda + (I_{n-s} - A_2^\dagger A_2)W_s$ with $W_s \in \mathbb{C}^{n-s\times s}$.

PROPOSITION 3.3. *Let A, B, T, D, D_λ and M_3 be matrices as represented above. Once an arbitrary Hermitian projection A of* rank s *is fixed and* $X = T\begin{pmatrix} I_s & 0 \\ M_3 & 0\end{pmatrix}T^{-1} \in \mathbb{C}_P^{n\times n}$, *we have that if $\lambda_1,\ldots,\lambda_r \in \mathbb{C}\setminus\{0,1\}$ and $s \leqslant n-s$, then $AX = D$ for some $D \in \mathbb{C}_D^{n\times n}\setminus\mathbb{C}_N^{n\times n}$ with $\{\lambda_1,\ldots,\lambda_r\} \subset \sigma(D)$, for any $W_s \in \mathbb{C}^{n-s\times s}$ and for any $A_2 \in \mathbb{C}^{s\times n-s}$ of* rank s.

Proof. Consider $A \in \mathbb{C}_{HP}^{n\times n}$ and $B \in \mathbb{C}_P^{n\times n}\setminus\mathbb{C}_{HP}^{n\times n}$ with $\operatorname{rank}(A) = \operatorname{rank}(B) = s$. Then there are $U_A \in \mathbb{C}_U^{n\times n}$, $V_B \in \mathbb{C}^{n\times n}\setminus\mathbb{C}_U^{n\times n}$ and nonsingular so that $U_A^* A U_A = \begin{pmatrix} I_s & 0 \\ 0 & 0\end{pmatrix} = V_B B V_B^{-1}$, that is, $(U_A V_B)^{-1} A U_A V_B = B = \begin{pmatrix} I_s & A_2 \\ 0 & 0\end{pmatrix}$, and so $A = T\begin{pmatrix} I_s & A_2 \\ 0 & 0\end{pmatrix}T^{-1} \in \mathbb{C}_{HP}^{n\times n}$ for any $A_2 \in \mathbb{C}^{s\times n-s}$, but $A_2 \neq 0$ and $U_A V_B = T \notin \mathbb{C}_U^{n\times n}$. Now, consider the matricial equation

$$AX = D, \tag{2}$$

where $D = T\operatorname{diag}(\lambda_1,\ldots,\lambda_r,0,\ldots,0)T^{-1} \in \mathbb{C}^{n\times n}$, $\lambda_1,\ldots,\lambda_r \in \mathbb{C}\setminus\{0,1\}$ and $r \leqslant s$. Keeping in mind that $A \in \mathbb{C}_{HP}^{n\times n}$, we have that $AA^\dagger = A^\dagger A = A = A^\dagger$, which implies $AA^\dagger D = AD = T\begin{pmatrix} I_s & A_2 \\ 0 & 0\end{pmatrix}T^{-1}T\begin{pmatrix} D_s & 0 \\ 0 & 0\end{pmatrix}T^{-1} = T\begin{pmatrix} D_s & 0 \\ 0 & 0\end{pmatrix}T^{-1} = D$, where $D_s = \operatorname{diag}(\lambda_1,\ldots,\lambda_r,0,\ldots,0) \in \mathbb{C}^{s\times s}$. This implies that $AA^\dagger D = D$, and by [2, p. 42], (2) is solvable for any $A_2 \in \mathbb{C}^{s\times n-s}$ and $A_2 \neq 0$. Again by [2, p. 42], in (2) the general solution is given by $X = A^\dagger D + (I - A^\dagger A)M = D + (I-A)M = T\begin{pmatrix} D_s & 0 \\ 0 & 0\end{pmatrix}T^{-1} + \left[\begin{pmatrix} I_s & 0 \\ 0 & I_{n-s}\end{pmatrix} - T\begin{pmatrix} I_s & A_2 \\ 0 & 0\end{pmatrix}T^{-1}\right]TT^{-1}M$ for any $M \in \mathbb{C}^{n\times n}$. Hence, $T^{-1}XT = \begin{pmatrix} D_s & 0 \\ 0 & 0\end{pmatrix} + \begin{pmatrix} 0 & -A_2 \\ 0 & I_{n-s}\end{pmatrix}\begin{pmatrix} M_1 & M_2 \\ M_3 & M_4\end{pmatrix} = \begin{pmatrix} D_s & 0 \\ 0 & 0\end{pmatrix} + \begin{pmatrix} -A_2M_3 & -A_2M_4 \\ M_3 & M_4\end{pmatrix} = \begin{pmatrix} D_s - A_2M_3 & -A_2M_4 \\ M_3 & M_4\end{pmatrix}$, where $T^{-1}MT = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4\end{pmatrix}$ and $M_1 \in \mathbb{C}^{s\times s}$. Note that if $M_4 = 0$ and $D_s - A_2M_3 = I_s$, then $X \in \mathbb{C}_P^{n\times n}$. Thus, take $M_4 = 0$ and $D_s - A_2M_3 = I_s$, that is,

$$A_2M_3 = \operatorname{diag}(\lambda_1 - 1,\ldots,\lambda_r - 1, -1,\ldots,-1) = D_\lambda. \tag{3}$$

Let $s \leqslant n-s$. Since $\lambda_i \neq 1$ for each $i \in \{1,\ldots,r\}$, it follows that $\operatorname{rank}(D_\lambda) = s$, which implies $\operatorname{rank}(A_2) = \operatorname{rank}(A_2A_2^\dagger) = s$, hence $A_2A_2^\dagger = I_s$, that is, $A_2A_2^\dagger D_\lambda = D_\lambda$, and so (3) is solvable for any $A_2 \in \mathbb{C}^{s\times n-s}$ of rank s. In this case, the general solution of (3) is given by $M_3 = A_2^\dagger D_\lambda + (I_{n-s} - A_2^\dagger A_2)W_s$ for any $W_s \in \mathbb{C}^{n-s\times s}$. Therefore, if $X = T\begin{pmatrix} I_s & 0 \\ M_3 & 0\end{pmatrix}T^{-1} \in \mathbb{C}_P^{n\times n}$, with $\lambda_1,\ldots,\lambda_r \in \mathbb{C}\setminus\{0,1\}$ and $s \leqslant n-s$, then X is a solution of (2) for any $W_s \in \mathbb{C}^{n-s\times s}$ and for any $A_2 \in \mathbb{C}^{s\times n-s}$ of rank s. $\square$

Let $A, B, C \in \mathbb{C}_P^{n\times n}$ be with representation in (1) and $\operatorname{rank}(A) \leqslant n/2$. In the next result, we shall make use of the submatrices of the projection C to provide a necessary and sufficient condition so that $AB \in \mathbb{C}_D^{n\times n}$, $\operatorname{rank}(AB) = \operatorname{rank}(A)$ and $\operatorname{Ker}(I - AB) = \{0\}$.

PROPOSITION 3.4. *Let $A,B,C \in \mathbb{C}_P^{n\times n}$ be with representation in (1) and $r \leqslant n/2$. Thus, $AB \in \mathbb{C}_D^{n\times n}$ with $\operatorname{rank}(AB) = r$ and $1 \notin \sigma(AB)$ if and only if $C_2C_3 \in \mathbb{C}_D^{r\times r}$ and nonsingular.*

Proof. Taking into account that $Y_AABY_A^{-1} = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix} = \begin{pmatrix} C_1 & C_2 \\ 0 & 0 \end{pmatrix}$ and that a square matrix is diagonalizable if and only if its minimal polynomial is a product of pairwise distinct monic linear polynomials, we have that if AB is diagonalizable, then so is C_1. Moreover, $1 \notin \sigma(AB)$ implies that $1 \notin \sigma(C_1)$, and also $\operatorname{rank}(AB) = r$ implies that $0 \notin \sigma(C_1)$ since $C_1 \in \mathbb{C}^{r\times r}$ and $\operatorname{Im}(C_2) \subset \operatorname{Im}(C_1)$, see [5, Theorem 2.11].

Since $\begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix}\begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix} = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix}$, it follows that

$$C_2C_3 = C_1 - C_1^2. \tag{4}$$

Hence $C_2C_3 \in \mathbb{C}_D^{r\times r}$ and nonsingular.

Conversely, consider $m_C(x) = (x-\lambda_1)\dots(x-\lambda_k)$ the minimal polynomial of C_2C_3. Thus, if $C_2C_3 \in \mathbb{C}_D^{r\times r}$ and nonsingular, then $\lambda_1,\dots,\lambda_k$ are distinct and nonzero, so $(x-\lambda_1)\dots(x-\lambda_k)(C_2C_3) = (x-\lambda_1)\dots(x-\lambda_k)x(1-x)(C_1) = 0$, and by (4), $\delta_{C_1} = r$ since C_2C_3 is nonsingular, and so we may conclude that $C_1 \in \mathbb{C}_D^{r\times r}$. Now, note that $\operatorname{Im}(C_2) \subset \operatorname{Im}(C_1)$ since C_1 is nonsingular, and therefore, by [5, Theorem 2.11], we conclude that $AB \in \mathbb{C}_D^{n\times n}$ with $\operatorname{rank}(AB) = r$ and $1 \notin \sigma(AB)$. □

The following Corollary provides a sufficient condition so that $AB \in \mathbb{C}_D^{n\times n}$, $\operatorname{rank}(AB) = \operatorname{rank}(A)$ and $\operatorname{Ker}(I-AB) = \{0\}$ whenever $A,B \in \mathbb{C}_P^{n\times n}$.

COROLLARY 3.5. *Let $A,B,C \in \mathbb{C}_P^{n\times n}$ be with representation in (1) and $\operatorname{rank}(A) = r < n/2$. Thus, if $(AB-BA)^2 \in \mathbb{C}_D^{n\times n}$ and $\operatorname{rank}(AB-BA)^2 = 2\operatorname{rank}(A)$, then $AB \in \mathbb{C}_D^{n\times n}$ with $\operatorname{rank}(AB) = r$ and $1 \notin \sigma(AB)$.*

Proof. We have that $Y_AABY_A^{-1} = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix} = \begin{pmatrix} C_1 & C_2 \\ 0 & 0 \end{pmatrix}$ and $Y_ABAY_A^{-1} = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix}\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} C_1 & 0 \\ C_3 & 0 \end{pmatrix}$, which implies $Y_A(AB-BA)^2Y_A^{-1} = \begin{pmatrix} -C_2C_3 & 0 \\ 0 & -C_3C_2 \end{pmatrix}$.

Clearly, if $(AB-BA)^2$ is diagonalizable with $\operatorname{rank}(AB-BA)^2 = 2\operatorname{rank}(A) = 2r$, then $-C_2C_3$ and $-C_3C_2$ are diagonalizable too with $\operatorname{rank}(-C_2C_3) + \operatorname{rank}(-C_3C_2) = 2r$, and as $-C_2C_3$ and $-C_3C_2$ have the same nonzero eigenvalues, it follows that $\operatorname{rank}(-C_2C_3) = \operatorname{rank}(-C_3C_2) = r$, so C_2C_3 is nonsingular, and by Proposition 3.4, we may conclude that $AB \in \mathbb{C}_D^{n\times n}$ with $\operatorname{rank}(AB) = r$ and $1 \notin \sigma(AB)$. □

REMARK 8. Regarding proposition 2.10, the condition $AB-BA$ being nonsingular to imply that $\delta_{AB} = \operatorname{Tr}(A)$ is not necessary because, according to Corollary 3.5, we have the following:

Let $A,B \in \mathbb{C}_P^{n\times n}$ be with $\operatorname{rank}(A) = \operatorname{Tr}(A) = r < n/2$. Thus, if $C^2 = (AB-BA)^2 \in \mathbb{C}_D^{n\times n}$ and $\operatorname{rank}(C^2) = 2\operatorname{rank}(A) = 2r < n$, then $AB \in \mathbb{C}_D^{n\times n}$ with $\operatorname{rank}(AB) = \operatorname{Tr}(A)$ and $1 \notin \sigma(AB)$. Hence, clearly, C is singular and $\delta_{AB} = \operatorname{Tr}(A)$.

From now on, once an arbitrary projection A is fixed, we shall show projections B so that $AB \in \mathbb{C}_D^{n\times n}$ with $\sigma(AB) \subset [0,1]$. Particularly, concerninig Lemma 3.6, Proposition 3.7 and Proposition 3.8, we shall need the following information:

We define k functions f_k by $f_k(t_k,\alpha_1,\ldots,\alpha_k) = \{0,\ldots,0,\alpha_1,\ldots,\alpha_k,1,\ldots,1\}$, where $\alpha_1,\ldots,\alpha_k \in (0,1)$, the number of nonzero elements of $f_k(t_k,\alpha_1,\ldots,\alpha_k)$ is equal to t_k, the number of zero elements of $f_k(t_k,\alpha_1,\ldots,\alpha_k)$ is equal to $n-t_k$ and $0 \leqslant k \leqslant n-t_k$. Hence, according to this definition for f_k, $k \leqslant t_k \leqslant n-k$ and $0 \leqslant k \leqslant n/2$. Then, for every k, $0 \leqslant k \leqslant n/2$; for every $\alpha_i \in (0,1)$, $i = 1,\ldots,k$, and for every t_k, $k \leqslant t_k \leqslant n-k$, there are Hermitian projections P and Q so that $\sigma(PQ) = f_k(t_k,\alpha_1,\ldots,\alpha_k)$, with $\operatorname{rank}(P) = r$, $\operatorname{rank}(Q) = s$ and $t_k = \min\{r,s\}$. Note that $k \leqslant n-t_k$ is a necessary condition for $\sigma(PQ) = f_k(t_k,\alpha_1,\ldots,\alpha_k)$ because $\delta_{PQ} \leqslant \min\{\dim\operatorname{Ker}(P),\dim\operatorname{Ker}(Q)\}$. Moreover, PQ is a diagonalizable matrix, see [6, p. 144], which implies $\alpha_{PQ} = \dim\operatorname{Ker}(PQ)$, and so $\alpha_{PQ} = \dim\operatorname{Ker}(PQ) \geqslant \dim\operatorname{Ker}(Q) \geqslant \delta_{PQ}$.

We should also consider Lemma 3.6, see proof in [4, Lemma 2.4].

LEMMA 3.6. *For every k, $0 \leqslant k \leqslant n/2$; for every $\alpha_i \in (0,1)$, $i = 1,\ldots,k$ and for every t_k, $k \leqslant t_k \leqslant n-k$, there are $E,F \in \mathbb{C}_{HP}^{n\times n}$ which are of block diagonal form with diagonal blocks of order 2 and of order 1 so that $\sigma(EF) = f_k(t_k,\alpha_1,\ldots,\alpha_k)$, with* $\operatorname{rank}(E) = r$, $\operatorname{rank}(F) = s$ *and* $t_k = \min\{r,s\}$.

Given projections A and B, in our next result we provide a necessary and sufficient condition so that AB is diagonalizable with $\sigma(AB) \subset [0,1]$.

PROPOSITION 3.7. *Let $A,B \in \mathbb{C}_P^{n\times n}$. Then $AB \in \mathbb{C}_D^{n\times n}$ with $\sigma(AB) \subset [0,1]$ if and only if AB is similar to PQ for some $P,Q \in \mathbb{C}_{HP}^{n\times n}$.*

Proof. Let $X \in \mathbb{C}^{n\times n}$ be nonsingular so that $X^{-1}ABX = PQ$, where $P,Q \in \mathbb{C}_{HP}^{n\times n}$. Thus, by [9, p. 143 and 144], we may conclude that $AB \in \mathbb{C}_D^{n\times n}$ with $\sigma(AB) \subset [0,1]$.

Conversely, if $AB \in \mathbb{C}_D^{n\times n}$ with $\sigma(AB) \subset [0,1]$, then there is some $Y \in \mathbb{C}^{n\times n}$ nonsingular so that $Y^{-1}ABY = \operatorname{diag}(\lambda_1,\ldots,\lambda_k,1,\ldots,1,0,\ldots,0)$, where $\lambda_1,\ldots,\lambda_k \in (0,1)$ and $k = \delta_{AB} \leqslant \min\{\dim\operatorname{Ker}(A),\dim\operatorname{Ker}(B)\} \leqslant \dim\operatorname{Ker}(AB)$. Hence, according to Lemma 3.6, there are $P,Q \in \mathbb{C}_{HP}^{n\times n}$ so that $Z^{-1}PQZ = \operatorname{diag}(\lambda_1,\ldots,\lambda_k,1,\ldots,1,0,\ldots,0)$ to some $Z \in \mathbb{C}^{n\times n}$ nonsingular, which implies $Y^{-1}ABY = Z^{-1}PQZ$, and therefore $(YZ^{-1})^{-1}ABYZ^{-1} = PQ$. $\square$

Let $E_j,F_j \in \mathbb{C}_{HP}^{2\times 2}$ be with the following entries: $E_j = \begin{pmatrix} \frac{1}{2} & b_1+b_2i \\ b_1-b_2i & \frac{1}{2} \end{pmatrix}$ and $F_j = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ with $b_1,b_2 \in \mathbb{R}$, $i = \sqrt{-1}$ and $j = 1,\ldots,k$, where $b_1^2+b_2^2 = \frac{1}{4}$ since E_j is singular. Moreover, for any $\alpha_j \in (0,1)$, there are $E_j,F_j \in \mathbb{C}_{HP}^{2\times 2}$ so that α_j is the eigenvalue of E_jF_j different of 0 and of 1, where $b_1 = \alpha_j - \frac{1}{2}$, see proof in [9, Lemma 2.4]. Now, let $E = \operatorname{diag}(E_1,\ldots,E_k,1,\ldots,1,0,\ldots,0)$ and $F = \operatorname{diag}(F_1,\ldots,F_k,1,\ldots,1,0,\ldots,0)$ be Hermitian projections of order n and with $\operatorname{rank}(E) = r$ and $\operatorname{rank}(F) = s$. Considering, also, the decompositions given below for the projections A, E and B:

$$Y_AAY_A^{-1} = Y_EEY_E^{-1} = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad Y_BBY_B^{-1} = \begin{pmatrix} I_s & 0 \\ 0 & 0 \end{pmatrix}, \tag{5}$$

where Y_A, Y_E and Y_B are nonsingular matrices, $\operatorname{rank}(A) = r$ and $\operatorname{rank}(B) = s$. Moreover, U_AP_A and U_BP_B are the polar decompositions of Y_A and Y_B, respectively, with $P_A > 0$ and $P_B > 0$.

Given $\alpha_1, \ldots, \alpha_k \in (0,1)$, $1 \leqslant k \leqslant n/2$, once an arbitrary projection A of rank r is fixed, in our next result, we shall identify projections B of rank s so that AB is a diagonalizable matrix with $\{\alpha_1, \ldots, \alpha_k\} \subset \sigma(AB) \subset [0,1]$, where $k \leqslant \min\{\dim \operatorname{Ker}(A), \dim \operatorname{Ker}(B)\}$ and $\operatorname{rank}(AB) = \min\{r,s\}$.

PROPOSITION 3.8. *Let $A, E \in \mathbb{C}_P^{n\times n}$ be with representation in (5). Once an arbitrary projection A is fixed, for any Y_A, for any Y_E and for any E, if $B = Y_A^{-1}Y_EF(Y_A^{-1}Y_E)^{-1}$, then $AB \in \mathbb{C}_D^{n\times n}$ with $\sigma(AB) = f_k(t_k, \alpha_1, \ldots, \alpha_k)$ for any $\alpha_i \in (0,1)$, $i = 1, \ldots, k$, $t_k = \min\{r,s\}$ and $\delta_{AB} = k$.*

Proof. Since $\operatorname{rank}(E) = \operatorname{rank}(A)$, it follows that $Y_EEY_E^{-1} = Y_AAY_A^{-1}$, and so $Y_A^{-1}Y_EEY_E^{-1}Y_A = A$. Consider that $X = Y_A^{-1}Y_E$ and $B = XFX^{-1}$. Hence, by Lemma 3.6, for any Y_A, for any Y_E and for any E, $AB = XEX^{-1}XFX^{-1} = XEFX^{-1} \in \mathbb{C}_D^{n\times n}$ with $\sigma(AB) = f_k(t_k, \alpha_1, \ldots, \alpha_k)$ for any $\alpha_i \in (0,1)$, $i = 1, \ldots, k$, $t_k = min\{r,s\}$ and $\delta_{AB} = k$. $\square$

In [5, Theorem 3.15], we have proved that if $P_A = P_B$, then $AB \in \mathbb{C}_D^{n\times n}$ with $\sigma(AB) \subset [0,1]$ for some Y_A and Y_B with representation in (1), but the converse does not hold. However, the following two Lemmas are useful to the Propositions presented shortly thereafter.

LEMMA 3.9. *Let $A, B \in \mathbb{C}_P^{n\times n}$. Then $P_A = P_B$ if and only if $Y_B = UY_A$ for some $U \in \mathbb{C}_U^{n\times n}$.*

Proof. If $P_A = P_B$, then $Y_B = U_BP_A = U_BU_A^*Y_A$, where $U = U_BU_A^* \in \mathbb{C}_U^{n\times n}$. Conversely, if $Y_B = UY_A$ for some $U \in \mathbb{C}_U^{n\times n}$, then $Y_B = U_BP_B = UU_AP_A$, which implies $P_A = U_A^*U^*U_BP_B$, and so by the uniqueness of the polar decomposition of P_A, we may conclude that $U_A^*U^*U_B = I$ and $P_A = P_B$. $\square$

LEMMA 3.10. *Let $A, B \in \mathbb{C}^{n\times n}$. If $A^*A = B^*B$, then $A = UB$ for some $U \in \mathbb{C}_U^{n\times n}$.*

Proof. Let $A = U_AP_A$ and $B = U_BP_B$ be the polar decompositions of A and B, respectively. Hence, if $A^*A = B^*B$, then $P_AU_A^*U_AP_A = P_A^2 = P_BU_B^*U_BP_B = P_B^2$, which implies $P_A = P_B$, and so $A = U_AP_A = U_AP_B = U_AU_B^*B$, where $U_AU_B^* \in \mathbb{C}_U^{n\times n}$. $\square$

Let $A, B, C \in \mathbb{C}_P^{n\times n}$ be with representation in (1). The next result provides a necessary and sufficient condition for C to be a Hermitian projection.

PROPOSITION 3.11. *Let $A, B, C \in \mathbb{C}_P^{n\times n}$ be with representation in (1). Then $C = C^*$ if and only if $Y_B = UY_A$ for some $U \in \mathbb{C}_U^{n\times n}$.*

Proof. Since $Y_ABY_A^{-1} = C$, it follows that $Y_B = Y_CY_A$. Thus, if $Y_B = UY_A = Y_CY_A$, then $Y_C = U$, and so $C = C^*$.

Conversely, if $C = C^*$, then there is some $Y_C = U \in \mathbb{C}_U^{n\times n}$ so that $Y_CCY_C^{-1} = \operatorname{diag}(I_s, 0)$, hence $Y_B = UY_A$ for some $U \in \mathbb{C}_U^{n\times n}$. $\square$

Now, we shall prove two results which provide sufficient conditions for AB to be a diagonalizable matrix with $\sigma(AB) \subset [0,1]$, once an arbitrary projection A is fixed and for some projection B.

PROPOSITION 3.12. *Let $A,B \in \mathbb{C}_P^{n\times n}$. Consider also Y_A and Y_B with representation in (1), $U \in \mathbb{C}_U^{n\times n}$ and $D = \operatorname{diag}(D_1,D_2) \in \mathbb{C}^{n\times n}$, where D_1 and D_2 are non-singular matrices with D_1 of order r. Thus, if $Y_A = DUY_B$, then $AB \in \mathbb{C}_D^{n\times n}$ with $\sigma(AB) \subset [0,1]$.*

Proof. According to (1),

$$\begin{aligned}
&AB = Y_A^{-1}\begin{pmatrix} I_r & 0\\ 0 & 0\end{pmatrix} Y_A Y_B^{-1}\begin{pmatrix} I_s & 0\\ 0 & 0\end{pmatrix} Y_B \Rightarrow\\
&Y_A ABY_A^{-1} = \begin{pmatrix} I_r & 0\\ 0 & 0\end{pmatrix}\\
&Y_A Y_B^{-1}\begin{pmatrix} I_s & 0\\ 0 & 0\end{pmatrix}(Y_A Y_B^{-1})^{-1} = \begin{pmatrix} I_r & 0\\ 0 & 0\end{pmatrix} DU\begin{pmatrix} I_s & 0\\ 0 & 0\end{pmatrix} U^{-1}D^{-1} \Rightarrow\\
&(Y_A^{-1}D)^{-1}ABY_A^{-1}D = D^{-1}\begin{pmatrix} I_r & 0\\ 0 & 0\end{pmatrix}\\
&DU\begin{pmatrix} I_s & 0\\ 0 & 0\end{pmatrix} U^{-1} = \begin{pmatrix} I_r & 0\\ 0 & 0\end{pmatrix} U\begin{pmatrix} I_s & 0\\ 0 & 0\end{pmatrix} U^{*}.
\end{aligned}$$

Taking into account that $\begin{pmatrix} I_r & 0\\ 0 & 0\end{pmatrix}, U\begin{pmatrix} I_s & 0\\ 0 & 0\end{pmatrix} U^{*} \in \mathbb{C}_{HP}^{n\times n}$, we may conclude that $AB \in \mathbb{C}_D^{n\times n}$ with $\sigma(AB) \subset [0,1]$. □

PROPOSITION 3.13. *Let $A,B \in \mathbb{C}_P^{n\times n}$ be with representation in (1). Thus, if $C = C^*$, then $AB \in \mathbb{C}_D^{n\times n}$ with $\sigma(AB) \subset [0,1]$.*

Proof. According to Proposition 3.11, if $C = C^*$, then $Y_B = UY_A$ for some $U \in \mathbb{C}_U^{n\times n}$, and by Lemma 3.9, $P_A = P_B$, and therefore $AB \in \mathbb{C}_D^{n\times n}$ with $\sigma(AB) \subset [0,1]$, see [5, Theorem 3.15]. □

REMARK 9. On the other hand, concerning Proposition 3.13, it may occur that $AB \in \mathbb{C}_D^{n\times n}$ with $\sigma(AB) \subset [0,1]$, but $C \neq C^*$ for some Y_A. Indeed, it suffices to keep in mind the following example:

Let $A,B,C \in \mathbb{C}_P^{3\times 3}$ and $Y_A \in \mathbb{C}^{3\times 3}$ be so that

$$A = \begin{pmatrix} 1 & 0 & -0.5\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix},\ B = \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 1 & 0 & 0\end{pmatrix},\ AB = \begin{pmatrix} 0.5 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix} \text{ and } Y_A = \begin{pmatrix} 1 & 0 & 0.4472\\ 0 & 1 & 0\\ 0 & 0 & 0.8944\end{pmatrix}.$$

Thus,

$$Y_A^{-1}AY_A = \begin{pmatrix} I_2 & 0\\ 0 & 0\end{pmatrix} \quad \text{and} \quad Y_A^{-1}BY_A = C = \begin{pmatrix} 0.5 & 0 & 0.2236\\ 0 & 1 & 0\\ 1.1180 & 0 & 0.5\end{pmatrix}.$$

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