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BEREZIN NUMBER INEQUALITIES VIA POSITIVITY OF 2×2 BLOCK MATRICES

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Abstract. Suppose $\mathscr{B}(\mathscr{H}(\Omega))$ is the set of all bounded linear operators acting on a reproducing kernel Hilbert space $\mathscr{H}(\Omega)$. Applying the positivity criteria of 2×2 block matrices, we develop several new upper bounds for the Berezin number of operators in $\mathscr{B}(\mathscr{H}(\Omega))$ involving Berezin norm, which are better than the earlier ones. Among other results, we obtain that if $T,S \in \mathscr{B}(\mathscr{H}(\Omega))$ and $0<\alpha<1$, then

$$\mathbf{ber}^p(T\pm S) \leqslant \frac{1}{2}\left\| \left(|T^*|^{2\alpha}+|S^*|^{2\alpha}\right)^p + \left(|T|^{2(1-\alpha)}+|S|^{2(1-\alpha)}\right)^p \right\|_{ber}, \text{ for all } p \geqslant 1.$$

By letting $S=0$, it follows that $\mathbf{ber}^p(T) \leqslant \frac{1}{2}\left\| |T^*|^{2\alpha p}+|T|^{2(1-\alpha)p}\right\|_{ber}$, for all $p\geqslant 1$.

1. Introduction

Let $\mathscr{B}(\mathscr{H})$ be the C^*- algebra of all bounded linear operators acting on a complex Hilbert space $\mathscr{H}$ with inner product $\langle .,. \rangle$ and associated norm $\|\cdot\|$. An operator $T \in \mathscr{B}(\mathscr{H})$ is called positive if $\langle Tx,x\rangle \geqslant 0$ for all $x\in\mathscr{H}$, and we then write $T\geqslant 0$. Let Ω be a nonempty set. A functional Hilbert space $\mathscr{H}(\Omega)$ is a Hilbert space of complex valued functions on Ω, which has the property that point evaluations are continuous, i.e., for each $\lambda\in\Omega$ the map $f\longmapsto f(\lambda)$ is a continuous linear functional on $\mathscr{H}(\Omega)$. The Riesz representation theorem ensues that for each $\lambda\in\Omega$ there exists a unique element $k_\lambda\in\mathscr{H}$ such that $f(\lambda)=\langle f,k_\lambda\rangle$ for all $f\in\mathscr{H}(\Omega)$. The set $\{k_\lambda:\lambda\in\Omega\}$ is called the reproducing kernel of the space $\mathscr{H}(\Omega)$. If $\{e_n\}_{n\geqslant 0}$ is an orthonormal basis for a functional Hilbert space $\mathscr{H}(\Omega)$, then the reproducing kernel of $\mathscr{H}(\Omega)$ is given by $k_\lambda(z)=\sum\limits_{n=0}^{\infty}\overline{e_n(\lambda)}e_n(z)$. For $\lambda\in\Omega$, let $\hat{k}_\lambda=\frac{k_\lambda}{\|k_\lambda\|}$ be the normalized reproducing kernel of $\mathscr{H}(\Omega)$. Let T be a bounded linear operator on $\mathscr{H}(\Omega)$, i.e., let $T\in\mathscr{B}(\mathscr{H}(\Omega))$. The Berezin symbol of T, is the function $\tilde{T}$ on Ω, defined by

$$\tilde{T}(\lambda) := \left\langle T\hat{k}_\lambda,\hat{k}_\lambda\right\rangle.$$

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The Berezin set and the Berezin number of the operator T are defined respectively by

$$\mathbf{Ber}(T) := \left\{ \left\langle T\hat{k}_\lambda, \hat{k}_\lambda \right\rangle : \lambda \in \Omega \right\}$$

and

$$\mathbf{ber}(T) := \sup \left\{ \left| \left\langle T\hat{k}_\lambda, \hat{k}_\lambda \right\rangle \right| : \lambda \in \Omega \right\}.$$

It is clear that the Berezin symbol $\tilde{T}$ is the bounded function on Ω, whose value lies in the numerical range of T. Therefore,

$$\mathbf{Ber}(T) \subseteq W(T) \quad \text{and} \quad \mathbf{ber}(T) \leqslant \omega(T),$$

where

$$W(T) = \{\langle Tx, x\rangle : x \in \mathscr{H}(\Omega),\ \|x\| = 1\}$$

is the numerical range of T and

$$\omega(T) = \sup\{|\langle Tx, x\rangle| : x \in \mathscr{H}(\Omega),\ \|x\| = 1\}$$

is the numerical radius of T. For more on numerical range and numerical radius inequalities readers can see the recent books [7, 21]. The Berezin number of an operator $T \in \mathscr{B}(\mathscr{H}(\Omega))$ satisfies the following properties:

(i) $\mathbf{ber}(T) \leqslant \|T\|$.

(ii) $\mathbf{ber}(\alpha T) = |\alpha|\, \mathbf{ber}(T)$ for all $\alpha \in \mathbb{C}$.

(iii) $\mathbf{ber}(T+S) \leqslant \mathbf{ber}(T) + \mathbf{ber}(S)$ for all $T, S \in \mathscr{B}(\mathscr{H}(\Omega))$.

The Berezin symbol of an operator provides important information about that operator. This has been studied in detail for Toeplitz and Hankel operators on Hardy and Bergman spaces. On the most familiar reproducing kernel Hilbert spaces (including Hardy and Bergman spaces), the Berezin symbol uniquely determines the operator, i.e., if $\tilde{T}(\lambda) = \tilde{S}(\lambda)$ for all $\lambda \in \Omega$, then $T = S$. This property is known as the "Ber" property. Therefore, the Berezin number, $\mathbf{ber}(\cdot)$ defines a norm on $\mathscr{B}(\mathscr{H}(\Omega))$ when $\mathscr{H}(\Omega)$ has the Ber property. For example, the Hardy space $H^2(D)$ is a reproducing kernel Hilbert space of analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ defined on the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$ such that $\sum_{n=0}^{\infty} |a_n|^2 < \infty$, with reproducing kernel $k_\lambda(z) = \sum_{n=0}^{\infty} \overline{\lambda^n} z^n = 1/(1 - \overline{\lambda} z)$.

Due to importance of the concept of Berezin symbol, the Berezin symbol and Berezin number have been studied by many mathematicians over the years, see [1, 2, 3, 4, 6, 9, 10, 11, 12, 13, 15, 18, 19, 22]. Now, here for our purpose we recall the Berezin norm of an operator $T \in \mathscr{B}(\mathscr{H}(\Omega))$. The Berezin norm of T, denoted as $\|T\|_{ber}$, is defined by

$$\|T\|_{ber} := \sup \left\{ \left| \left\langle T\hat{k}_\lambda, \hat{k}_\mu \right\rangle \right| : \lambda, \mu \in \Omega \right\}.$$

In this article, considering 2×2 positive block matrices, we obtain various inequalities involving the Berezin norm and Berezin number of bounded linear operators defined on $\mathscr{H}(\Omega)$. The inequalities involving sum and product of operators are also provided.

2. Main results

In order to prove our results we state the following lemmas.

LEMMA 1. [14, p. 26] *Let $a,b \geqslant 0$ and $0 \leqslant \alpha \leqslant 1$. Then*

$$a^{\alpha}b^{1-\alpha} \leqslant \alpha a+(1-\alpha)b \leqslant (\alpha a^{r}+(1-\alpha)b^{r})^{\frac{1}{r}},$$

for all $r \geqslant 1$.

LEMMA 2. [17] *Let T be a positive operator in $\mathscr{B}(\mathscr{H})$. Then for any unit vector $x \in \mathscr{H}$,*

$$\langle Tx,x\rangle^{p} \leqslant \langle T^{p}x,x\rangle, \text{ for all } \quad p \geqslant 1.$$

LEMMA 3. [16] *Let $T,S,R \in \mathscr{B}(\mathscr{H})$ with T and R are positive operators. Then $\begin{pmatrix} T & S^{*} \\ S & R \end{pmatrix}$ is positive in $\mathscr{B}(\mathscr{H}\oplus\mathscr{H})$ if and only if*

$$|\langle Sx,y\rangle|^{2} \leqslant \langle Tx,x\rangle \langle Ry,y\rangle, \text{ for all } x,y \in \mathscr{H}.$$

LEMMA 4. [16] *Let $T \in \mathscr{B}(\mathscr{H})$ and let $0<\alpha<1$. Then*

$$\begin{pmatrix} |T^{*}|^{2\alpha} & T^{*} \\ T & |T|^{2(1-\alpha)} \end{pmatrix} \geqslant 0,$$

where $|T|=(T^{*}T)^{1/2}$ and $|T^{*}|=(TT^{*})^{1/2}$.

First we prove the following known proposition (recently proved in [5]) by using block matrix approach.

PROPOSITION 1. *If $T \in \mathscr{B}(\mathscr{H}(\Omega))$ is positive, then*

$$\mathbf{ber}(T)=\|T\|_{ber}.$$

Proof. Since T is positive, $\begin{pmatrix} T & T \\ T & T \end{pmatrix} \in \mathscr{B}(\mathscr{H}(\Omega)\oplus\mathscr{H}(\Omega))$ is also positive. So, from Lemma 3 we have

$$\left|\langle T\hat{k}_{\lambda},\hat{k}_{\mu}\rangle\right|^{2} \leqslant \langle T\hat{k}_{\lambda},\hat{k}_{\lambda}\rangle \langle T\hat{k}_{\mu},\hat{k}_{\mu}\rangle,$$

for all normalized reproducing kernels $\hat{k}_{\lambda},\hat{k}_{\mu} \in \mathscr{H}(\Omega)$. This implies $\mathbf{ber}(T) \geqslant \|T\|_{ber}$. Also, $\mathbf{ber}(T) \leqslant \|T\|_{ber}$ is trivial. □

Following Proposition 1 it is easy to observe that when $T \in \mathscr{B}(\mathscr{H}(\Omega))$ is a self-adjoint operator, then

$$\mathbf{ber}^{1/2}(T^{2})=\|T^{2}\|_{ber}^{1/2}=\sup_{\lambda\in\Omega}\|T\hat{k}_{\lambda}\|. \tag{1}$$

Now, we prove our first result.

THEOREM 1. *Let $T,S,R \in \mathscr{B}(\mathscr{H}(\Omega))$ with T and R are positive operators. If $\begin{pmatrix} T & S^* \\ S & R \end{pmatrix}$ is positive, then*

$$\mathbf{ber}^p(S) \leqslant \frac{1}{2}\|T^p + R^p\|_{ber}, \quad \textit{for all} \quad p \geqslant 1.$$

Proof. Let $\hat{k}_\lambda$ be the normalized reproducing kernel of $\mathscr{H}(\Omega)$. Then it follows from Lemma 3 that

$$\begin{aligned}
\left|\langle S\hat{k}_\lambda, \hat{k}_\lambda\rangle\right| &\leqslant \langle T\hat{k}_\lambda, \hat{k}_\lambda\rangle^{\frac{1}{2}} \langle R\hat{k}_\lambda, \hat{k}_\lambda\rangle^{\frac{1}{2}} \\
&\leqslant \frac{1}{2}\left(\langle T\hat{k}_\lambda, \hat{k}_\lambda\rangle + \langle R\hat{k}_\lambda, \hat{k}_\lambda\rangle\right) \\
&\leqslant \left(\frac{\langle T\hat{k}_\lambda, \hat{k}_\lambda\rangle^p + \langle R\hat{k}_\lambda, \hat{k}_\lambda\rangle^p}{2}\right)^{\frac{1}{p}} \text{ (by Lemma 1)} \\
&\leqslant \left(\frac{\langle T^p\hat{k}_\lambda, \hat{k}_\lambda\rangle + \langle R^p\hat{k}_\lambda, \hat{k}_\lambda\rangle}{2}\right)^{\frac{1}{p}} \text{ (by Lemma 2).}
\end{aligned}$$

Consequently, we infer that

$$\begin{aligned}
\left|\langle S\hat{k}_\lambda, \hat{k}_\lambda\rangle\right|^p &\leqslant \frac{1}{2}\langle (T^p + R^p)\hat{k}_\lambda, \hat{k}_\lambda\rangle \\
&\leqslant \frac{1}{2}\mathbf{ber}(T^p + R^p) \\
&= \frac{1}{2}\|T^p + R^p\|_{ber} \text{ (by Proposition 1).}
\end{aligned}$$

Thus,

$$\left|\langle S\hat{k}_\lambda, \hat{k}_\lambda\rangle\right|^p \leqslant \frac{1}{2}\|T^p + R^p\|_{ber}.$$

By taking supremum over $\lambda \in \Omega$, we get the desired inequality. □

Considering $p = 1$ in Theorem 1, we have the following corollary.

COROLLARY 1. *Let $T,S,R \in \mathscr{B}(\mathscr{H}(\Omega))$ with T and R are positive operators. If $\begin{pmatrix} T & S^* \\ S & R \end{pmatrix}$ is positive, then*

$$\mathbf{ber}(S) \leqslant \frac{1}{2}\|T + R\|_{ber}.$$

An application of Theorem 1 leads to the following result.

THEOREM 2. *Let* $T,S,R,X,Y \in \mathscr{B}(\mathscr{H}(\Omega))$ *with* T *and* R *are positive operators. If* $\begin{pmatrix} T & S^* \\ S & R \end{pmatrix}$ *is positive, then*

$$\mathbf{ber}^p(YSX) \leqslant \frac{1}{2}\left\| \left(X^*TX\right)^p + \left(YRY^*\right)^p \right\|_{ber}, \text{ for all } \quad p \geqslant 1.$$

Proof. Since $\begin{pmatrix} T & S^* \\ S & R \end{pmatrix} \geqslant 0$,

$$\begin{pmatrix} X & 0 \\ 0 & Y^* \end{pmatrix}^* \begin{pmatrix} T & S^* \\ S & R \end{pmatrix} \begin{pmatrix} X & 0 \\ 0 & Y^* \end{pmatrix} = \begin{pmatrix} X^*TX & (YSX)^* \\ YSX & YRY^* \end{pmatrix} \geqslant 0.$$

Applying Theorem 1, we obtain

$$\mathbf{ber}^p(YSX) \leqslant \frac{1}{2}\left\| \left(X^*TX\right)^p + \left(YRY^*\right)^p \right\|_{ber},$$

as desired. □

As another application of Theorem 1 we get the following theorem.

THEOREM 3. *Let* $T,S \in \mathscr{B}(\mathscr{H}(\Omega))$ *and let* $0 < \alpha < 1$. *Then*

$$\mathbf{ber}^p(T \pm S) \leqslant \frac{1}{2}\left\| \left(|T^*|^{2\alpha} + |S^*|^{2\alpha}\right)^p + \left(|T|^{2(1-\alpha)} + |S|^{2(1-\alpha)}\right)^p \right\|_{ber},$$

for all $p \geqslant 1$.

Proof. Since the sum of two positive operator matrices is also positive, by using Lemma 4, we have

$$\begin{pmatrix} |T^*|^{2\alpha} + |S^*|^{2\alpha} & T^* + S^* \\ T + S & |T|^{2(1-\alpha)} + |S|^{2(1-\alpha)} \end{pmatrix} \geqslant 0.$$

Now, by applying Theorem 1, we obtain

$$\mathbf{ber}^p(T + S) \leqslant \frac{1}{2}\left\| \left(|T^*|^{2\alpha} + |S^*|^{2\alpha}\right)^p + \left(|T|^{2(1-\alpha)} + |S|^{2(1-\alpha)}\right)^p \right\|_{ber}. \tag{2}$$

By replacing S by $-S$ in (2), we have

$$\mathbf{ber}^p(T - S) \leqslant \frac{1}{2}\left\| \left(|T^*|^{2\alpha} + |S^*|^{2\alpha}\right)^p + \left(|T|^{2(1-\alpha)} + |S|^{2(1-\alpha)}\right)^p \right\|_{ber}. \tag{3}$$

From the inequalities (2) and (3), we obtain

$$\mathbf{ber}^p(T \pm S) \leqslant \frac{1}{2}\left\| \left(|T^*|^{2\alpha} + |S^*|^{2\alpha}\right)^p + \left(|T|^{2(1-\alpha)} + |S|^{2(1-\alpha)}\right)^p \right\|_{ber},$$

as desired. □

By letting $\alpha = \frac{1}{2}$ in Theorem 3, we get the following inequality.

COROLLARY 2. *Let* $T,S \in \mathscr{B}(\mathscr{H}(\Omega))$. *Then*

$$\mathbf{ber}^p(T \pm S) \leqslant \frac{1}{2}\|(|T^*|+|S^*|)^p + (|T|+|S|)^p\|_{ber}, \text{ for all } \ p \geqslant 1.$$

Taking $S=0$ in Theorem 3, we get the following corollary.

COROLLARY 3. *Let* $T \in \mathscr{B}(\mathscr{H}(\Omega))$ *and let* $0<\alpha<1$. *Then*

$$\mathbf{ber}^p(T) \leqslant \frac{1}{2}\left\||T^*|^{2\alpha p} + |T|^{2(1-\alpha)p}\right\|_{ber}, \text{ for all } \ p \geqslant 1.$$

In particular, by taking $\alpha = \frac{1}{2}$ in Corollary 3, we get

$$\mathbf{ber}^p(T) \leqslant \frac{1}{2}\||T^*|^p + |T|^p\|_{ber},$$

see also in [20].

Again, as an application of Theorem 1 we get the following theorem.

THEOREM 4. *Let* $T,S,R,Q \in \mathscr{B}(\mathscr{H}(\Omega))$. *Then*

$$\mathbf{ber}^p\left(TR^* \pm SQ^*\right) \leqslant \frac{1}{2}\left\|\left(TT^*+SS^*\right)^p + \left(RR^*+QQ^*\right)^p\right\|_{ber}, \text{ for all } \ p \geqslant 1.$$

Proof. Let $\begin{pmatrix} T & S \\ R & Q \end{pmatrix} \in \mathscr{B}(\mathscr{H}(\Omega)\oplus\mathscr{H}(\Omega))$, then $\begin{pmatrix} T & S \\ R & Q \end{pmatrix}\begin{pmatrix} T & S \\ R & Q \end{pmatrix}^*$ is positive. Now, it can be verified that

$$\begin{aligned}\begin{pmatrix} T & S \\ R & Q \end{pmatrix}\begin{pmatrix} T & S \\ R & Q \end{pmatrix}^* &= \begin{pmatrix} TT^*+SS^* & TR^*+SQ^* \\ RT^*+QS^* & RR^*+QQ^* \end{pmatrix} \\ &= \begin{pmatrix} TT^*+SS^* & \left(RT^*+QS^*\right)^* \\ RT^*+QS^* & RR^*+QQ^* \end{pmatrix}.\end{aligned}$$

Since TT^*+SS^* and RR^*+QQ^* are positive operators, applying Theorem 1, we get for all $p \geqslant 1$,

$$\mathbf{ber}^p\left(RT^*+QS^*\right) \leqslant \frac{1}{2}\left\|\left(TT^*+SS^*\right)^p + \left(RR^*+QQ^*\right)^p\right\|_{ber}.$$

Moreover, we have

$$\mathbf{ber}\left(RT^*+QS^*\right) = \mathbf{ber}\left((TR^*+SQ^*)^*\right) = \mathbf{ber}(TR^*+SQ^*).$$

Consequently, we get

$$\mathbf{ber}^p\left(TR^*+SQ^*\right) \leqslant \frac{1}{2}\left\|\left(TT^*+SS^*\right)^p + \left(RR^*+QQ^*\right)^p\right\|_{ber}, \text{ for all } \ p \geqslant 1.$$

Replacing S by $-S$ in the above inequality, we have

$$\mathbf{ber}^p\left(TR^* - SQ^*\right) \leqslant \frac{1}{2}\left\|\left(TT^* + SS^*\right)^p + \left(RR^* + QQ^*\right)^p\right\|_{ber}, \text{ for all } \ p \geqslant 1.$$

Therefore, we infer that

$$\mathbf{ber}^p\left(TR^* \pm SQ^*\right) \leqslant \frac{1}{2}\left\|\left(TT^* + SS^*\right)^p + \left(RR^* + QQ^*\right)^p\right\|_{ber}, \text{ for all } \ p \geqslant 1.$$

This completes the proof. □

REMARK 1. Let $T,S \in \mathscr{B}(\mathscr{H}(\Omega))$.

(i) By letting $p = 1, R = S$ and $Q = \pm T$ in Theorem 4, we get the following inequality:

$$\mathbf{ber}\left(TS^* \pm ST^*\right) \leqslant \|TT^* + SS^*\|_{ber}.$$

(ii) Considering the case $p = 1, R^* = S$ and $Q^* = T$ in Theorem 4, we obtain the following inequality:

$$\mathbf{ber}(TS \pm ST) \leqslant \frac{1}{2}\|T^*T + TT^* + S^*S + SS^*\|_{ber}.$$

We next obtain the following corollary.

COROLLARY 4. *Let* $T,R \in \mathscr{B}(\mathscr{H}(\Omega))$. *Then for all* $p \geqslant 1$,

$$\mathbf{ber}^p(RT) \leqslant \frac{1}{2}\|(T^*T)^p + (RR^*)^p\|_{ber}.$$

Proof. Putting $S = Q = 0$ in the Theorem 4, we obtain

$$\mathbf{ber}^p\left(TR^*\right) \leqslant \frac{1}{2}\left\|\left(TT^*\right)^p + \left(RR^*\right)^p\right\|_{ber}, \text{ for all } \ p \geqslant 1.$$

Replacing T by T^* in the above inequality, we get

$$\mathbf{ber}^p(T^*R^*) \leqslant \frac{1}{2}\|(T^*T)^p + (RR^*)^p\|_{ber}.$$

Since, $\mathbf{ber}(T^*R^*) = \mathbf{ber}\left((RT)^*\right) = \mathbf{ber}(RT)$,

$$\mathbf{ber}^p(RT) \leqslant \frac{1}{2}\|(T^*T)^p + (RR^*)^p\|_{ber}, \text{ for all } \ p \geqslant 1,$$

as desired. □

REMARK 2. (i) Let $T \in \mathscr{B}(\mathscr{H}(\Omega))$. Then it follows from Corollaries 3 and 4 that

$$\max\left\{\mathbf{ber}^2(T), \mathbf{ber}\left(T^2\right)\right\} \leqslant \frac{1}{2}\|T^*T + TT^*\|_{ber}. \tag{4}$$

(ii) Let $R, T \in \mathscr{B}(\mathscr{H}(\Omega))$. By letting $p = 1$ in Corollary 4, we get

$$\mathbf{ber}(RT) \leqslant \frac{1}{2}\|T^*T + RR^*\|_{ber},$$

see also in [20].

Next inequalities read as follows.

COROLLARY 5. *Let $T, S, R, Q, X, Y \in \mathscr{B}(\mathscr{H}(\Omega))$. Then, for all $p \geqslant 1$,*

$$\begin{aligned}
(i)\quad & \mathbf{ber}^p(Y(RT^* + QS^*)X) \\
&\leqslant \frac{1}{2}\|(X^*(TT^* + SS^*)X)^p + (Y(RR^* + QQ^*)Y^*)^p\|_{ber}, \\
(ii)\quad & \mathbf{ber}^p(Y(TT^* \pm SS^*)X) \\
&\leqslant \frac{1}{2}\|(X^*(TT^* + SS^*)X)^p + (Y(TT^* + SS^*)Y^*)^p\|_{ber}.
\end{aligned}$$

Proof. (i) The proof follows from Theorem 2 by using the fact

$$\begin{pmatrix} TT^* + SS^* & (RT^* + QS^*)^* \\ RT^* + QS^* & RR^* + QQ^* \end{pmatrix}$$

is positive (see Theorem 4).

(ii) Taking $R = T$ and $Q = \pm S$ in (i) we get the desired inequality. $\square$

In order to prove our next results we need the following lemma.

LEMMA 5. [8] *Let $x, y, z \in \mathscr{H}$ with $\|z\| = 1$. Then*

$$|\langle x, z\rangle||\langle z, y\rangle| \leqslant \frac{1}{2}(\|x\|\|y\| + |\langle x, y\rangle|).$$

Now we prove the following theorem.

THEOREM 5. *Let $T, S, R \in \mathscr{B}(\mathscr{H}(\Omega))$ with T and R are positive. If $\begin{pmatrix} T & S^* \\ S & R \end{pmatrix}$ is positive, then*

$$\mathbf{ber}^{2p}(S) \leqslant \frac{1}{4}\left\|T^{2p} + R^{2p}\right\|_{ber} + \frac{1}{2}\mathbf{ber}^p(RT), \text{ for all } p \geqslant 1.$$

Proof. Let $\hat{k}_\lambda$ be the normalized reproducing kernel of $\mathscr{H}(\Omega)$. From Lemma 3, we have

$$\begin{aligned}
&\left|\left\langle S\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\right|^2\\
&\leqslant \left\langle T\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\left\langle R\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\\
&\leqslant \frac{1}{2}\|T\hat{k}_\lambda\|\|R\hat{k}_\lambda\|+\frac{1}{2}|\langle T\hat{k}_\lambda,R\hat{k}_\lambda\rangle| \text{ (by Lemma 5)}\\
&\leqslant \left(\frac{\|T\hat{k}_\lambda\|^p\|R\hat{k}_\lambda\|^p+|\langle T\hat{k}_\lambda,R\hat{k}_\lambda\rangle|^p}{2}\right)^{\frac{1}{p}} \text{ (by Lemma 1)}\\
&\leqslant \left(\frac{1}{4}(\|T\hat{k}_\lambda\|^{2p}+\|R\hat{k}_\lambda\|^{2p})+\frac{1}{2}|\langle T\hat{k}_\lambda,R\hat{k}_\lambda\rangle|^p\right)^{\frac{1}{p}}\\
&=\left(\frac{1}{4}(\langle T^2\hat{k}_\lambda,\hat{k}_\lambda\rangle^p+\langle R^2\hat{k}_\lambda,\hat{k}_\lambda\rangle^p)+\frac{1}{2}|\langle RT\hat{k}_\lambda,\hat{k}_\lambda\rangle|^p\right)^{\frac{1}{p}}\\
&\leqslant \left(\frac{1}{4}(\langle T^{2p}\hat{k}_\lambda,\hat{k}_\lambda\rangle+\langle R^{2p}\hat{k}_\lambda,\hat{k}_\lambda\rangle)+\frac{1}{2}|\langle RT\hat{k}_\lambda,\hat{k}_\lambda\rangle|^p\right)^{\frac{1}{p}} \text{ (by Lemma 2)}\\
&\leqslant \left(\frac{1}{4}\mathbf{ber}\left(T^{2p}+R^{2p}\right)+\frac{1}{2}\mathbf{ber}^p(RT)\right)^{\frac{1}{p}}\\
&=\left(\frac{1}{4}\left\|T^{2p}+R^{2p}\right\|_{ber}+\frac{1}{2}\mathbf{ber}^p(RT)\right)^{\frac{1}{p}}.
\end{aligned}$$

Consequently, we infer that

$$\left|\left\langle S\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\right|^{2p}\leqslant \frac{1}{4}\left\|T^{2p}+R^{2p}\right\|_{ber}+\frac{1}{2}\mathbf{ber}^p(RT).$$

Taking supremum over $\lambda\in\Omega$, we get the desired inequality. □

Considering $p=1$ in Theorem 5 we obtain the following corollary.

COROLLARY 6. *Let $T,S,R\in\mathscr{B}(\mathscr{H}(\Omega))$ with T and R are positive. If $\begin{pmatrix} T & S^* \\ S & R\end{pmatrix}$ is positive, then*

$$\mathbf{ber}^2(S)\leqslant \frac{1}{4}\left\|T^2+R^2\right\|_{ber}+\frac{1}{2}\mathbf{ber}(RT).$$

REMARK 3. Let $T\in\mathscr{B}(\mathscr{H}(\Omega))$. Then using Lemma 4 in Corollary 6 we obtain

$$\mathbf{ber}^2(T)\leqslant \frac{1}{4}\left\||T|^2+|T^*|^2\right\|_{ber}+\frac{1}{2}\mathbf{ber}(|T||T^*|). \tag{5}$$

It is easy to verify that $\mathbf{ber}(|T||T^*|)\leqslant \frac{1}{2}\left\||T|^2+|T^*|^2\right\|_{ber}$. Therefore, the bound (5) is stronger than the bound obtained in Corollary 3 (for $p=2$).

Next theorem is as follows.

THEOREM 6. *Let $T,S,R \in \mathscr{B}(\mathscr{H}(\Omega))$ with T and R are positive. If $\begin{pmatrix} T & S^* \\ S & R \end{pmatrix}$ is positive, then*

$$\mathbf{ber}^2(S) \leqslant \frac{1}{2}\left\|T^2\right\|_{ber}^{1/2}\left\|R^2\right\|_{ber}^{1/2} + \frac{1}{2}\mathbf{ber}(RT).$$

Proof. Let $\hat{k}_\lambda$ be the normalized reproducing kernel of $\mathscr{H}(\Omega)$. Then, it follows from Lemma 3 that

$$\begin{aligned}
\left|\langle S\hat{k}_\lambda, \hat{k}_\lambda\rangle\right|^2 &\leqslant \langle T\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle R\hat{k}_\lambda, \hat{k}_\lambda\rangle \\
&\leqslant \frac{1}{2}\|T\hat{k}_\lambda\|\|R\hat{k}_\lambda\| + \frac{1}{2}|\langle T\hat{k}_\lambda, R\hat{k}_\lambda\rangle| \text{ (by Lemma 5)} \\
&= \frac{1}{2}\|T\hat{k}_\lambda\|\|R\hat{k}_\lambda\| + \frac{1}{2}|\langle RT\hat{k}_\lambda, \hat{k}_\lambda\rangle_A| \\
&\leqslant \frac{1}{2}\left\|T^2\right\|_{ber}^{1/2}\left\|R^2\right\|_{ber}^{1/2} + \frac{1}{2}\mathbf{ber}(RT) \text{ (by (1))}.
\end{aligned}$$

By taking supremum over $\lambda \in \Omega$, we get the desired result. □

As an application of Theorem 6 we obtain the following corollary.

COROLLARY 7. *Let $T,S \in \mathscr{B}(\mathscr{H}(\Omega))$. Then*

$$\mathbf{ber}^2(ST) \leqslant \frac{1}{2}\left\|(T^*T)^2\right\|_{ber}^{1/2}\left\|(SS^*)^2\right\|_{ber}^{1/2} + \frac{1}{2}\mathbf{ber}\left(S(TS)^*T\right).$$

Proof. In view of Lemma 3 it is not difficult to verify that $\begin{pmatrix} T^*T & (ST)^* \\ ST & SS^* \end{pmatrix}$ is positive. By applying Theorem 6 we obtain that

$$\begin{aligned}
\mathbf{ber}^2(ST) &\leqslant \frac{1}{2}\left\|(T^*T)^2\right\|_{ber}^{1/2}\left\|(SS^*)^2\right\|_{ber}^{1/2} + \frac{1}{2}\mathbf{ber}\left(SS^*T^*T\right) \\
&= \frac{1}{2}\left\|(T^*T)^2\right\|_{ber}^{1/2}\left\|(SS^*)^2\right\|_{ber}^{1/2} + \frac{1}{2}\mathbf{ber}\left(S(TS)^*T\right),
\end{aligned}$$

as desired. □

Also, by applying Corollary 6 we obtain the following.

COROLLARY 8. *Let $T,S \in \mathscr{B}(\mathscr{H}(\Omega))$. Then*

$$\mathbf{ber}^2(ST) \leqslant \frac{1}{4}\left\|(T^*T)^2 + (SS^*)^2\right\|_{ber} + \frac{1}{2}\mathbf{ber}\left(S(TS)^*T\right).$$

Proof. Since $\begin{pmatrix} T^*T & (ST)^* \\ ST & SS^* \end{pmatrix}$ is positive, it follows from Corollary 6 that

$$\mathbf{ber}^2(ST) \leqslant \frac{1}{4}\left\|(T^*T)^2+(SS^*)^2\right\|_{ber}+\frac{1}{2}\mathbf{ber}\left(S(TS)^*T\right),$$

as required. $\square$

REMARK 4. By Remark 2 (ii), we have

$$\mathbf{ber}\left(S(TS)^*T\right) \leqslant \frac{1}{2}\left\|(T^*T)^2+(SS^*)^2\right\|_{ber}.$$

Therefore, Corollary 8 refines the following existing inequality (see [20, Theorem 3.6])

$$\mathbf{ber}^2(ST) \leqslant \frac{1}{2}\left\|(T^*T)^2+(SS^*)^2\right\|_{ber}.$$

Another application of Theorem 6 leads to the following corollary.

COROLLARY 9. *Let $T,S,R,X,Y \in \mathscr{B}(\mathscr{H}(\Omega))$ with T and R are positive. If $\begin{pmatrix} T & S^* \\ S & R \end{pmatrix}$ is positive, then*

$$\mathbf{ber}^2\left(YSX\right) \leqslant \frac{1}{2}\left\|(X^*TX)^2\right\|_{ber}^{1/2}\left\|(YRY^*)^2\right\|_{ber}^{1/2}+\frac{1}{2}\mathbf{ber}\left(YR(XY)^*TX\right).$$

Proof. Since $\begin{pmatrix} T & S^* \\ S & R \end{pmatrix}$ is positive, $\begin{pmatrix} X^*TX & (YSX)^* \\ YSX & YRY^* \end{pmatrix}$ is also positive (see Theorem 2), and so by applying Theorem 6, we have

$$\begin{aligned}
\mathbf{ber}^2\left(YSX\right) &\leqslant \frac{1}{2}\left\|(X^*TX)^2\right\|_{ber}^{1/2}\left\|(YRY^*)^2\right\|_{ber}^{1/2}+\frac{1}{2}\mathbf{ber}\left(YRY^*X^*TX\right)\\
&= \frac{1}{2}\left\|(X^*TX)^2\right\|_{ber}^{1/2}\left\|(YRY^*)^2\right\|_{ber}^{1/2}+\frac{1}{2}\mathbf{ber}\left(YR(XY)^*TX\right). \quad \square
\end{aligned}$$

Also, another application of Theorem 6 gives the following.

COROLLARY 10. *Let $T,S,R,Q \in \mathscr{B}(\mathscr{H}(\Omega))$. Then*

$$\begin{aligned}
&\mathbf{ber}^2\left(TR^* \pm SQ^*\right)\\
&\leqslant \frac{1}{2}\left\|(TT^*+SS^*)^2\right\|_{ber}^{1/2}\left\|(RR^*+QQ^*)^2\right\|_{ber}^{1/2}+\frac{1}{2}\mathbf{ber}\left((RR^*+QQ^*)(TT^*+SS^*)\right).
\end{aligned}$$

Proof. The proof follows from Theorem 6 and by using the fact

$$\begin{pmatrix} TT^*+SS^* & (RT^*+QS^*)^* \\ RT^*+QS^* & RR^*+QQ^* \end{pmatrix}$$

is positive (see Theorem 4). $\square$

Finally, as an immediate consequence of Corollary 10 we get the following result.

COROLLARY 11. *Let $T,S \in \mathscr{B}(\mathscr{H}(\Omega))$, then*

$$\mathbf{ber}^2(T \pm S) \leqslant 2\|TT^* + SS^*\|_{ber}.$$

Proof. The proof follows from Corollary 10 by putting $R = Q = I$. □

REFERENCES

[1] M. BAKHERAD, R. LASHKARIPOUR, M. HAJMOHAMADI AND U. YAMANCI, *Complete refinements of the Berezin number inequalities*, J. Math. Inequal. **13** (2019), no. 4, 1117–1128.

[2] M. BAKHERAD, M. HAJMOHAMADI, R. LASHKARIPOUR AND S. SAHOO, *Some extensions of Berezin number inequalities on operators*, Rocky Mountain J. Math. **51** (2021), no. 6, 1941–1951.

[3] P. BHUNIA, M. T. GARAYEV, K. PAUL AND R. TAPDIGOGLU, *Some new applications of Berezin symbols*, Complex Anal. Oper. Theory **17** (2023), no. 6, Paper No. 96, 15 pp.

[4] P. BHUNIA, M. GÜRDAL, K. PAUL, A. SEN AND R. TAPDIGOGLU, *On a new norm on the space of reproducing kernel Hilbert space operators and Berezin radius inequalities*, Numer. Funct. Anal. Optim. **44** (2023), no. 9, 970–986.

[5] P. BHUNIA, K. PAUL AND A. SEN, *Inequalities involving Berezin norm and Berezin number*, Complex Anal. Oper. Theory **17** (2023), no. 1, Paper No. 7, 15 pp.

[6] P. BHUNIA, A. SEN AND K. PAUL, *Development of the Berezin number inequalities*, Acta Math. Sin. (Engl. Ser.) **39** (2023), no. 7, 1219–1228.

[7] P. BHUNIA, S. S. DRAGOMIR, M. S. MOSLEHIAN AND K. PAUL, *Lectures on Numerical Radius Inequalities*, Infosys Science Foundation Series, Infosys Science Foundation Series in Mathematical Sciences, Springer Cham, (2022), https://doi.org/10.1007/978-3-031-13670-2.

[8] M. L. BUZANO, *Generalizzatione della diseguaglianza di Cauchy-Schwarz*, Rend. Sem. Mat. Univ. epolitech. Trimo **31** (1971/73), 405–409.

[9] C. CONDE, K. FEKI AND F. KITTANEH, *Berezin Number and Norm Inequalities for Operators in Hilbert and Semi-Hilbert Spaces*, In: Moslehian, M.S. (eds) Matrix and Operator Equations and Applications, Mathematics Online First Collections. Springer, Cham (2023), https://doi.org/10.1007/16618202355.

[10] M. T. GARAYEV, M. GÜRDAL AND A. OKUDAN, *Hardy-Hilbert's inequality and a power inequality for Berezin numbers for operators*, Math. Inequal. Appl., **19** (3) (2016), 883–891.

[11] M. T. GARAYEV AND M. GUESBA, *Refinements of some inequalities involving Berezin norms and Berezin number and related questions*, Ann. Univ. Ferrara (2023), https://doi.org/10.1007/s11565-023-00477-2.

[12] M. GUESBA, *Some Euclidean Berezin number inequalities of a pair of operators and their applications*, Filomat **37** (26) (2023), 8777–8790.

[13] M. HAJMOHAMADI, R. LASHKARIPOUR AND M. BAKHERAD, *Improvements of Berezin number inequalities*, Linear Multilinear Algebra **68** (2020), no. 6, 1218–1229.

[14] G. H. HARDY, J. E. LITTLEWOOD AND G. POLYA, *Inequalities*, 2nd ed., Cambridge Univ. Press, Cambridge, 1988.

[15] M. T. KARAEV, *Reproducing Kernels and Berezin symbols Techniques in Various Questions of Operator Theory*, Complex Anal. Oper. Theory **7** (2013), 983–1018.

[16] F. KITTANEH, *Notes on some inequalities for Hilbert space operators*, Publ. Res. Inst. Math. Sci. **24** (1988), no. 2, 283–293.

[17] C. A. MCCARTHY, C_p, Israel J. Math. **5** (1967), 249–271.

[18] S. SAHOO, N. DAS AND N. C. ROUT, *On Berezin number inequalities for operator matrices*, Acta Math. Sin. (Engl. Ser.) **37** (2021), no. 6, 873–892.

[19] A. SEN, P. BHUNIA AND K. PAUL, *Berezin number inequalities of operators on reproducing kernel Hilbert spaces*, Rocky Mountain J. Math. **52** (2022), no. 3, 1039–1046.

[20] A. TAGHAVI, T. A. ROUSHAN AND V. DARVISH, *Some upper bounds for the Berezin number of Hilbert space operators*, Filomat, **33** (2019), no. 14, 4353–4360.
[21] P. Y. WU AND H.-L. GAU, *Numerical ranges of Hilbert space operators*, Encyclopedia of Mathematics and its Applications, **179**, Cambridge University Press, Cambridge, 2021.
[22] U. YAMANCI AND M. GUESBA, *Refinements of some Berezin number inequalities and related questions*, J. Anal, **31** (2023), 539–549.

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