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ON THE PROPERTIES OF THE SYSTEMS OF ROOT VECTOR FUNCTIONS OF DIRAC–TYPE OPERATOR WITH SUMMABLE POTENTIAL

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Abstract. One-dimensional Dirac-type operator

$$Dy = By' + P(x)y, \; y = (y_1, y_2)^T,$$

is considered in this work, where $B = \begin{pmatrix} 0 & b_1 \\ b_2 & 0 \end{pmatrix}$, $b_2 < 0 < b_1$, $P(x) = diag\,(p_1(x), p_2(x))$ and $p_j(x)$, $j = 1,2$ are the complex-valued functions defined on the arbitrary finite interval $G = (a,b)$ of the real axis with $p_j(x) \in L_1(G)$, $j = 1,2$.

We establish antiapriori estimates for associated vector functions. We also prove criterion of Bessel property and unconditional basis property for the systems of root vector functions of the operator D in $L_2^2(G)$.

1. Main results

In this work, we study the one-dimensional Dirac-type operator with summable potential and establish antiapriori estimates for associated vector functions, criterion of Bessel property and unconditional basis property for the systems of root vector functions in $L_2^2(G)$. By root vector functions we mean those in a generalized sense, i.e. those with no regard to boundary conditions (see [8]). Note that for these generalized root functions, the necessary and sufficient conditions for unconditional basis property of the systems of root functions of the operator $Lu = -u'' + q(x)u$ in L_2, where $q(x) \in L_1(G)$, have been first found by V. A. Il'in [8]. Later in [2, 9, 11–13, 21, 32], these and other problems have been considered for the higher order ordinary differential operator and the criteria of Bessel property, Riesz property and unconditional basis property have been found. Criterion of Bessel property and unconditional basis property for Dirac and Dirac-type operators with potential $P(x) \in L_2(G)$ have been obtained in [14, 20], while the same criteria for Dirac operator with potential $P(x) \in L_1(G)$ have been found in [19]. Problems of componentwise uniform equiconvergence on a compact, uniform convergence, Riesz property of the system of root vector functions of Dirac operator have been treated in [3, 15–18].

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A lot of works [1, 4–7, 10, 22–31, 33] have been dedicated to basis property and other spectral properties of root vector functions of Dirac operator (with boundary conditions).

In [33], Riesz basis property has been established in case where the potential of Dirac operator belongs to L_2 and the boundary conditions are separated. In [4], a criterion of Riesz basis property for 2×2 Dirac operator with periodic (antiperiodic) boundary conditions was established. In [5], the case where the boundary conditions are regular and the potential belongs to the class L_2 has been considered. In the same work, Riesz basis property from subspaces and Riesz basis property in the case of strongly regular boundary conditions have been proved. Dirac operator with the potential from L_p, $p \geqslant 1$, has been considered in [30, 31], and Riesz basis property has been proved in case of regular boundary conditions. Moreover, in case of regular (but not strongly regular) boundary conditions, Riesz basis property of subspaces has been proved. In [22, 23], Dirac-type 2×2-system with the potentials from L_1 and strongly regular boundary conditions has been considered and Riesz basis property has been proved. A criterion of Bari basis property for 2×2 Dirac-type operators with strictly regular boundary conditions was established.

Note, that the Dirac-type case has several interesting features and more difficult for investigation when operators are considered with two-point boundary conditions. For instance, antiperiodic boundary conditions can be strictly regular for some pairs of $b_1,\ b_2,$ while it is never the case for Dirac operator (see [23]). Another contrasting feature is a description of strictly regular boundary conditions. It is trivial in the Dirac case, but a criterion of strict regularity for a general Dirac-type operator is unknown (several strong sufficient conditions are contained in [23]).

Let $L_p^2(G),\ p \geqslant 1,$ be a space of two-component vector functions

$$f(x) = (f_1(x), f_2(x))^T$$

equipped with the norm

$$\|f\|_{p,2} = \left[\int_G \left(|f_1(x)|^2 + |f_2(x)|^2\right)^{p/2} dx\right]^{1/p}.$$

In case $p=\infty$, the norm is defined by the equality

$$\|f\|_{\infty,2} = \sup_{x\in G} vrai\,|f(x)|\,.$$

It is clear that for arbitrary functions $f(x)\in L_p^2(G),\ g(x)\in L_q^2(G)$, with $p^{-1}+q^{-1}=1$, $1\leqslant p\leqslant\infty$, the scalar product $(f,g)=\int_G \sum_{j=1}^2 f_j(x)\overline{g_j(x)}dx$ is defined.

Consider the one-dimensional Dirac-type operator

$$Dy = B\frac{dy}{dx} + P(x)y, y(x) = (y_1(x), y_2(x))^T\,,$$

where $B=\begin{pmatrix} 0 & b_1 \\ b_2 & 0\end{pmatrix}$, $b_2<0<b_1$, $P(x)=diag\,(p_1(x),p_2(x))$, and $p_1(x),p_2(x)$ are the complex-valued functions defined on the arbitrary finite interval $G=(a,b)$ of the real axis.

Following [8], by the eigenfunction of the operator D corresponding to the complex eigenvalue λ, we will mean any complex-valued vector function $\overset{0}{u}(x)$ not identically zero, which is absolutely continuous on every closed subinterval of G and satisfies the equation $D\overset{o}{u} = \lambda \overset{o}{u}$ almost everywhere in G.

Similarly, by the associated function of degree ℓ, $\ell \geqslant 1$, corresponding to the same λ and the same eigenfunction $\overset{o}{u}(x)$, we will mean any complex-valued vector function $\overset{\ell}{u}(x)$, which is absolutely continuous on every closed subinterval of G and satisfies the equation $D\overset{\ell}{u} = \lambda \overset{\ell}{u} + \overset{\ell-1}{u}$ almost everywhere in G.

Let us denote by $L_1^{loc}(G)$ the class of summable functions on an arbitrary segment belonging to G.

THEOREM 1. *Let the functions $p_1(x), p_2(x)$ belong to the class $L_1^{loc}(G)$. Then, for every compact $K \subset G$ there exist the constants $C^i(K,\ell,b_1,b_2)$, $i = 1,2$, $\ell = 0,1,2,\ldots$, independent of λ, such that the estimates*

$$\left\| \overset{\ell-1}{u} \right\|_{L^2_\infty(K)} \leqslant C^1(K,\ell,b_1,b_2)(1+|\mathrm{Im}\,\lambda|) \left\| \overset{\ell}{u} \right\|_{L^2_\infty(K)}, \tag{1}$$

$$\left\| \overset{\ell}{u} \right\|_{L^2_\infty(K)} \leqslant C^2(K,\ell,b_1,b_2)(1+|\mathrm{Im}\,\lambda|)^{1/p} \left\| \overset{\ell}{u} \right\|_{L^2_p(K)}, \quad 1 \leqslant p < \infty, \tag{2}$$

hold.

REMARK 1. If G is a finite interval, $p_1(x)$ and $p_2(x)$ belong to the class $L_1(G)$, then the estimates (1) and (2) hold in case $K = \overline{G}$, too. Note that for $b_1 = -b_2 = 1$ the estimates (1) and (2) have been proved in [16].

Let $\{u_k(x)\}_{k=1}^\infty$ be an arbitrary system consisting of eigenfunctions and associated vector functions of the operator D, and $\{\lambda_k\}_{k=1}^\infty$ be the corresponding system of eigenvalues. Besides, let the function $u_k(x)$ belong to the system $\{u_k(x)\}_{k=1}^\infty$ together with all corresponding associated functions of a lesser degree. This means $Du_k = \lambda_k u_k + \theta_k u_{k-1}$, where θ_k is equal to either 0 (in this case, $u_k(x)$ is an eigen vector function) or 1 (in this case, $u_k(x)$ is an associated vector function with $\lambda_k = \lambda_{k-1}$).

DEFINITION 1. We will say that the Bessel inequality is true for the given system of functions $\varphi_k(x) \in L_2^2(G)$, if there exists a constant M, independent of $f(x)$, such that the relation

$$\sum_{k=1}^{\infty} |(\varphi_k, f)|^2 \leqslant M \|f\|_{2,2}^2$$

holds for every vector function $f(x) \in L_2^2(G)$.

THEOREM 2. (Criterion of Bessel property) *Let $P(x) \in L_1(G)$, the lengths of the chains of root vector functions be uniformly bounded and there exist a constant C_0 such that*

$$|\mathrm{Im}\,\lambda_k| \leqslant C_0, \quad k = 1,2,\ldots \tag{3}$$

Then, for the system $\left\{u_k(x)\,\|u_k\|_{2,2}^{-1}\right\}_{k=1}^{\infty}$ *to satisfy the Bessel inequality in* $L_2^2(G)$*, it is necessary and sufficient that there exist a constant* M_1 *such that*

$$\sum_{|Re\lambda_k-\tau|\leqslant 1} 1 \leqslant M_1, \tag{4}$$

where τ *is an arbitrary real number.*

Denote by D^* the formal adjoint of the operator D:

$$D^* = -B^*\frac{d}{dx} + P^*(x),$$

where $P^*(x)$ is a conjugate of the matrix $P(x)$.

Let $\{\vartheta_k(x)\}$ be a biorthogonal conjugate of $\{u_k(x)\}$ in $L_2^2(G)$ and consist of root vector functions of the operator D^* (i.e. $D^*\vartheta_k = \overline{\lambda_k}\vartheta_k + \theta_{k+1}\vartheta_{k+1}$).

THEOREM 3. (On unconditional basis property) *Let* $P(x)\in L_1(G)$*, the lengths of the chains of root vector functions be uniformly bounded, one of the systems* $\{u_k(x)\}_{k=1}^{\infty}$ *and* $\{\vartheta_k(x)\}_{k=1}^{\infty}$ *be complete in* $L_2^2(G)$ *and the condition* (3) *be satisfied. Then, for each of these systems to form an unconditional basis for* $L_2^2(G)$*, it is necessary and sufficient that there exist the constants* M_1 *and* M_2 *such that the inequality* (4) *holds and*

$$\|u_k\|_{2,2}\,\|\vartheta_k\|_{2,2} \leqslant M_2, \;\; k=1,2,\ldots\,. \tag{5}$$

REMARK 2. Note that under the conditions of Theorem 3 the validity of the inequalities (4) and (5) is a necessary and sufficient condition for the Riesz basis property of each of the systems $\left\{u_k(x)\,\|u_k\|_{2,2}^{-1}\right\}_{k=1}^{\infty}$ and $\left\{\vartheta_k(x)\,\|\vartheta_k\|_{2,2}^{-1}\right\}_{k=1}^{\infty}$ for $L_2^2(G)$.

Note that the system of root functions of the operator Dirac considered with regular two-point boundary conditions is always complete. This result was first established by V. A. Marchenko in [29] in the case of Dirac operators with continuous potential $P(x)$, and later generalized to arbitrary $n\times n$ Dirac-type operators with summable potential by M. M. Malamud and L. L. Oridoroga in [26].

DEFINITION 2. The system $\{\tau_k(x)\}_{k=1}^{\infty}\subset L_2^2(G)$ is quadratically close to the system $\{u_k(x)\}_{k=1}^{\infty}\subset L_2^2(G)$ if

$$\sum_{k=1}^{\infty}\|\tau_k-u_k\|_{2,2}^2<\infty.$$

THEOREM 4. (On equivalent basis property) *Let* $P(x)\in L_1(G)$*, the lengths of the chains of root vector functions be uniformly bounded, conditions* (3)–(5) *be satisfied and the system* $\left\{u_k(x)\,\|u_k\|_{2,2}^{-1}\right\}_{k=1}^{\infty}$ *be quadratically close to some basis* $\{\psi_k(x)\}_{k=1}^{\infty}$ *of the space* $L_2^2(G)$*. Then the systems* $\left\{u_k(x)\,\|u_k\|_{2,2}^{-1}\right\}$ *and* $\left\{\vartheta_k(x)\,\|u_k\|_{2,2}\right\}_{k=1}^{\infty}$ *form bases for* $L_2^2(G)$*, and they are equivalent to the basis* $\{\psi_k(x)\}_{k=1}^{\infty}$ *and its biorthogonal conjugate, respectively.*

2. Some auxiliary lemmas

To prove Theorem 1, we need some auxiliary lemmas.

LEMMA 1. (Shift formula) *If the functions* $p_1(x)$ *and* $p_2(x)$ *belong to the class* $L_1^{loc}(G)$ *and the points* $x-t$, x, $x+t$ *lie in the domain* G, *then the following formulas hold:*

$$\overset{\ell}{u}(x+t)=\left[\cos\frac{\lambda}{\sqrt{|b_1b_2|}}tI-\sin\frac{\lambda}{\sqrt{|b_1b_2|}}t\frac{B}{\sqrt{|b_1b_2|}}\right]\overset{\ell}{u}(x)+$$

$$+B^{-1}\int_x^{x+t}\left(\sin\frac{\lambda}{\sqrt{|b_1b_2|}}(t-\xi+x)\frac{B}{\sqrt{|b_1b_2|}}-\cos\frac{\lambda}{\sqrt{|b_1b_2|}}(t-\xi+x)I\right)\times \tag{6}$$

$$\times\left[P(\xi)\overset{\ell}{u}(\xi)-\overset{\ell-1}{u}(\xi)\right]d\xi,$$

$$\overset{\ell}{u}(x-t)=\left[\cos\frac{\lambda}{\sqrt{|b_1b_2|}}tI+\sin\frac{\lambda}{\sqrt{|b_1b_2|}}t\frac{B}{\sqrt{|b_1b_2|}}\right]\overset{\ell}{u}(x)+$$

$$+B^{-1}\int_{x-t}^{x}\left(\sin\frac{\lambda}{\sqrt{|b_1b_2|}}(t+\xi-x)\frac{B}{\sqrt{|b_1b_2|}}+\cos\frac{\lambda}{\sqrt{|b_1b_2|}}(t+\xi-x)I\right)\times \tag{7}$$

$$\times\left[P(\xi)\overset{\ell}{u}(\xi)-\overset{\ell-1}{u}(\xi)\right]d\xi,$$

$$\overset{\ell}{u}(x+t)+\overset{\ell}{u}(x-t)=2\overset{\ell}{u}(x)\cos\frac{\lambda}{\sqrt{|b_1b_2|}}t+$$

$$+B^{-1}\int_{x-t}^{x+t}\left(\sin\frac{\lambda}{\sqrt{|b_1b_2|}}(t-|x-\xi|)\frac{B}{\sqrt{|b_1b_2|}}-\right.$$

$$\left.-sign(\xi-x)\cos\frac{\lambda}{\sqrt{|b_1b_2|}}(t-|x-\xi|)I\right)\times\left[P(\xi)\overset{\ell}{u}(\xi)-\overset{\ell-1}{u}(\xi)\right]d\xi, \tag{8}$$

where I *is a unit operator in* E^2, *and* E^2 *is a two-dimensional Euclidean space.*

Proof. To obtain formulas (6) and (7), it suffices to apply the operator

$$\cos\frac{\lambda}{\sqrt{|b_1b_2|}}(t-|x-\xi|)I-sign(\xi-x)\sin\frac{\lambda}{\sqrt{|b_1b_2|}}(t-|\xi-x|)\frac{B}{\sqrt{|b_1b_2|}}$$

to the equation $D\overset{\ell}{u}(\xi)=\lambda\overset{\ell}{u}(\xi)+\overset{\ell-1}{u}(\xi)$, integrate with respect to the parameter ξ from x to $x+t$ (from $x-t$ to x), and then, having combined similar terms, integrate by parts in the expression of the form

$$\int_x^{x+t}\left(\cos\frac{\lambda}{\sqrt{|b_1b_2|}}(t-\xi+x)I-\sin\frac{\lambda}{\sqrt{|b_1b_2|}}(t-\xi+x)\frac{B}{\sqrt{|b_1b_2|}}\right)Bd\overset{\ell}{u}(\xi)$$

$$\left(\int_{x-t}^{x}\left(\cos\frac{\lambda}{\sqrt{|b_1b_2|}}(t+\xi-x)I+\sin\frac{\lambda}{\sqrt{|b_1b_2|}}(t+\xi-x)\frac{B}{\sqrt{|b_1b_2|}}\right)Bd\overset{\ell}{u}(\xi)\right).$$

Formula (8) follows from (6) and (7). Lemma 1 is proved. □

LEMMA 2. *If $p_1(x)$ and $p_2(x)$ are the functions from the class $L_1^{loc}(G)$ and the points $x-t,\ x,\ x+t$ lie in the domain G, then the following formula is true:*

$$\frac{2t}{\sqrt{|b_1b_2|}}\sin\frac{\lambda t}{\sqrt{|b_1b_2|}}\overset{\ell-1}{u}(x)=2\cos\frac{\lambda t}{\sqrt{|b_1b_2|}}\overset{\ell}{u}(x)-\overset{\ell}{u}(x+t)-\overset{\ell}{u}(x-t)+$$

$$+\frac{1}{\sqrt{|b_1b_2|}}\int_{x-t}^{x+t}\left[\sin\frac{\lambda}{\sqrt{|b_1b_2|}}(t-|x-\xi|)I+\right.$$

$$\left.+sign\,(\xi-x)\cos\frac{\lambda}{\sqrt{|b_1b_2|}}(t-|x-\xi|)\frac{B}{\sqrt{|b_1b_2|}}\right]P(\xi)\overset{\ell}{u}(\xi)d\xi+$$

$$+\frac{1}{\sqrt{|b_1b_2|}}\int_{x-t}^{x+t}(t-|x-\xi|)\left[\sin\frac{\lambda}{\sqrt{|b_1b_2|}}(t-|x-\xi|)I-\right.$$

$$\left.-sign\,(\xi-x)\sin\frac{\lambda}{\sqrt{|b_1b_2|}}(t-|x-\xi|)\frac{B}{\sqrt{|b_1b_2|}}\right]\times$$

$$\times\left(P(\xi)\overset{\ell-1}{u}(\xi)-\overset{\ell-2}{u}(\xi)\right)d\xi. \tag{9}$$

Proof. Subtracting the equality (7) from the equality (6), we obtain

$$\overset{\ell}{u}(x+t)-\overset{\ell}{u}(x-t)=-2\sin\frac{\lambda t}{\sqrt{|b_1b_2|}}\frac{B}{\sqrt{|b_1b_2|}}\overset{\ell}{u}(x)+$$

$$+B^{-1}\int_{x-t}^{x+t}\left(sign\,(\xi-x)\sin\frac{\lambda}{\sqrt{|b_1b_2|}}(t-|x-\xi|)\frac{B}{\sqrt{|b_1b_2|}}-\right.$$

$$\left.-\cos\frac{\lambda}{\sqrt{|b_1b_2|}}(t-|x-\xi|)I\right)\times\left[P(\xi)\overset{\ell}{u}(\xi)-\overset{\ell-1}{u}(\xi)\right]d\xi. \tag{10}$$

Rewrite the formula (8) in the following form:

$$\overset{\ell}{u}(x+t)+\overset{\ell}{u}(x-t)=2\cos\frac{\lambda t}{\sqrt{|b_1b_2|}}\overset{\ell}{u}(x)+$$

$$+B^{-1}\int\limits_{x-t}^{x+t}\left[\sin\frac{\lambda}{\sqrt{|b_1b_2|}}(t-|x-\xi|)\frac{B}{\sqrt{|b_1b_2|}}-sign\,(\xi-x)\cos\frac{\lambda}{\sqrt{|b_1b_2|}}(t-|x-\xi|)I\right]\times$$

$$\times P(\xi)\overset{\ell}{u}(\xi)\,d\xi - B^{-1}\int_0^t \sin\frac{\lambda}{\sqrt{|b_1b_2|}}(t-r)\frac{B}{\sqrt{|b_1b_2|}}\left\{\overset{\ell-1}{u}(x+r)+\overset{\ell-1}{u}(x-r)\right\}dr+$$

$$+B^{-1}\int_0^t \cos\frac{\lambda}{\sqrt{|b_1b_2|}}(t-r)I\left\{\overset{\ell-1}{u}(x+r)-\overset{\ell-1}{u}(x-r)\right\}dr.$$

Substituting the formulas (8) and (10) for $(\ell-1)$ into the last equality, changing the order of integration in double integrals and taking into account the relations $B^{-1}=-\frac{1}{|b_1b_2|}B$ and $\left(B^{-1}\right)^2=-\frac{1}{|b_1b_2|}I$, we obtain

$$\overset{\ell}{u}(x+t)+\overset{\ell}{u}(x-t)=2\cos\frac{\lambda t}{\sqrt{|b_1b_2|}}\overset{\ell}{u}(x)+$$

$$+\frac{1}{\sqrt{|b_1b_2|}}\int\limits_{x-t}^{x+t}\left[\sin\frac{\lambda}{\sqrt{|b_1b_2|}}(t-|x-\xi|)I+sign(\xi-x)\cos\frac{\lambda}{\sqrt{|b_1b_2|}}(t-|x-\xi|)\frac{B}{\sqrt{|b_1b_2|}}\right]\times$$

$$\times P(\xi)\overset{\ell}{u}(\xi)\,d\xi-\frac{2}{\sqrt{|b_1b_2|}}\int_0^t\left[\cos\frac{\lambda}{\sqrt{|b_1b_2|}}r\sin\frac{\lambda}{\sqrt{|b_1b_2|}}(t-r)+\right.$$

$$\left.+\sin\frac{\lambda}{\sqrt{|b_1b_2|}}r\cos\frac{\lambda}{\sqrt{|b_1b_2|}}(t-r)\right]dr\times$$

$$\times\overset{\ell-1}{u}(x)-\int_{x-t}^{x+t}d\xi\int_{|x-\xi|}^t\left[\sin\frac{\lambda}{\sqrt{|b_1b_2|}}(t-r)\sin\frac{\lambda}{\sqrt{|b_1b_2|}}(r-|x-\xi|)\frac{I}{|b_1b_2|}+\right.$$

$$+sign(\xi-x)\sin\frac{\lambda}{\sqrt{|b_1b_2|}}(t-r)\cos\frac{\lambda}{\sqrt{|b_1b_2|}}(r-|x-\xi|)\frac{B}{|b_1b_2|^{3/2}}+$$

$$+sign(\xi-x)\cos\frac{\lambda}{\sqrt{|b_1b_2|}}(t-r)\sin\frac{\lambda}{\sqrt{|b_1b_2|}}(r-|x-\xi|)\frac{B}{|b_1b_2|^{3/2}}-$$

$$\left.-\cos\frac{\lambda}{\sqrt{|b_1b_2|}}(t-r)\cos\frac{\lambda}{\sqrt{|b_1b_2|}}(r-|x-\xi|)\frac{I}{|b_1b_2|}\right]\left(P(\xi)\overset{\ell-1}{u}(\xi)-\overset{\ell-2}{u}(\xi)\right)dr.$$

Hence, in turn, we get the formula

$$\overset{\ell}{u}(x+t)+\overset{\ell}{u}(x-t)=2\cos\frac{\lambda t}{\sqrt{|b_1b_2|}}\overset{\ell}{u}(x)-\frac{2t}{\sqrt{|b_1b_2|}}\sin\frac{\lambda t}{\sqrt{|b_1b_2|}}\overset{\ell-1}{u}(x)+$$

$$+\frac{1}{\sqrt{|b_1b_2|}}\int\limits_{x-t}^{x+t}\left[\sin\frac{\lambda}{\sqrt{|b_1b_2|}}(t-|x-\xi|)I+sign(\xi-x)\cos\frac{\lambda}{\sqrt{|b_1b_2|}}(t-|x-\xi|)\frac{B}{\sqrt{|b_1b_2|}}\right]\times$$

$$\times P(\xi)\overset{\ell-1}{u}(\xi)\,d\xi+\int_{x-t}^{x+t}(t-|x-\xi|)\left[\cos\frac{\lambda}{\sqrt{|b_1b_2|}}(t-|x-\xi|)\frac{I}{|b_1b_2|}-\right.$$

$$-sign\,(\xi-x)\sin\frac{\lambda}{\sqrt{|b_1b_2|}}\,(t-|x-\xi|)\,\frac{B}{|b_1b_2|^{3/2}}\Bigg]\left(P(\xi)\overset{\ell-1}{u}(\xi)-\overset{\ell-2}{u}(\xi)\right)d\xi.$$

Formula (9) follows from the last equality. Lemma 2 is proved. □

LEMMA 3. *The following formulas are true under the conditions of Lemma 2:*

$$\overset{\ell}{u}(x+t)=\overset{\ell}{u}(x)+\frac{B}{|b_1b_2|}\int_x^{x+t}(P(\xi)-\lambda I)\overset{\ell}{u}(\xi)d\xi-\frac{B}{|b_1b_2|}\int_x^{x+t}\overset{\ell-1}{u}(\xi)d\xi, \qquad (11)$$

$$\overset{\ell}{u}(x-t)=\overset{\ell}{u}(x)-\frac{B}{|b_1b_2|}\int_{x-t}^{x}(P(\xi)-\lambda I)\overset{\ell}{u}(\xi)d\xi+\frac{B}{|b_1b_2|}\int_{x-t}^{x}\overset{\ell-1}{u}(\xi)d\xi, \qquad (12)$$

$$2t\overset{\ell-1}{u}(x)=B\left\{\overset{\ell}{u}(x+t)-\overset{\ell}{u}(x-t)\right\}+\int_{x-t}^{x+t}(P(\xi)-\lambda I)\overset{\ell}{u}(\xi)d\xi-$$

$$-\frac{B}{|b_1b_2|}\int_0^t dr\int_{x-r}^{x+r}sign\,(\xi-x)\left\{(P(\xi)-\lambda I)\overset{\ell-1}{u}(\xi)-\overset{\ell-2}{u}(\xi)\right\}d\xi. \qquad (13)$$

Proof. Integrate the equation $D\overset{\ell}{u}(\xi)=\lambda\overset{\ell}{u}(\xi)+\overset{\ell-1}{u}(\xi)$ with respect to ξ from x to $x+t$:

$$B\int_x^{x+t}d\overset{\ell}{u}(\xi)+\int_x^{x+t}P(\xi)\overset{\ell}{u}(\xi)d\xi=\lambda\int_x^{x+t}\overset{\ell}{u}(\xi)d\xi+\int_x^{x+t}\overset{\ell-1}{u}(\xi)d\xi.$$

Integrating by parts in the first integral on the left-hand side, we have

$$B\left[\overset{\ell}{u}(x+t)-\overset{\ell}{u}(x)\right]=-\int_x^{x+t}(P(\xi)-\lambda I)\overset{\ell}{u}(\xi)d\xi+\int_x^{x+t}\overset{\ell-1}{u}(\xi)d\xi.$$

Hence, in turn, we get the equality

$$\overset{\ell}{u}(x+t)-\overset{\ell}{u}(x)=-B^{-1}\int_x^{x+t}(P(\xi)-\lambda I)\overset{\ell}{u}(\xi)d\xi+B^{-1}\int_x^{x+t}\overset{\ell-1}{u}(\xi)d\xi.$$

As $B^{-1}=-\frac{1}{|b_1b_2|}B$, from the last equality we obtain (11).

Formula (12) can be proved similarly.

Now let's prove the formula (13). To do so, we rewrite this formula in the following form:

$$\overset{\ell}{u}(x+t)=\overset{\ell}{u}(x)+\frac{B}{|b_1b_2|}\int_x^{x+t}(P(\xi)-\lambda I)\overset{\ell}{u}(\xi)d\xi-\frac{B}{|b_1b_2|}\int_0^t\overset{\ell-1}{u}(x+r)dr.$$

Considering the value of $\overset{\ell-1}{u}(x+r)$ from (10) in the last integral, we obtain

$$\overset{\ell}{u}(x+t)=\overset{\ell}{u}(x)+\frac{B}{|b_1b_2|}\int_x^{x+t}(P(\xi)-\lambda I)\overset{\ell}{u}(\xi)d\xi-\frac{B}{|b_1b_2|}\overset{\ell-1}{u}(x)t-$$

$$-\frac{B^2}{|b_1b_2|^2}\int_0^t dr\int_x^{x+r}(P(\xi)-\lambda I)\overset{\ell-1}{u}(\xi)d\xi+\frac{B^2}{|b_1b_2|^2}\int_0^t dr\int_x^{x+r}\overset{\ell-2}{u}(\xi)d\xi.$$

Taking into account that $B^2=-|b_1b_2|I,$ from the last equality we get

$$\overset{\ell}{u}(x+t)=\overset{\ell}{u}(x)+\frac{B}{|b_1b_2|}\int_x^{x+t}(P(\xi)-\lambda I)\overset{\ell}{u}(\xi)d\xi-\frac{B}{|b_1b_2|}\overset{\ell-1}{u}(x)t+$$

$$+\frac{1}{|b_1b_2|}\int_0^t dr\int_x^{x+r}(P(\xi)-\lambda I)\overset{\ell-1}{u}(\xi)d\xi-\frac{1}{|b_1b_2|}\int_0^t dr\int_x^{x+r}\overset{\ell-2}{u}(\xi)d\xi. \tag{14}$$

Similarly,

$$\overset{\ell}{u}(x-t)=\overset{\ell}{u}(x)-\frac{B}{|b_1b_2|}\int_{x-t}^{x}(P(\xi)-\lambda I)\overset{\ell}{u}(\xi)d\xi+\frac{B}{|b_1b_2|}\overset{\ell-1}{u}(x)t+$$

$$+\frac{1}{|b_1b_2|}\int_0^t dr\int_{x-r}^{x}(P(\xi)-\lambda I)\overset{\ell-1}{u}(\xi)d\xi-\frac{1}{|b_1b_2|}\int_0^t dr\int_{x-r}^{x}\overset{\ell-2}{u}(\xi)d\xi. \tag{15}$$

Subtracting (15) from (14), we find

$$\overset{\ell}{u}(x+t)-\overset{\ell}{u}(x-t)=\frac{B}{|b_1b_2|}\int_{x-t}^{x+t}(P(\xi)-\lambda I)\overset{\ell}{u}(\xi)d\xi-\frac{2B}{|b_1b_2|}\overset{\ell-1}{u}(x)t+$$

$$+\frac{1}{|b_1b_2|}\int_0^t dr\int_{x-r}^{x+r}sign\,(\xi-x)\left\{(P(\xi)-\lambda I)\overset{\ell-1}{u}(\xi)-\overset{\ell-2}{u}(\xi)\right\}d\xi.$$

The last relation implies (13). □

3. Proof of the Theorem 1

Let $K=[a',b']\subset G$. We are going to prove the estimate (1) by the method of mathematical induction.

As $\overset{-1}{u}\equiv 0$, the estimate (1) will be true for $\ell=0$ with the constant $C^1(K,0,b_1,b_2)=1$. Assume that the estimate (1) holds for $\ell=k$. As $p_1(x),p_2(x)\in L_1(K)$, we can choose the number R_1 such that for every set $E\subset K$, $mes\,E\leqslant 2\max\left\{1,\sqrt{|b_1b_2|}\right\}R_1$, the inequality

$$\max\left\{\frac{1}{\sqrt{|b_1b_2|}}\int_E(|p_1(\xi)|+|p_2(\xi)|)d\xi,\frac{1}{|b_1b_2|}\int_E(|b_1p_2(\xi)|+|b_2p_1(\xi)|)d\xi\right\}\leqslant\frac{1}{240} \tag{16}$$

is valid. Let's choose the numbers h and h_{k+1} as follows:

$$0<h\leqslant\frac{1}{\max\left\{1,\sqrt{|b_1b_2|}\right\}}\min\left\{\frac{b'-a'}{4},R_1,\frac{1}{|\mathrm{Im}\,\lambda|}\right\},$$

$$0<h_{k+1}=\frac{1}{\max\left\{1,\sqrt{|b_1b_2|}\right\}}\times$$

$$\times \min\left\{\frac{1}{120C^1(K,k,b_1,b_2)\left(\frac{1}{\sqrt{|b_1b_2|}}+\frac{|b_1|+|b_2|}{|b_1b_2|}\right)(1+|\mathrm{Im}\lambda|)},\frac{b'-a'}{4},R_1,\frac{1}{|\mathrm{Im}\lambda|}\right\}$$

and denote $h\sqrt{|b_1b_2|}=\delta,\ h_{k+1}\sqrt{|b_1b_2|}=\delta_{k+1}$.

Considering the formula (8) for $l=k$ at the points $x,x+t,x+2t$, where $t=\delta$, $x\in\left[a',\frac{a'+b'}{2}\right]$, by the inequalities

$$|\sin z|,\ |\cos z|\leqslant 2\ \text{ for }\ |\mathrm{Im}\,z|\leqslant 1 \tag{17}$$

we get

$$\left|\overset{k}{u}(x)\right|\leqslant 5\max_{x\in[a'+\delta,b'-\delta]}\left|\overset{k}{u}(x)\right|+2\left\{\frac{1}{\sqrt{b_1b_2}}\int_x^{x+2\delta}\{|p_1(\xi)|+|p_2(\xi)|\}\,d\xi+\right.$$
$$\left.+\frac{1}{|b_1b_2|}\int_x^{x+2\delta}\{|b_1p_2(\xi)|+|b_2p_1(\xi)|\}\,d\xi\right\}\left\|\overset{k}{u}\right\|_{L^2_\infty(K)}+$$
$$+4\delta\left\{\frac{1}{\sqrt{|b_1b_2|}}+\frac{|b_1|+|b_2|}{|b_1b_2|}\right\}\left\|\overset{k-1}{u}\right\|_{L^2_\infty(K)}.$$

Hence, due to (16), for every $x\in\left[a',\frac{a'+b'}{2}\right]$ we have

$$\left|\overset{k}{u}(x)\right|\leqslant 5\max_{x\in[a'+\delta,b'-\delta]}\left|\overset{k}{u}(x)\right|+\frac{1}{2}\left\|\overset{k}{u}\right\|_{L^2_\infty(K)}+$$
$$+4\delta\left\{\frac{1}{\sqrt{|b_1b_2|}}+\frac{|b_1|+|b_2|}{|b_1b_2|}\right\}\left\|\overset{k-1}{u}\right\|_{L^2_\infty(K)} \tag{18}$$

From (8) we similarly obtain the inequality (18) for $x\in\left[\frac{a'+b'}{2},b'\right]$. So, for $x\in K$ we have

$$\frac{1}{2}\left\|\overset{k}{u}\right\|_{L^2_\infty(K)}-4\delta\left\{\frac{1}{\sqrt{|b_1b_2|}}+\frac{|b_1|+|b_2|}{|b_1b_2|}\right\}\left\|\overset{k-1}{u}\right\|_{L^2_\infty(K)}\leqslant 5\max_{x\in[a'+\delta,b'-\delta]}\left|\overset{k}{u}(x)\right|.$$

Hence,

$$\frac{1}{10}\left[\left\|\overset{k}{u}\right\|_{L^2_\infty(K)}-8\delta\left\{\frac{1}{\sqrt{|b_1b_2|}}+\frac{|b_1|+|b_2|}{|b_1b_2|}\right\}\left\|\overset{k-1}{u}\right\|_{L^2_\infty(K)}\right]\leqslant$$
$$\leqslant\max_{x\in[a'+\delta,b'-\delta]}\left|\overset{k}{u}(x)\right|. \tag{19}$$

Due to (17), from (9) for $t=\delta,\ x\in[a'+\delta,b'-\delta],\ \ell=k+1$ we have

$$2\delta|\sin\lambda\delta|\left|\overset{k}{u}(x)\right|\leqslant 6\left\|\overset{k+1}{u}\right\|_{L^2_\infty(K)}+2\left\{\frac{1}{\sqrt{b_1b_2}}\int_{x-\delta}^{x+\delta}\{|p_1(\xi)|+|p_2(\xi)|\}\,d\xi+\right.$$

$$+\frac{1}{|b_1 b_2|}\int_{x-\delta}^{x+\delta}\{|b_1 p_2(\xi)|+|b_2 p_1(\xi)|\}\,d\xi\Bigg\}\left\|\overset{k}{u}\right\|_{L^2_\infty(K)}+$$

$$+2\delta\left\{\frac{1}{\sqrt{|b_1 b_2|}}\int_{x-\delta}^{x+\delta}\{|p_1(\xi)|+|p_2(\xi)|\}\,d\xi+\right.$$

$$\left.+\frac{1}{|b_1 b_2|}\int_{x-\delta}^{x+\delta}\{|b_1 p_2(\xi)|+|b_2 p_1(\xi)|\}\,d\xi\right\}\left\|\overset{k}{u}\right\|_{L^2_\infty(K)}+$$

$$+4\delta^2\left\{\frac{1}{\sqrt{|b_1 b_2|}}+\frac{|b_1|+|b_2|}{|b_1 b_2|}\right\}\left\|\overset{k-1}{u}\right\|_{L^2_\infty(K)}.$$

Then, by the inequality (16) and the arbitrariness of the point $x\in[a'+\delta,b'-\delta]$, we find

$$2\delta|\sin\lambda\delta|\max_{x\in[a'+\delta,b'-\delta]}\left|\overset{k}{u}(x)\right|\leqslant 7\left\|\overset{k+1}{u}\right\|_{L^2_\infty(K)}+\frac{\delta}{60}\left\|\overset{k}{u}\right\|_{L^2_\infty(K)}+$$

$$+4\delta^2\left\{\frac{1}{\sqrt{|b_1 b_2|}}+\frac{|b_1|+|b_2|}{|b_1 b_2|}\right\}\left\|\overset{k-1}{u}\right\|_{L^2_\infty(K)},$$

or

$$|\sin\lambda\delta|\max_{x\in[a'+\delta,b'-\delta]}\left|\overset{k}{u}(x)\right|\leqslant\frac{4}{\delta}\left\|\overset{k+1}{u}\right\|_{L^2_\infty(K)}+\frac{1}{120}\left\|\overset{k}{u}\right\|_{L^2_\infty(K)}+$$

$$+2\delta\left\{\frac{1}{\sqrt{|b_1 b_2|}}+\frac{|b_1|+|b_2|}{|b_1 b_2|}\right\}\left\|\overset{k-1}{u}\right\|_{L^2_\infty(K)}.$$

Combining the last inequality with (19), we obtain

$$\frac{|\sin\lambda\delta|}{10}\left[\left\|\overset{k}{u}\right\|_{L^2_\infty(K)}-8\delta\left\{\frac{1}{\sqrt{|b_1 b_2|}}+\frac{|b_1|+|b_2|}{|b_1 b_2|}\right\}\left\|\overset{k-1}{u}\right\|_{L^2_\infty(K)}\right]-$$

$$-\frac{1}{120}\left\|\overset{k}{u}\right\|_{L^2_\infty(K)}-2\delta\left\{\frac{1}{\sqrt{|b_1 b_2|}}+\frac{|b_1|+|b_2|}{|b_1 b_2|}\right\}\left\|\overset{k-1}{u}\right\|_{L^2_\infty(K)}\leqslant\frac{4}{\delta}\left\|\overset{k+1}{u}\right\|_{L^2_\infty(K)}.$$

By the assumption of validity of the estimate (1) for $\ell=k$, we have

$$\left\{\frac{|\sin\lambda\delta|}{10}\left[1-8\delta\left\{\frac{1}{\sqrt{|b_1 b_2|}}+\frac{|b_1|+|b_2|}{|b_1 b_2|}\right\}C^1(K,k,b_1,b_2)(1+|\mathrm{Im}\,\lambda|)\right]-\right.$$

$$\left.-\frac{1}{120}-2\delta\left\{\frac{1}{\sqrt{|b_1 b_2|}}+\frac{|b_1|+|b_2|}{|b_1 b_2|}\right\}C^1(K,k,b_1,b_2)(1+|\mathrm{Im}\,\lambda|)\right\}\left\|\overset{k}{u}\right\|_{L^2_\infty(K)}\leqslant$$

$$\leqslant\frac{4}{\delta}\left\|\overset{k+1}{u}\right\|_{L^2_\infty(K)}\tag{20}$$

Consider the case

$$\lambda \geqslant 2\max\left\{1,\sqrt{|b_1b_2|}\right\}\max\left\{\frac{4}{b'-a'},\frac{1}{R_1},120C^1\left(K,k,b_1,b_2\right)\times\right.$$

$$\left.\times\left(\frac{1}{\sqrt{|b_1b_2|}}+\frac{|b_1|+|b_2|}{|b_1b_2|}\right)\right\}(1+|\mathrm{Im}\,\lambda|).$$

Then, by the definition of the number δ_{k+1}, we have $|\delta_{k+1}\lambda|\geqslant 2$, $|\delta_{k+1}\mathrm{Im}\,\lambda|\leqslant 1$. We need the following elementary inequality: $\sup\limits_{\alpha\in\left(\frac{1}{2},1\right)}|\sin\alpha z|>\frac{1}{3}$ for $|\mathrm{Im}\,z|\leqslant 1$ and $|z|\geqslant 2$. Let's choose the number $\alpha\in\left(\frac{1}{2},1\right)$ such that $|\sin\alpha\lambda\,\delta_{k+1}|\geqslant\frac{1}{3}$. Then from (20) for $\delta=\alpha\delta_{k+1}$ we obtain

$$\left\{\frac{1}{30}\left[1-\frac{8}{120}\right]-\frac{1}{120}-\frac{1}{60}\right\}\left\|\overset{k}{u}\right\|_{L^2_\infty(K)}\leqslant\frac{4}{\delta}\left\|\overset{k+1}{u}\right\|_{L^2_\infty(K)}.$$

Consequently,

$$\left\|\overset{k}{u}\right\|_{L^2_\infty(K)}\leqslant\frac{1400}{\delta_{k+1}}\left\|\overset{k+1}{u}\right\|_{L^2_\infty(K)}.$$

By the definition of the number δ_{k+1}, we have

$$\left\|\overset{k}{u}\right\|_{L^2_\infty(K)}\leqslant 1400\max\left\{\frac{4}{b'-a'},\frac{1}{R_1},120C^1\left(K,k,b_1,b_2\right)\left(\frac{1}{\sqrt{|b_1b_2|}}+\frac{|b_1|+|b_2|}{|b_1b_2|}\right)\right\}\times$$

$$\times(1+|\mathrm{Im}\,\lambda|)\left\|\overset{k+1}{u}\right\|_{L^2_\infty(K)}. \tag{21}$$

Consider the case

$$|\lambda|<2\max\left\{1,\sqrt{|b_1b_2|}\right\}\max\left\{\frac{4}{b'-a'},\frac{1}{R_1},120C^1\left(K,k,b_1,b_2\right)\times\right.$$

$$\left.\times\left(\frac{1}{\sqrt{|b_1b_2|}}+\frac{|b_1|+|b_2|}{|b_1b_2|}\right)\right\}(1+|\mathrm{Im}\,\lambda|).$$

Choose the number

$$S_k\geqslant 2\max\left\{1,\sqrt{|b_1b_2|}\right\}\max\left\{\frac{4}{b'-a'},\frac{1}{R_1},120C^1\left(K,k,b_1,b_2\right)\left(\frac{1}{\sqrt{|b_1b_2|}}+\frac{|b_1|+|b_2|}{|b_1b_2|}\right)\right\}$$

such that

$$\frac{1}{|b_1b_2|}\int_E\{|b_1p_2(\xi)|+|b_2p_1(\xi)|+|\lambda|(|b_1|+|b_2|)\}\,d\xi<\frac{1}{8},\forall E\subset K,\ mes\,E\leqslant\frac{1}{S_k(1+|\mathrm{Im}\,\lambda|)}.$$

This is possible due to the summability of the functions $p_1(\xi)$ and $p_2(\xi)$ on K. Define the number $\delta^{k+1}=\frac{\sqrt{|b_1b_2|}}{2S_k(1+|\mathrm{Im}\lambda|)}$. Then, from the formulas (11) and (12) for $\ell=k$ we find

$$\left\|\overset{k}{u}\right\|_{L^2_\infty(K)}\leqslant 2\max_{x\in\left[a'+\delta^{k+1},b'-\delta^{k+1}\right]}\left|\overset{k}{u}(x)\right|+\frac{1}{8}\left\|\overset{k}{u}\right\|_{L^2_\infty(K)}+2\frac{|b_1|+|b_2|}{|b_1b_2|}\delta^{k+1}\left\|\overset{k-1}{u}\right\|_{L^2_\infty(K)}.$$

Consequently,

$$\frac{7}{16}\left\|\overset{k}{u}\right\|_{L^2_\infty(K)}-\frac{|b_1|+|b_2|}{|b_1b_2|}\delta^{k+1}\left\|\overset{k-1}{u}\right\|_{L^2_\infty(K)}\leqslant\max_{x\in\left[a'+\delta^{k+1},b'-\delta^{k+1}\right]}\left|\overset{k}{u}(x)\right|. \tag{22}$$

From (13) for $t=\delta^{k+1}$ and $\ell=k+1$ it follows

$$2\delta^{k+1}\max_{x\in\left[a'+\delta^{k+1},b'-\delta^{k+1}\right]}\left|\overset{k}{u}(x)\right|\leqslant 2\left(|b_1|+|b_2|\right)\left\|\overset{k+1}{u}\right\|_{L^2_\infty(K)}+$$
$$+\frac{1}{8}\left\|\overset{k+1}{u}\right\|_{L^2_\infty(K)}+\frac{1}{8}\delta^{k+1}\left\|\overset{k}{u}\right\|_{L^2_\infty(K)}+\frac{|b_1|+|b_2|}{|b_1b_2|}\left(\delta^{k+1}\right)^2\left\|\overset{k-1}{u}\right\|_{L^2_\infty(K)}.$$

Hence,

$$\max_{x\in\left[a'+\delta^{k+1},b'-\delta^{k+1}\right]}\left|\overset{k}{u}(x)\right|\leqslant\frac{|b_1|+|b_2|+\frac{1}{16}}{\delta^{k+1}}\left\|\overset{k+1}{u}\right\|_{L^2_\infty(K)}+\frac{1}{16}\left\|\overset{k}{u}\right\|_{L^2_\infty(K)}+$$
$$+\frac{|b_1|+|b_2|}{2\,|b_1b_2|}\delta^{k+1}\left\|\overset{k-1}{u}\right\|_{L^2_\infty(K)}.$$

Taking into account the inequality (22), from the last relation we have

$$\frac{3}{8}\left\|\overset{k}{u}\right\|_{L^2_\infty(K)}\leqslant\frac{3}{2}\frac{|b_1|+|b_2|}{|b_1b_2|}\delta^{k+1}\left\|\overset{k-1}{u}\right\|_{L^2_\infty(K)}+\left(|b_1|+|b_2|+\frac{1}{16}\right)\frac{1}{\delta^{k+1}}\left\|\overset{k+1}{u}\right\|_{L^2_\infty(K)}.$$

Due to the definitions of the numbers δ^{k+1} and S_k, we obtain the estimate

$$\left\|\overset{k}{u}\right\|_{L^2_\infty(K)}\leqslant\frac{16S_k\left(|b_1|+|b_2|+\frac{1}{16}\right)}{3\sqrt{|b_1b_2|}-6\left(|b_1|+|b_2|\right)\frac{1}{S_k}C^1\left(K,k,b_1,b_2\right)}(1+|\mathrm{Im}\lambda|)\left\|\overset{k+1}{u}\right\|_{L^2_\infty(K)}. \tag{23}$$

The estimates (21) and (23) imply (1) for $\ell=k+1$.

Now let's prove the estimate (2). Rewrite the formula (7) at the points x, $x+t$, $x\in\left[a',\frac{a'+b'}{2}\right]$, $0\leqslant t\leqslant\delta_{\ell+1}$, as follows:

$$\overset{\ell}{u}(x)=\left[\cos\frac{\lambda}{\sqrt{|b_1b_2|}}tI+\sin\frac{\lambda}{\sqrt{|b_1b_2|}}t\frac{B}{\sqrt{|b_1b_2|}}\right]\overset{\ell}{u}(x+t)+$$

$$+B^{-1}\int_x^{x+t}\left(\sin\frac{\lambda}{\sqrt{|b_1b_2|}}(\xi-x)\frac{B}{\sqrt{|b_1b_2|}}+\right.$$

$$\left.+\cos\frac{\lambda}{\sqrt{|b_1b_2|}}(\xi-x)I\right)\left(P(\xi)\overset{\ell}{u}(\xi)-\overset{\ell-1}{u}(\xi)\right)d\xi.$$

Integrating the last equality with respect to t from 0 to $\delta_{\ell+1}$ and using the inequality (17), we find

$$\delta_{\ell+1}\left|\overset{\ell}{u}(x)\right|\leqslant 2\left(1+\frac{|b_1|+|b_2|}{\sqrt{|b_1b_2|}}\right)\int_0^{\delta_{\ell+1}}\left|\overset{\ell}{u}(x+t)\right|dt+$$

$$+2\delta_{\ell+1}\left\{\frac{1}{\sqrt{|b_1b_2|}}\int_x^{x+\delta_{\ell+1}}(|p_1(\xi)|+|p_2(\xi)|)\,d\xi+\right.$$

$$\left.+\frac{1}{|b_1b_2|}\int_x^{x+\delta_{\ell+1}}(|b_1p_2(\xi)|+|b_2p_1(\xi)|)\,d\xi\right\}\left\|\overset{\ell}{u}\right\|_{L^2_\infty(K)}+\frac{2}{\sqrt{|b_1b_2|}}\delta^2_{\ell+1}\left\|\overset{\ell-1}{u}\right\|_{L^2_\infty(K)}+$$

$$+2\frac{|b_1|+|b_2|}{\sqrt{|b_1b_2|}}\delta^2_{\ell+1}\left\|\overset{\ell-1}{u}\right\|_{L^2_\infty(K)}.$$

Applying Hölder's inequality and the estimate (1), from the last relation, by (16), we obtain

$$\delta_{\ell+1}\left|\overset{\ell}{u}(x)\right|\leqslant 2\left(1+\frac{|b_1|+|b_2|}{\sqrt{|b_1b_2|}}\right)\delta_{\ell+1}^{1-\frac{1}{p}}\left\|\overset{\ell}{u}\right\|_{L^2_p(K)}+\frac{1}{60}\delta_{\ell+1}\left\|\overset{\ell}{u}\right\|_{L^2_p(K)}+$$

$$+2\delta^2_{\ell+1}\left\{\frac{1}{\sqrt{|b_1b_2|}}+\frac{|b_1|+|b_2|}{\sqrt{|b_1b_2|}}\right\}C^1(K,\ell,b_1,b_2)(1+|\mathrm{Im}\,\lambda|)\left\|\overset{\ell}{u}\right\|_{L^2_p(K)}.$$

Hence, in turn, it follows

$$\left|\overset{\ell}{u}(x)\right|\leqslant 2\left(1+\frac{|b_1|+|b_2|}{\sqrt{|b_1b_2|}}\right)\delta_{\ell+1}^{-\frac{1}{p}}\left\|\overset{\ell}{u}\right\|_{L^2_p(K)}+\left[\frac{1}{60}+2\delta_{\ell+1}\left\{\frac{1}{\sqrt{|b_1b_2|}}+\frac{|b_1|+|b_2|}{|b_1b_2|}\right\}\times\right.$$

$$\left.\times C^1(K,\ell,b_1,b_2)(1+|\mathrm{Im}\,\lambda|)\right]\left\|\overset{\ell}{u}\right\|_{L^2_\infty(K)}\leqslant$$

$$\leqslant 2\left(1+\frac{|b_1|+|b_2|}{\sqrt{|b_1b_2|}}\right)\delta_{\ell+1}^{-\frac{1}{p}}\left\|\overset{\ell}{u}\right\|_{L^2_p(K)}+\frac{1}{30}\left\|\overset{\ell}{u}\right\|_{L^2_\infty(K)}.$$

Using formula (6), we obtain the similar formula for $x\in\left[\frac{a'+b'}{2},b'\right]$, too. Consequently,

$$\left\|\overset{\ell}{u}\right\|_{L^2_\infty(K)}\leqslant 3\left(1+\frac{|b_1|+|b_2|}{\sqrt{|b_1b_2|}}\right)\left[\max\left\{\frac{4}{b'-a'},\frac{1}{R_1},120C^1(K,k,b_1,b_2)\times\right.\right.$$

$$\times\left(\frac{1}{\sqrt{|b_1b_2|}}+\frac{|b_1|+|b_2|}{|b_1b_2|}\right)\Bigg\}\Bigg]^{\frac{1}{p}}(1+|\mathrm{Im}\,\lambda|)^{\frac{1}{p}}\left\|\overset{\ell}{u}\right\|_{L^2_\infty(K)}.$$

The validity of the estimate (2) is proved. Theorem 1 is proved.

4. Proof of the Bessel property criterion

Necessity. Let $G=(0,2\pi)$ and τ be an arbitrary real number. Introduce an index set

$$I_\tau=\{k:|Re\lambda_k-\tau|\leqslant 1,\ |\mathrm{Im}\,\lambda_k|\leqslant C_0\},$$

where C_0 is a constant defined in the condition (3). Let's choose the positive number R such that the inequality $\omega(R)=\sup\limits_{E\subset\overline{G}}\left\{\|P\|_{1,E}\right\}\leqslant L^{-1}$ holds for every set $E\subset\overline{G}$, $mes\,E\leqslant 2R$, where L is a positive number to be defined later and

$$\|P\|_{1,E}=\int_E\{|p_1(x)|+|p_2(x)|\}\,dx.$$

Let $k\in I_\tau$ and $x\in[0,\pi]$. Consider the formula of average value (8) at the points $x,\ x+t,\ x+2t$ for $t\in[0,R]$:

$$u_k(x)=2u_k(x+t)\cos\frac{\lambda_k}{\sqrt{|b_1b_2|}}t-u_k(x+2t)+$$

$$+B^{-1}\int_x^{x+2t}\left\{\sin\frac{\lambda_k}{\sqrt{|b_1b_2|}}(t-|x+t-\xi|)\frac{B}{\sqrt{|b_1b_2|}}-\right.$$

$$\left.-sign\,(\xi-x-t)\cos\frac{\lambda_k}{\sqrt{|b_1b_2|}}(t-|x+t-\xi|)I\right\}[P(\xi)u_k(\xi)-\theta_ku_{k-1}(\xi)]\,d\xi.$$

Adding and subtracting $2u_k(x+t)\cos\frac{\tau}{\sqrt{|b_1b_2|}}t$ on the right-hand side of this equality and integrating the obtained relation with respect to t from 0 to R, we get

$$u_k(x)=2R^{-1}\int_0^R u_k(x+t)\cos\frac{\tau}{\sqrt{|b_1b_2|}}t\,dt-R^{-1}\int_0^R u_k(x+2t)dt+$$

$$+4R^{-1}\int_0^R u_k(x+t)\sin\frac{\lambda_k+\tau}{2\sqrt{|b_1b_2|}}t\times$$

$$\times\sin\frac{\tau-\lambda_k}{2\sqrt{|b_1b_2|}}t\,dt+R^{-1}B^{-1}\int_0^R\int_x^{x+2t}\left\{\sin\left(\frac{\lambda_k}{\sqrt{|b_1b_2|}}(t-|x+t-\xi|)\right)\frac{B}{\sqrt{|b_1b_2|}}-\right.$$

$$\left.-sign\,(\xi-x-t)\cos\left(\frac{\lambda_k}{\sqrt{|b_1b_2|}}(t-|x+t-\xi|)\right)I\right\}[P(\xi)u_k(\xi)-\theta_ku_{k-1}(\xi)]\,d\xi\,dt.$$

Applying formula (6) to the third term, we obtain

$$u_k(x) = R^{-1}\int_0^{2\pi} u_k(t)w(t)dt + 4R^{-1}\int_0^R \left(\cos\frac{\lambda_k}{\sqrt{|b_1b_2|}}tI - \sin\frac{\lambda}{\sqrt{|b_1b_2|}}t\frac{B}{\sqrt{|b_1b_2|}}\right)\times$$

$$\times\sin\frac{\lambda_k+\tau}{2\sqrt{|b_1b_2|}}t\sin\frac{\tau-\lambda_k}{2\sqrt{|b_1b_2|}}t\,dt\,u_k(x)+$$

$$+4R^{-1}B^{-1}\int_0^R\int_x^{x+t}\left\{\left(\sin\frac{\lambda_k}{\sqrt{|b_1b_2|}}(t-|\xi-x|)\right)\frac{B}{\sqrt{|b_1b_2|}}-\right.$$

$$\left.-\left(\cos\frac{\lambda_k}{\sqrt{|b_1b_2|}}(t-|\xi-x|)\right)I\right\}[P(\xi)u_k(\xi)-\theta_k u_{k-1}(\xi)]\,d\xi\sin\frac{\tau+\lambda_k}{2\sqrt{|b_1b_2|}}t\sin\frac{\tau-\lambda_k}{2\sqrt{|b_1b_2|}}t\,dt+$$

$$+R^{-1}B^{-1}\int_0^R\int_x^{x+2t}\left\{\left(\sin\frac{\lambda_k}{\sqrt{|b_1b_2|}}(t-|x+t-\xi|)\right)\frac{B}{\sqrt{|b_1b_2|}}-\right.$$

$$\left.-\left(\cos\frac{\lambda_k}{\sqrt{|b_1b_2|}}(t-|x+t-\xi|)\right)I\right\}[P(\xi)u_k(\xi)-\theta_k u_{k-1}(\xi)]\,d\xi\,dt=$$

$$=R^{-1}\int_0^{2\pi}u_k(t)w(t)dt+J_1+J_2+J_3, \tag{24}$$

where $w(t)=2\cos\frac{\tau}{\sqrt{|b_1b_2|}}(x-t)-\frac{1}{2}$ for $x\leqslant t\leqslant x+R$, $w(t)=-\frac{1}{2}$ for $x+R<t\leqslant x+2R$ and $w(t)=0$ for $t\notin[x,x+2R]$.

Taking into account $k\in J_k$ and using the inequalities $|\sin z|\leqslant 2$, $|\cos z|\leqslant 2$, $|\sin z|\leqslant 2|z|$ for $|\mathrm{Im}\,z|\leqslant 1$, we obtain the following estimates for J_1, J_2, J_3:

$$|J_1|\leqslant 8R\left(\frac{2}{\sqrt{|b_1b_2|}}+\frac{|b_1|+|b_2|}{|b_1b_2|}\right)|\tau-\lambda_k|\,|u_k(x)|\leqslant$$

$$\leqslant 8R\left(\frac{2}{\sqrt{|b_1b_2|}}+\frac{|b_1|+|b_2|}{|b_1b_2|}\right)(1+|\mathrm{Im}\,\lambda_k|)\,|u_k(x)|\leqslant$$

$$\leqslant 8R\left(\frac{2}{\sqrt{|b_1b_2|}}+\frac{|b_1|+|b_2|}{|b_1b_2|}\right)(1+C_0)\,\|u_k(x)\|_{\infty,2};$$

$$|J_2|\leqslant 32\left(\frac{2}{\sqrt{|b_1b_2|}}+\frac{|b_1|+|b_2|}{|b_1b_2|}\right)\left(\omega(R)\,\|u_k\|_{\infty,2}+\frac{R}{2}\,\|\theta_k u_{k-1}\|_{\infty,2}\right);$$

$$|J_3|\leqslant 2\left(\frac{2}{\sqrt{|b_1b_2|}}+\frac{|b_1|+|b_2|}{|b_1b_2|}\right)\left(\omega(R)\,\|u_k\|_{\infty,2}+R\,\|\theta_k u_{k-1}\|_{\infty,2}\right).$$

Considering these estimates in (24), we arrive at the inequality

$$|u_k(x)| \leqslant R^{-1}\left|\int_0^{2\pi} u_k(t)w(t)dt\right| + 8\left(\frac{2}{\sqrt{|b_1b_2|}} + \frac{|b_1|+|b_2|}{|b_1b_2|}\right)\times$$

$$\times\left(R(1+C_0)+5\omega(R)\right)\|u_k\|_{\infty,2} + 18R\left(\frac{2}{\sqrt{|b_1b_2|}} + \frac{|b_1|+|b_2|}{|b_1b_2|}\right)\|\theta_k u_{k-1}\|_{\infty,2}. \quad (25)$$

In case $x\in[\pi,2\pi]$, the inequality (25) can be proved similarly. In this case the function $w(t)$ is defined as follows: $w(t)=-\frac{1}{2}$ for $x-2R\leqslant t<x-R$, $w(t)=2\cos\frac{\tau}{\sqrt{|b_1b_2|}}(x-t)-\frac{1}{2}$ for $x-R\leqslant t\leqslant x$, and $w(t)=0$ for $t\notin[x-2R,x]$. Consequently, the inequality (25) is true for every $x\in[0,2\pi]$.

Applying the estimates (1), (2) and taking into account the relation $1+|\operatorname{Im}\lambda_k|\leqslant 1+C_0$, from (25) we obtain

$$|u_k(x)| \leqslant R^{-1}\left|\int_0^{2\pi} u_k(t)w(t)dt\right| +$$

$$+\left(\frac{2}{\sqrt{|b_1b_2|}} + \frac{|b_1|+|b_2|}{|b_1b_2|}\right)\Big\{40\omega(R)C^2(G,n_k,b_1,b_2)(1+C_0)^{\frac{1}{2}} +$$

$$+8RC^2(G,n_k,b_1,b_2)(1+C_0)^{\frac{3}{2}} +$$

$$+18RC^1(G,n_k,b_1,b_2)C^2(n_k,G,b_1,b_2)\theta_k(1+C_0)^{\frac{3}{2}}\Big\}\|u_k\|_{2,2}.$$

Due to the uniform boundedness of the lengths of chains,

$$\left(\frac{2}{\sqrt{|b_1b_2|}} + \frac{|b_1|+|b_2|}{|b_1b_2|}\right)C^2(G,n_k,b_1,b_2) \leqslant \gamma_1 = const,$$

$$\left(\frac{2}{\sqrt{|b_1b_2|}} + \frac{|b_1|+|b_2|}{|b_1b_2|}\right)C^1(G,n_k,b_1,b_2)C^2(G,n_k,b_1,b_2) \leqslant \gamma_2 = const.$$

Consequently,

$$|u_k(x)| \leqslant R^{-1}\left|\int_0^{2\pi} u_k(t)w(t)dt\right| +$$

$$+\Big\{40\omega(R)\gamma_1(1+C_0)^{\frac{1}{2}} + 8R\gamma_1(1+C_0)^{\frac{3}{2}} + 18R\gamma_2\theta_k(1+C_0)^{\frac{3}{2}}\Big\}\|u_k\|_{2,2}.$$

By multiplying both sides of this inequality by $\|u_k\|_{2,2}^{-1}$, squaring, and applying the inequality $\left|\sum\limits_{i=1}^m a_i\right|^2 \leqslant m\sum\limits_{i=1}^m |a_i|^2$, we find

$$|u_k(x)|^2\|u_k\|_{2,2}^{-2} \leqslant 3R^{-2}\left\{\left|\int_0^{2\pi} u_k^1(t)w(t)dt\right|^2\|u_k\|_{2,2}^{-2} + \left|\int_0^{2\pi} u_k^2(t)w(t)dt\right|^2\|u_k\|_{2,2}^{-2}\right\}+$$

$$+3\left\{40L^{-1}\gamma_1\left(1+C_0\right)^{\frac{1}{2}}+8R\gamma_1\left(1+C_0\right)^{\frac{3}{2}}+18R\gamma_2\theta_k\left(1+C_0\right)^{\frac{3}{2}}\right\}^2,$$

where $u_k(t)=\left(u_k^1(t),u_k^2(t)\right)^T$.

By Bessel's inequality, we get the validity of the inequality

$$\sum_{k\in J}\left|u_k(x)\right|\left\|u_k\right\|_{2,2}^{-2}\leqslant 6MR^{-2}\left\|w\right\|_2^2+$$

$$+3\left\{40L^{-1}\gamma_1\left(1+C_0\right)^{\frac{1}{2}}+8R\gamma_1\left(1+C_0\right)^{\frac{3}{2}}+18R\gamma_2\theta_k\left(1+C_0\right)^{\frac{3}{2}}\right\}^2\sum_{k\in J}1 \tag{26}$$

for every finite set $J\subset I_\tau$. Taking into account the equality $\|w\|_2^2=O(R)$, choosing R (consequently, the number L^{-1} too) small enough to have an estimate that

$$3\left\{40L^{-1}\gamma_1\left(1+C_0\right)^{\frac{1}{2}}+8R\gamma_1\left(1+C_0\right)^{\frac{3}{2}}+18R\gamma_2\theta_k\left(1+C_0\right)^{\frac{3}{2}}\right\}^2\leqslant\frac{1}{4\pi}.$$

Integrating (26), we obtain

$$\sum_{k\in J}1\leqslant const\,R^{-1}=const.$$

Due to the arbitrariness of the set $J\subset I_\tau$, we get the validity of (4). The necessity of the inequality (4) is proved.

Sufficiency. For definiteness, we consider $G=(0,2\pi)$. Rewriting the formula (6) for $u_k(x+t)$ with $x=0$ and multiplying it scalarly by the vector function $f(t)=(f_1(t),f_2(t))^T\in L_2^2(0,2\pi)$, we arrive at the conclusion that to prove the validity of Bessel's inequality for the system $\varphi_k(t)=u_k(t)\left\|u_k\right\|_{2,2}^{-1}$, $k=1,2,\ldots$, it suffices to prove the validity of the following inequalities:

$$\sum_{k=1}^{\infty}\left|\int_0^{2\pi}\overline{f_i(t)}\cos\frac{\lambda_k}{\sqrt{|b_1b_2|}}t\,dt\right|^2\left|\varphi_k^i(0)\right|^2\leqslant C\left\|f\right\|_{2,2}^2,\; i=1,2; \tag{27}$$

$$\sum_{k=1}^{\infty}\left|\int_0^{2\pi}\overline{f_i(t)}\sin\frac{\lambda_k}{\sqrt{|b_1b_2|}}t\,dt\right|^2\left|\varphi_k^{3-i}(0)\right|^2\leqslant C\left\|f\right\|_{2,2},\; i=1,2; \tag{28}$$

$$\sum_{k=1}^{\infty}\left|\int_0^{2\pi}\overline{f_1(t)}\int_0^t p_1(\xi)\varphi_k^1(\xi)\sin\frac{\lambda_k}{\sqrt{|b_1b_2|}}(t-\xi)\,d\xi\,dt\right|^2\leqslant C\left\|f\right\|_{2,2}^2, \tag{29}$$

$$\sum_{k=1}^{\infty}\left|\int_0^{2\pi}\overline{f_1(t)}\int_0^t p_2(\xi)\varphi_k^2(\xi)\cos\frac{\lambda_k}{\sqrt{|b_1b_2|}}(t-\xi)\,d\xi\,dt\right|^2\leqslant C\left\|f\right\|_{2,2}^2, \tag{30}$$

$$\sum_{k=1}^{\infty}\left|\int_0^{2\pi}\overline{f_2(t)}\int_0^t p_1(\xi)\varphi_k^1(\xi)\cos\frac{\lambda_k}{\sqrt{|b_1b_2|}}(t-\xi)\,d\xi\,dt\right|^2\leqslant C\left\|f\right\|_{2,2}^2, \tag{31}$$

$$\sum_{k=1}^{\infty}\left|\int_0^{2\pi}\overline{f_2(t)}\int_0^t p_2(\xi)\varphi_k^2(\xi)\sin\frac{\lambda_k}{\sqrt{|b_1b_2|}}(t-\xi)d\xi dt\right|^2\leqslant C\|f\|_{2,2}^2, \tag{32}$$

$$\sum_{k=1}^{\infty}\left|\theta_k\int_0^{2\pi}\overline{f_i(t)}\int_0^t \frac{u_{k-1}^i(\xi)}{\|u_k\|_{2,2}}\sin\frac{\lambda_k}{\sqrt{|b_1b_2|}}(t-\xi)d\xi dt\right|^2\leqslant C\|f\|_{2,2}^2,\ i=1,2; \tag{33}$$

$$\sum_{k=1}^{\infty}\left|\theta_k\int_0^{2\pi}\overline{f_i(t)}\int_0^t \frac{u_{k-1}^{3-i}(\xi)}{\|u_k\|_{2,2}}\cos\frac{\lambda_k}{\sqrt{|b_1b_2|}}(t-\xi)d\xi dt\right|^2\leqslant C\|f\|_{2,2}^2,\ i=1,2; \tag{34}$$

where $\varphi_k^i(\xi)=u_k^i(\xi)\,\|u_k\|_{2,2}^{-1}$.

Let's prove the estimate (27). By the estimate (2) and the conditions (3), (4), we have

$$\left|\varphi_k^i(0)\right|=\left|u_k^i(0)\right|\|u_k\|_{2,2}^{-1}\leqslant\|u_k\|_{\infty,2}\|u_k\|_{2,2}^{-1}\leqslant$$

$$\leqslant C^2(G,n_k,b_1,b_2)(1+C_0)^{\frac{1}{2}}\|u_k\|_{2,2}\|u_k\|_{2,2}^{-1}\leqslant C^2(G,n_k,b_1,b_2)(1+C_0)^{\frac{1}{2}}=const,$$

because the sequence $C^2(G,n_k,b_1,b_2)$ is bounded due to the condition (4). Therefore, for (27) to hold, it is sufficient that

$$\sum_{k=1}^{\infty}\left|\int_0^{2\pi}\overline{f_i(t)}\cos\frac{\lambda_k}{\sqrt{|b_1b_2|}}t dt\right|^2\leqslant C\|f\|_{2,2}^2,\ i=1,2. \tag{35}$$

Under the conditions (3) and (4), the validity of the inequality (35) for $\tau\geqslant 1$ has been proved in [8], which implies the validity of (35) for $Re\,\lambda_k\in(-\infty,+\infty)$, $|\mathrm{Im}\,\lambda_k|\leqslant C_0$, because, by the conditions of Theorem 1.2, the condition (4) is satisfied for every $\tau\in(-\infty,+\infty)$. The inequality (28) can be proved in a similar way.

Let's make sure that the inequalities (29)–(32) are true. They all are proved in the same way. Therefore, we restrict ourselves to proving only the inequality (29). Denote

$$g_i(t,\xi)=\begin{cases} f_i(t+\xi), & 0\leqslant t\leqslant 2\pi-\xi,\\ 0, & 2\pi-\xi<t\leqslant 2\pi,\end{cases}$$

where $\xi\in[0,2\pi]$, $i=1,2$. Then, by the estimate (2) and the conditions (3), (4), we obtain

$$J_k=\left|\int_0^{2\pi}\overline{f_1(t)}\int_0^t p_1(\xi)\varphi_k^1(\xi)\sin\frac{\lambda_k}{\sqrt{|b_1b_2|}}(t-\xi)d\xi dt\right|^2=$$

$$=\int_0^{2\pi}\overline{f_1(t)}\int_0^t p_1(\xi)\varphi_k^1(\xi)\sin\frac{\lambda_k}{\sqrt{|b_1b_2|}}(t-\xi)d\xi dt\times$$

$$\times\int_0^{2\pi}f_1(t)\int_0^t\overline{p_1(\xi)\varphi_k^1(\xi)\sin\frac{\lambda_k}{\sqrt{|b_1b_2|}}(t-\xi)}d\xi dt=$$

$$=\int_0^{2\pi}p_1(\xi)\varphi_k^1(\xi)\int_0^{2\pi}\overline{g_1(t,\xi)}\sin\frac{\lambda_k}{\sqrt{|b_1b_2|}}t dt d\xi\times$$

$$\times \int_0^{2\pi} \overline{p(\tau)\varphi_k^1(\tau)} \int_0^{2\pi} g_1(r,\tau) \overline{\sin \frac{\lambda_k}{\sqrt{|b_1 b_2|}} r} dr d\tau =$$

$$= \int_0^{2\pi} \int_0^{2\pi} p_1(\xi) \overline{p_1(\tau)} \varphi_k^1(\xi) \overline{\varphi_k^1(\tau)} \int_0^{2\pi} \overline{g_1(t,\xi)} \sin \frac{\lambda_k}{\sqrt{|b_1 b_2|}} t dt \times$$

$$\times \int_0^{2\pi} g_1(r,\tau) \overline{\sin \frac{\lambda_k}{\sqrt{|b_1 b_2|}} r} dr d\xi d\tau \leqslant$$

$$\leqslant \left[C^2 (G, n_k, b_1, b_2)\right]^2 (1 + C_0) \int_0^{2\pi} \int_0^{2\pi} |p_1(\xi)| |p_1(\tau)| \left| \int_0^{2\pi} \overline{g_1(t,\xi)} \sin \frac{\lambda_k}{\sqrt{|b_1 b_2|}} t dt \right| \times$$

$$\times \left| \int_0^{2\pi} g_1(r,t) \overline{\sin \frac{\lambda_k}{\sqrt{|b_1 b_2|}} r} dr \right| d\xi d\tau \leqslant$$

$$\leqslant const \int_0^{2\pi} \int_0^{2\pi} |p_1(\xi)| |p_1(\tau)| \left| \int_0^{2\pi} \overline{g_1(t,\xi)} \sin \frac{\lambda_k}{\sqrt{|b_1 b_2|}} t dt \right| \times$$

$$\times \left| \int_0^{2\pi} \overline{g_1(r,\tau)} \sin \frac{\lambda_k}{\sqrt{|b_1 b_2|}} r dr \right| d\xi d\tau.$$

Hence, for an arbitrary positive integer N we find

$$\sum_{k=1}^{N} J_k \leqslant const \int_0^{2\pi} \int_0^{2\pi} |p_1(\xi)| |p_1(\tau)| \left(\sum_{k=1}^{N} \left| \int_0^{2\pi} \overline{g_1(t,\xi)} \sin \frac{\lambda_k}{\sqrt{|b_1 b_2|}} t dt \right| \times \right.$$

$$\left. \times \left| \int_0^{2\pi} \overline{g_1(r,\tau)} \sin \frac{\lambda_k}{\sqrt{|b_1 b_2|}} r dr \right| \right) d\xi d\tau \leqslant$$

$$\leqslant const \int_0^{2\pi} \int_0^{2\pi} |p_1(\xi)| |p_1(\tau)| \|g_1(\cdot,\xi)\|_2 \|g_1(\cdot,\tau)\|_2 d\xi d\tau.$$

Taking into account that for every fixed $\xi \in [0, 2\pi]$ the inequality $\|g_1(\cdot,\xi)\|_2 \leqslant \|f_1\|_2$ holds, we obtain

$$\sum_{k=1}^{N} J_k \leqslant const \|p_1\|_1^2 \|f_1\|_2^2 \leqslant const \|f\|_{2,2}^2.$$

Hence, as the number N is arbitrary, we get the validity of the inequality (28).

Now let's prove the validity of (33). By the estimates (1), (2) and the conditions (3), (4), we have

$$\theta_k \left|u_{k-1}^i(\xi)\right| \|u_k\|_{2,2}^{-1} \leqslant \theta_k C^1 \left(G, n_{k,}, b_1, b_2\right) C^2 \left(G, n_{k,}, b_1, b_2\right) (1 + C_0)^{\frac{3}{2}} \times$$

$$\times \|u_k\|_{2,2} \|u_k\|_{2,2}^{-1} = \theta_k C^1 \left(G, n_{k,}, b_1, b_2\right) C^2 \left(G, n_{k,} b_1, b_2\right) (1 + C_0)^{\frac{3}{2}} \leqslant C = const.$$

After changing the order of integration, the left-hand side of the inequality (33) gets majorized from above by the series

$$C\sum_{k=1}^{\infty}\int_0^{2\pi}\left|\int_0^{2\pi}\overline{g_i(t,\xi)}\sin\frac{\lambda_k}{\sqrt{|b_1b_2|}}t\,dt\right|^2 d\xi.$$

This series converges and its some does not exceed the value $C\|f\|_{2,2}^2$. The inequality (33) is proved. The inequality (34) can be proved similarly. Theorem 2 is proved.

The proofs of Theorems 3 and 4 are similar to the proofs of Theorems 1.6 and 1.7, respectively, in [19, pp. 951–953].

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