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FURTHER PROPERTIES OF PPT AND (α,β)–NORMAL MATRICES

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Abstract. This paper discusses further properties of positive partial transpose matrices, with applications towards normal and (α,β)-normal matrices. The obtained results present extensions and improvements of many results in the literature.

1. Introduction and preliminaries

In the sequel, $\mathscr{M}_n$ denotes the algebra of all $n\times n$ complex matrices, with identity I. When $A\in\mathscr{M}_n$ is such that $\langle Ax,x\rangle\geqslant 0$ for all $x\in\mathbb{C}^n$, we say that A is positive semidefinite, and we write $A\geqslant O$. On the other hand, if $\langle Ax,x\rangle>0$ for all nonzero $x\in\mathbb{C}^n$, A is said to be positive definite, and we write $A>O$. Here, O denotes the zero element of $\mathscr{M}_n$.

Given $A,B,X\in\mathscr{M}_n$, the matrix $\begin{bmatrix} A & X^*\\ X & B\end{bmatrix}$ is in $\mathscr{M}_{2n}$. It is well known that if $\begin{bmatrix} A & X^*\\ X & B\end{bmatrix}\geqslant O$, then

$$\left\|\begin{bmatrix} A & X^*\\ X & B\end{bmatrix}\right\|\leqslant\|A\|+\|B\|, \tag{1.1}$$

where $\|\cdot\|$ is the usual operator norm. Indeed, (1.1) follows from the following useful decomposition [5, Lemma 3.4]: For every matrix $\begin{bmatrix} A & X^*\\ X & B\end{bmatrix}\geqslant O$, we have

$$\begin{bmatrix} A & X^*\\ X & B\end{bmatrix}=U\begin{bmatrix} A & O\\ O & O\end{bmatrix}U^*+V\begin{bmatrix} O & O\\ O & B\end{bmatrix}V^*$$

for some unitaries U,V.

However, if $\begin{bmatrix} A & X^*\\ X & B\end{bmatrix}\geqslant O$ and the off-diagonal block X is Hermitian, Hiroshima [12] showed a stronger inequality than (1.1), as follows

$$\left\|\begin{bmatrix} A & X\\ X & B\end{bmatrix}\right\|\leqslant\|A+B\|. \tag{1.2}$$

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The 2×2 block matrix $\begin{bmatrix} A & X^* \\ X & B \end{bmatrix}$ has received the attention of numerous researchers in the literature due to its usability and applications. We refer the reader to [13, 16, 17, 19, 20, 22] as a recent list of references treating and using block matrices.

The 2×2 block matrix $\begin{bmatrix} A & X^* \\ X & B \end{bmatrix}$ is said to be positive partial transpose (PPT) if both $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$ and $\begin{bmatrix} A & X^* \\ X & B \end{bmatrix}$ are positive semidefinite.

If $\begin{bmatrix} A & X^* \\ X & B \end{bmatrix}$ is PPT, Lee [16, Theorem 2.1] showed that

$$|X| \leqslant \frac{A\sharp B + U^*(A\sharp B)U}{2}, \tag{1.3}$$

where $X = U|X|$ is the polar decomposition of X, and $\sharp$ denotes the matrix geometric mean. Recall, here, that if $A,B > O$ and $0 \leqslant t \leqslant 1$, the weighted geometric mean of A and B is defined by

$$A\sharp_t B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^t A^{\frac{1}{2}}.$$

If $A \geqslant O$ or $B \geqslant O$, the geometric mean is found via a limiting process. More precisely,

$$A\sharp_t B = \lim_{n\to\infty} A_n^{\frac{1}{2}}\left(A_n^{-\frac{1}{2}}B_nA_n^{-\frac{1}{2}}\right)^t A_n^{\frac{1}{2}},$$

where $A_n = A + \frac{1}{n}I$ and $B_n = B + \frac{1}{n}I$.

When $t = \frac{1}{2}$, we simply write $A\sharp B$ instead of $A\sharp_{\frac{1}{2}} B$. The geometric mean $\sharp$ was first introduced by Pusz and Woronowicz [18], which was further developed into a general theory of operator means by Kubo and Ando [15].

Fu et al. [6, Theorem 2.3] improved (1.3) as follows

$$|X| \leqslant (A\sharp B)\sharp(U^*(A\sharp B)U). \tag{1.4}$$

The fact that (1.4) improves (1.3) follows from the arithmetic-geometric mean inequality that states

$$A\sharp B \leqslant \frac{A+B}{2}, \tag{1.5}$$

for any $A,B \in \mathscr{M}_n$ with $A,B \geqslant O$.

Among those useful characterizations of the geometric mean, we have [4, (4.15)])

$$X\sharp Y = \max\left\{Z:\ Z = Z^*, \begin{bmatrix} X & Z \\ Z & Y \end{bmatrix} \geqslant O\right\};\ X,Y > O. \tag{1.6}$$

Recently, the following lemma has been shown in [10, Theorem 2.1].

LEMMA 1.1. *If* $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$ *is PPT, then so is* $\begin{bmatrix} A\sharp_t B & X \\ X^* & A\sharp_{1-t}B \end{bmatrix}$, $0 \leqslant t \leqslant 1$.

Further, it has been shown in [16, Corollary 2.2] that

$$\lambda_j\left(2|X| - A\sharp_t B\right) \leqslant \lambda_j\left(A\sharp_{1-t}B\right), \tag{1.7}$$

when $\begin{bmatrix} A & X^* \\ X & B \end{bmatrix}$ is PPT. Here λ_j denotes the j^{th} largest eigenvalue.

This paper discusses extensions of (1.4) and (1.7), where we extend both inequalities to the weighted geometric mean. We will also extend [6, Theorem 2.2], where we show that when $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \geqslant O$, then $|X| \leqslant \left(A\sharp_t\left(U^*BU\right)\right)\sharp\left(A\sharp_{1-t}\left(U^*BU\right)\right)$, for example. Further, we give an improvement of (1.1) that is different from (1.2). Many other consequences for PPT matrices and positive semidefinite matrices of the form $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$ will be presented.

After that, we present some applications that include normal and (α,β)-normal matrices. Here we recall that if $T \in \mathscr{M}_n$ is such that $|T^*|^2 \leqslant |T|^2$, then T is said to be hyponormal. If T satisfies the weaker condition $|T^*| \leqslant |T|$, then T is said to be semi-hyponormal. In fact, semi-hyponormal matrices are normal. The notion of semi-hyponormality becomes significant when we deal with the C^* algebra of bounded linear operators on an infinite dimensional complex Hilbert space. More generally, if $0 \leqslant \alpha \leqslant 1 \leqslant \beta$ are such that $\alpha^2|T|^2 \leqslant |T^*|^2 \leqslant \beta^2|T|^2$, then T is said to be (α,β)-normal. For example, we will show that

$$\begin{bmatrix} |T| & T^* \\ T & |T| \end{bmatrix} \geqslant O$$

if and only if T is normal. Many other results and consequences will be shown for these classes. As a consequence, we will be able to present a possible reverse for the inequality $\|T^2\| \leqslant \|T\|^2$, when T is (α,β)-normal.

2. Main results for positive and PPT block matrices

Our first result is an extension of [6, Theorem 2.2], which states that if $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \geqslant O$, then

$$|X| \leqslant A\sharp\left(U^*BU\right) \quad \text{and} \quad |X^*| \leqslant \left(UAU^*\right)\sharp B.$$

Once this has been shown, we use it to extend some results about PPT matrices.

THEOREM 2.1. *Let* $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$ *be PPT with* $A,B,X \in \mathscr{M}_n$ *and let* $X = U|X|$ *be the polar decomposition of* X. *Then for any* $0 \leqslant t \leqslant 1$,

$$|X| \leqslant \left(A\sharp_t\left(U^*BU\right)\right)\sharp\left(A\sharp_{1-t}\left(U^*BU\right)\right) \tag{2.1}$$

and

$$|X^*| \leqslant ((UAU^*)\sharp_t B)\sharp((UAU^*)\sharp_{1-t}B). \tag{2.2}$$

In particular,

$$|X| \leqslant A\sharp(U^*BU) \quad \text{and} \quad |X^*| \leqslant (UAU^*)\sharp B.$$

Proof. We prove (2.1). Since $X = U|X|$ is the polar decomposition of X, we have

$$\begin{bmatrix} I & O \\ O & U^* \end{bmatrix} \begin{bmatrix} A & X^* \\ X & B \end{bmatrix} \begin{bmatrix} I & O \\ O & U \end{bmatrix} = \begin{bmatrix} A & X^*U \\ U^*X & U^*BU \end{bmatrix} \geqslant O,$$

which implies

$$\begin{bmatrix} A & |X| \\ |X| & U^*BU \end{bmatrix} \geqslant O,$$

since $X^*U = U^*X = |X|$. Lemma 1.1 implies

$$\begin{bmatrix} A\sharp_t(U^*BU) & |X| \\ |X| & A\sharp_{1-t}(U^*BU) \end{bmatrix} \geqslant O.$$

Now, the result follows from (1.6). To prove (2.2), we have

$$\begin{aligned} |X^*| &= U|X|U^* \quad \text{(by [9, p. 58])} \\ &\leqslant U((A\sharp_t U^*BU)\sharp(A\sharp_{1-t}U^*BU))U^* \\ &= (U(A\sharp_t U^*BU)U^*)\sharp(U(A\sharp_{1-t}U^*BU)U^*) \\ &= (UAU^*\sharp_t BU)\sharp(UAU^*\sharp_{1-t}B). \end{aligned}$$

This proves (2.2). Letting $t = \frac{1}{2}$ in (2.1) and (2.2), and noting that $T\sharp T = T$, when $T > O$, yield $|X| \leqslant A\sharp(U^*BU)$ and $|X^*| \leqslant (UAU^*)\sharp B$. This completes the proof. □

Using Theorem 2.1, we present the following extension of (1.4).

COROLLARY 2.1. *Let $A, B, X \in \mathscr{M}_n$ be such that $\begin{bmatrix} A & X^* \\ X & B \end{bmatrix}$ is PPT, and let $X = U|X|$ be the polar decomposition of X. Then for any $0 \leqslant t \leqslant 1$,*

$$|X| \leqslant (A\sharp_t B)\sharp(U^*(A\sharp_{1-t}B)U),$$

and

$$|X^*| \leqslant (U(A\sharp_t B)U^*)\sharp(A\sharp_{1-t}B).$$

Proof. Lemma 1.1 ensures that $\begin{bmatrix} A\sharp_t B & X^* \\ X & A\sharp_{1-t}B \end{bmatrix}$ is also PPT, for any $0 \leqslant t \leqslant 1$. By Theorem 2.1, we conclude the first desired result.

The second inequality can be shown using the method we used to show (2.2). This completes the proof. □

Interestingly, Corollary 2.1 can be used to present a weighted version of (1.7), as follows.

COROLLARY 2.2. *Let* $A,B,X \in \mathscr{M}_n$ *be such that* $\begin{bmatrix} A & X^* \\ X & B \end{bmatrix}$ *is PPT. Then for any* $0 \leqslant t \leqslant 1$,

$$\lambda_j\left(2|X| - A\sharp_t B\right) \leqslant \lambda_j\left(A\sharp_{1-t}B\right)$$

and

$$\lambda_j\left(2|X^*| - A\sharp_{1-t}B\right) \leqslant \lambda_j\left(A\sharp_t B\right)$$

for all $j = 1,2,\ldots,n$.

Proof. Corollary 2.1 means that

$$\begin{aligned} |X| &\leqslant (A\sharp_t B)\sharp\left(U^*(A\sharp_{1-t}B)U\right) \\ &\leqslant \frac{A\sharp_t B + U^*(A\sharp_{1-t}B)U}{2}. \end{aligned}$$

Thus,

$$2|X| - A\sharp_t B \leqslant U^*(A\sharp_{1-t}B)U.$$

Therefore,

$$\lambda_j\left(2|X| - A\sharp_t B\right) \leqslant \lambda_j\left(A\sharp_{1-t}B\right)$$

as desired. □

Related to the discussion, we employ (1.2) to obtain another refinement of (1.1), as follows.

THEOREM 2.2. *Let* $A,B,X \in \mathscr{M}_n$ *be such that* $\begin{bmatrix} A & X^* \\ X & B \end{bmatrix} \geqslant O$ *and let* $X = U|X|$ *be the polar decomposition of* X. *Then*

$$\left\| \begin{bmatrix} A & X^* \\ X & B \end{bmatrix} \right\| \leqslant \|A + U^*BU\|.$$

Proof. If $X = U|X|$ is the polar decomposition of X, then

$$\begin{bmatrix} I & O \\ O & U^* \end{bmatrix}\begin{bmatrix} I & O \\ O & U \end{bmatrix} = \begin{bmatrix} I & O \\ O & U \end{bmatrix}\begin{bmatrix} I & O \\ O & U^* \end{bmatrix} = \begin{bmatrix} I & O \\ O & I \end{bmatrix},$$

since U is unitary. The fact that $\begin{bmatrix} A & X^* \\ X & B \end{bmatrix} \geqslant O$ implies

$$\begin{aligned} \left\| \begin{bmatrix} A & X^* \\ X & B \end{bmatrix} \right\| &= \left\| \begin{bmatrix} I & O \\ O & U^* \end{bmatrix}\begin{bmatrix} A & X^* \\ X & B \end{bmatrix}\begin{bmatrix} I & O \\ O & U \end{bmatrix} \right\| \\ &= \left\| \begin{bmatrix} A & |X| \\ |X| & U^*BU \end{bmatrix} \right\| \\ &\leqslant \|A + U^*BU\|, \end{aligned}$$

where we have used (1.2) to obtain the last inequality. This completes the proof. □

The following is an interesting characterization related to PPT matrices.

THEOREM 2.3. *Let $A,B,X \in \mathcal{M}_n$ be such that $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$ is PPT and let $0 \leqslant t \leqslant 1$. Then there are some isometries $\widetilde{U},\widetilde{V} \in \mathcal{M}_{2n,n}$ (depending on t), such that*

$$\begin{bmatrix} A\sharp_t B & X \\ X^* & A\sharp_{1-t}B \end{bmatrix} = \widetilde{U}\left(A\sharp_t B\right)\widetilde{U}^* + \widetilde{V}\left(A\sharp_{1-t}B\right)\widetilde{V}^*.$$

Proof. By Lemma 1.1, $\begin{bmatrix} A\sharp_t B & X \\ X & A\sharp_{1-t}B \end{bmatrix}$ is PPT. From [5, Lemma 3.4], there are two unitaries $U,V \in \mathcal{M}_{2n}$ partitioned into equally sized matrices,

$$U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \text{ and } V = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}$$

such that

$$\begin{bmatrix} A\sharp_t B & X \\ X^* & A\sharp_{1-t}B \end{bmatrix} = U\begin{bmatrix} A\sharp_t B & O \\ O & O \end{bmatrix}U^* + V\begin{bmatrix} O & O \\ O & A\sharp_{1-t}B \end{bmatrix}V^*.$$

Hence,

$$\begin{bmatrix} A\sharp_t B & X \\ X^* & A\sharp_{1-t}B \end{bmatrix} = \widetilde{U}\left(A\sharp_t B\right)\widetilde{U}^* + \widetilde{V}\left(A\sharp_{1-t}B\right)\widetilde{V}^*,$$

where

$$\widetilde{U} = \begin{bmatrix} U_{11} \\ U_{21} \end{bmatrix} \text{ and } \widetilde{V} = \begin{bmatrix} V_{12} \\ V_{22} \end{bmatrix}$$

are isometries. This completes the proof. □

Theorem 2.3 implies the following remarkable result.

COROLLARY 2.3. *Let $A,B,X \in \mathcal{M}_n$ be such that $\begin{bmatrix} A & X^* \\ X & B \end{bmatrix}$ is PPT. Then for any $0 \leqslant t \leqslant 1$,*

$$\left\|\begin{bmatrix} A\sharp_t B & X \\ X^* & A\sharp_{1-t}B \end{bmatrix}\right\| \leqslant \|A\sharp_t B\| + \|A\sharp_{1-t}B\|.$$

In particular

$$\left\|\begin{bmatrix} A\sharp B & X \\ X^* & A\sharp B \end{bmatrix}\right\| \leqslant 2\|A\sharp B\|.$$

REMARK 2.1. Ando [1, Theorem 3.3] proved that if $\begin{bmatrix} A & X^* \\ X & B \end{bmatrix}$ is PPT, then

$$\|X\| \leqslant \|A\sharp B\|.$$

We know that [21] if $\begin{bmatrix} A & X^* \\ X & B \end{bmatrix} \geqslant O$, then

$$2\|X\| \leqslant \left\|\begin{bmatrix} A & X^* \\ X & B \end{bmatrix}\right\|.$$

Consequently, if $\begin{bmatrix} A & X^* \\ X & B \end{bmatrix}$ is PPT, then

$$\|X\| \leqslant \frac{1}{2}\left\|\begin{bmatrix} A\sharp B & X \\ X^* & A\sharp B \end{bmatrix}\right\| \leqslant \|A\sharp B\|.$$

REMARK 2.2. It is well-known that

$$\left|\left\langle T|T|^{\alpha+\beta-1}x,y\right\rangle\right|^2 \leqslant \left\langle |T|^{2\alpha}x,x\right\rangle \left\langle |T^*|^{2\beta}y,y\right\rangle; (\alpha,\beta\in[0,1],\alpha+\beta\geqslant 1)$$

for any T [8]. By [14, Lemma 1], we infer that

$$\begin{bmatrix} |T|^{2\alpha} & |T|^{\alpha+\beta-1}T^* \\ T|T|^{\alpha+\beta-1} & |T^*|^{2\beta} \end{bmatrix} \geqslant O.$$

So,

$$\begin{aligned} 2\left\|T|T|^{\alpha+\beta-1}\right\| &\leqslant \left\|\begin{bmatrix} |T|^{2\alpha} & |T|^{\alpha+\beta-1}T^* \\ T|T|^{\alpha+\beta-1} & |T^*|^{2\beta} \end{bmatrix}\right\| \\ &\leqslant \left\||T|^{2\alpha}+U^*|T^*|^{2\beta}U\right\| \\ &= 2\left\||T|^{2\alpha}+|T|^{2\beta}\right\|. \end{aligned}$$

That is,

$$\left\|T|T|^{\alpha+\beta-1}\right\| \leqslant \frac{1}{2}\left\|\begin{bmatrix} |T|^{2\alpha} & |T|^{\alpha+\beta-1}T^* \\ T|T|^{\alpha+\beta-1} & |T^*|^{2\beta} \end{bmatrix}\right\| \leqslant \left\||T|^{2\alpha}+|T|^{2\beta}\right\|.$$

In particular,

$$\|T\| = \frac{1}{2}\left\|\begin{bmatrix} |T^*| & T \\ T^* & |T| \end{bmatrix}\right\|.$$

We notice that when $\begin{bmatrix} A & X^* \\ X & B \end{bmatrix} \geqslant O$, both (1.1) and (1.2) give upper bounds of $\begin{bmatrix} A & X^* \\ X & B \end{bmatrix}$ in terms of A and B only. In the following result, we obtain an upper bound that involves X, as well. We emphasize here that for a general $T\in\mathscr{M}_n$, the inequality $T\leqslant|T|$ is not true. However, it becomes true when T is Hermitian. We refer the reader to [20] for a related discussion of this ordering.

THEOREM 2.4. *Let $A,B,X\in\mathscr{M}_n$ be such that $\begin{bmatrix} A & X^* \\ X & B \end{bmatrix} \geqslant O$. Then*

$$\left\|\begin{bmatrix} A & X^* \\ X & B \end{bmatrix}\right\| \leqslant \|A+B+|X|+|X^*|\|.$$

Proof. Indeed,

$$\begin{aligned}\left\|\begin{bmatrix} A & X^* \\ X & B \end{bmatrix}\right\| &= \left\|\begin{bmatrix} A & O \\ O & B \end{bmatrix} + \begin{bmatrix} O & X^* \\ X & O \end{bmatrix}\right\| \\ &\leqslant \left\|\begin{bmatrix} A & O \\ O & B \end{bmatrix} + \begin{bmatrix} |X| & O \\ O & |X^*| \end{bmatrix}\right\| \\ &= \left\|\begin{bmatrix} A+|X| & O \\ O & B+|X^*| \end{bmatrix}\right\| \\ &\leqslant \|A+B+|X|+|X^*|\|\end{aligned}$$

where the first inequality follows from the fact that $\begin{bmatrix} O & X^* \\ X & O \end{bmatrix}$ is Hermitian, and for any Hermitian matrix T, we have $T \leqslant |T|$. The second inequality is also obtained from (1.2). This completes the proof. □

3. Applications towards normal and (α,β)-normal matrices

This section presents several results on normal and (α,β)-normal matrices. While some of these results can be considered as applications of the results of the previous section, other results are related but independent.

THEOREM 3.1. *Let $T \in \mathscr{M}_n$. Then $\begin{bmatrix} |T| & T^* \\ T & |T| \end{bmatrix} \geqslant O$ if and only if T is normal.*

Proof. It is easy to show that [20] if T is normal, then

$$\begin{bmatrix} |T| & T^* \\ T & |T| \end{bmatrix} \geqslant O.$$

We show that if $\begin{bmatrix} |T| & T^* \\ T & |T| \end{bmatrix} \geqslant O$, then T is normal. Indeed, if $T = U|T|$ is the polar decomposition of T, then

$$\begin{bmatrix} |T| & T^*U \\ U^*T & U^*|T|U \end{bmatrix} = \begin{bmatrix} I & O \\ O & U^* \end{bmatrix}\begin{bmatrix} |T| & T^* \\ T & |T| \end{bmatrix}\begin{bmatrix} I & O \\ O & U \end{bmatrix} \geqslant O.$$

Since $T^*U = U^*T = |T|$, we get

$$\begin{bmatrix} |T| & |T| \\ |T| & U^*|T|U \end{bmatrix} \geqslant O.$$

From (1.6), we get

$$|T| \leqslant |T| \sharp (U^*|T|U).$$

By the definition of geometric mean, we have

$$|T| \leqslant |T|^{\frac{1}{2}}\left(|T|^{-\frac{1}{2}}U^*|T|U|T|^{-\frac{1}{2}}\right)^{\frac{1}{2}}|T|^{\frac{1}{2}}.$$

Multiplying both sides by $|T|^{-\frac{1}{2}}$, we infer that

$$I \leqslant \left(|T|^{-\frac{1}{2}}U^*|T|U|T|^{-\frac{1}{2}}\right)^{\frac{1}{2}}.$$

This implies

$$|T| \leqslant U^*|T|U.$$

Thus,

$$|T^*| = U|T|U^* \leqslant |T|,$$

which shows that T is semi-hyponormal. But semi-hyponormal matrices are indeed normal. This completes the proof. □

In [5, (2.11)], it has been shown that if A,B are normal, then

$$|A+B| \leqslant \frac{|A|+|B|+U^*(|A|+|B|)U}{2}, \tag{3.1}$$

where U is the unitary matrix in the polar decomposition of $A+B$. Since every normal matrix is necessarily semi-hyponormal, and because of (1.5), the following result significantly improves [5, (2.11)].

COROLLARY 3.1. *Let $A,B \in \mathcal{M}_n$ be normal and let U be the unitary part in the polar decomposition $A+B = U|A+B|$. Then*

$$|A+B| \leqslant (|A|+|B|)\sharp(U^*(|A|+|B|)U).$$

Proof. Using Theorem 3.1, we see that

$$\begin{bmatrix} |A|+|B| & A^*+B^* \\ A+B & |A|+|B| \end{bmatrix} \geqslant O. \tag{3.2}$$

By (3.2) and Theorem 2.1, we get the desired result. □

COROLLARY 3.2. *Let $A,B \in \mathcal{M}_n$ be normal. Then*

$$\|A+B\| \leqslant \||A|+|B|\|.$$

REMARK 3.1. We highlight that Corollary 3.2 is well-known, as one can see in [3, (1.42)].

THEOREM 3.2. *Let $T \in \mathcal{M}_n$. Then the following assertions are equivalent.*

(i) T *is* (α,β)*-normal.*

(ii) $\begin{bmatrix} \frac{1}{\alpha}|T^*|^2 & |T|^2 \\ |T|^2 & \frac{1}{\alpha}|T|^2 \end{bmatrix} \geqslant O$ *and* $\begin{bmatrix} \beta|T|^2 & |T^*|^2 \\ |T^*|^2 & \beta|T^*|^2 \end{bmatrix} \geqslant O.$

(iii) $\begin{bmatrix} \frac{1}{\alpha^2}|T^*|^2 & |T|^2 \\ |T|^2 & \frac{1}{\alpha^2}|T^*|^2 \end{bmatrix} \geqslant O$ *and* $\begin{bmatrix} \beta^2|T|^2 & |T^*|^2 \\ |T^*|^2 & \beta^2|T|^2 \end{bmatrix} \geqslant O.$

Proof. $(i) \Leftrightarrow (ii)$ Since T is (α,β)-normal, we have

$$\begin{aligned} &|T|^2 \leqslant \frac{1}{\alpha^2}|T^*|^2 \\ &\Leftrightarrow |T|^2\left(\frac{1}{\alpha}|T|^2\right)^{-1}|T|^2 \leqslant \frac{1}{\alpha}|T^*|^2 \\ &\Leftrightarrow \begin{bmatrix} \frac{1}{\alpha}|T^*|^2 & |T|^2 \\ |T|^2 & \frac{1}{\alpha}|T|^2 \end{bmatrix} \geqslant O \quad \text{(by [4, Theorem 1.3.3]).} \end{aligned}$$

Again, since T is (α,β)-normal, we have

$$\begin{aligned} &|T^*|^2 \leqslant \beta^2|T|^2 \\ &\Leftrightarrow |T^*|^2\left(\beta|T^*|^2\right)^{-1}|T^*|^2 \leqslant \beta|T|^2 \\ &\Leftrightarrow \begin{bmatrix} \beta|T|^2 & |T^*|^2 \\ |T^*|^2 & \beta|T^*|^2 \end{bmatrix} \geqslant O \quad \text{(by [4, Theorem 1.3.3]).} \end{aligned}$$

$(i) \Leftrightarrow (iii)$ See [20, Theorem 2.2]. □

We can establish the following theorem by employing the same arguments as in the proof of $(i) \Leftrightarrow (ii)$ in Theorem 3.2, but we present another proof.

THEOREM 3.3. *Let $T \in \mathscr{M}_n$ be (α,β)-normal. Then*

$$\begin{bmatrix} \frac{1}{\sqrt{\alpha}}|T^*| & |T| \\ |T| & \frac{1}{\sqrt{\alpha}}|T| \end{bmatrix} \geqslant O \ \ \textit{and} \ \ \begin{bmatrix} \sqrt{\beta}|T| & |T^*| \\ |T^*| & \sqrt{\beta}|T^*| \end{bmatrix} \geqslant O.$$

Proof. According to the assumption,

$$\langle |T|x,x\rangle \langle |T^*|y,y\rangle \leqslant \frac{1}{\alpha}\langle |T^*|x,x\rangle \langle |T^*|y,y\rangle$$

and

$$\langle |T|x,x\rangle \langle |T^*|y,y\rangle \leqslant \beta \langle |T|x,x\rangle \langle |T|y,y\rangle$$

for any $x,y \in \mathscr{H}$. On the other hand, we know that (see, e.g., [9, p. 216])

$$|\langle Tx,y\rangle|^2 \leqslant \langle |T|x,x\rangle \langle |T^*|y,y\rangle .$$

Consequently,

$$|\langle Tx,y\rangle|^2 \leqslant \frac{1}{\alpha}\langle |T^*|x,x\rangle \langle |T^*|y,y\rangle$$

and

$$|\langle Tx,y\rangle|^2 \leqslant \beta \langle |T|x,x\rangle \langle |T|y,y\rangle .$$

The last two inequalities are equivalent to

$$\begin{bmatrix} \frac{1}{\sqrt{\alpha}}|T^*| & T^* \\ T & \frac{1}{\sqrt{\alpha}}|T^*| \end{bmatrix} \geqslant O \text{ and } \begin{bmatrix} \sqrt{\beta}|T| & T^* \\ T & \sqrt{\beta}|T| \end{bmatrix} \geqslant O, \tag{3.3}$$

thanks to [14, Lemma 1].

Now assume that $T = U|T|$ is the polar decomposition of T. Then

$$\begin{bmatrix} I & O \\ O & U^* \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{\alpha}}|T^*| & T^* \\ T & \frac{1}{\sqrt{\alpha}}|T^*| \end{bmatrix} \begin{bmatrix} I & O \\ O & U \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{\alpha}}|T^*| & T^*U \\ U^*T & \frac{1}{\sqrt{\alpha}}U^*|T^*|U \end{bmatrix} \geqslant O. \tag{3.4}$$

On the other hand,

$$\begin{bmatrix} \sqrt{\beta}|T| & T^* \\ T & \sqrt{\beta}|T| \end{bmatrix} \geqslant O \Leftrightarrow \begin{bmatrix} \sqrt{\beta}|T| & T \\ T^* & \sqrt{\beta}|T| \end{bmatrix} \geqslant O.$$

So,

$$\begin{bmatrix} I & O \\ O & U \end{bmatrix} \begin{bmatrix} \sqrt{\beta}|T| & T \\ T^* & \sqrt{\beta}|T| \end{bmatrix} \begin{bmatrix} I & O \\ O & U^* \end{bmatrix} = \begin{bmatrix} \sqrt{\beta}|T| & TU^* \\ UT^* & \sqrt{\beta}U|T|U^* \end{bmatrix} \geqslant O. \tag{3.5}$$

One can easily check that $U^*T = T^*U = |T|$. Meanwhile, $|T^*| = U|T|U^*$ (see [9, p. 58]), so $U^*|T^*|U = |T|$. Hence, by (3.4) and (3.5), we obtain

$$\begin{bmatrix} \frac{1}{\sqrt{\alpha}}|T^*| & |T| \\ |T| & \frac{1}{\sqrt{\alpha}}|T| \end{bmatrix} \geqslant O \text{ and } \begin{bmatrix} \sqrt{\beta}|T| & |T^*| \\ |T^*| & \sqrt{\beta}|T^*| \end{bmatrix} \geqslant O,$$

as desired. □

The matrices in (3.3) are PPT Therefore, by Corollary 2.2, we have the following eigenvalue inequalities.

COROLLARY 3.3. *Let $T \in \mathscr{M}_n$ be (α,β)-normal. Then*

$$\lambda_j\left(2\sqrt{\alpha}|T| - |T^*|\right) \leqslant \lambda_j\left(|T^*|\right)$$

and

$$\lambda_j\left(\frac{2}{\sqrt{\beta}}|T^*| - |T|\right) \leqslant \lambda_j\left(|T|\right)$$

for $j = 1,2,\ldots,n$.

REMARK 3.2. It is well-known that for any $T \in \mathscr{M}_n$,

$$\|\,|T| - |T^*|\,\| \leqslant \|T\|.$$

From Corollary 3.3, we infer that if T is (α,β)-normal, then

$$\left\| 2\sqrt{\alpha}|T| - |T^*| \right\| \leqslant \|T\|,$$

and

$$\left\| \frac{2}{\sqrt{\beta}}|T^*| - |T| \right\| \leqslant \|T\|.$$

The inequality (1.5) is usually referred to as the operator arithmetic-geometric mean inequality. It is of great interest in the literature to find possible reverses for this inequality. Usually, such reverses are found under additional conditions, as seen in [7, 11]. In the following, we present a reverse of (1.5) for (α,β)-normal matrices.

PROPOSITION 3.1. *Let $T \in \mathscr{M}_n$ be (α,β)-normal. Then*

$$\frac{|T|+|T^*|}{2} \leqslant \min\left\{\frac{1}{\sqrt{\alpha}}, \sqrt{\beta}\right\}(|T|\sharp|T^*|).$$

Proof. Theorem 3.3 implies

$$|T| \leqslant \frac{1}{\sqrt{\alpha}}(|T|\sharp|T^*|) \quad \text{and} \quad |T^*| \leqslant \sqrt{\beta}(|T|\sharp|T^*|) \tag{3.6}$$

due to (1.6). Further,

$$|T^*| \leqslant \frac{1}{\sqrt{\alpha}}(|T|\sharp|T^*|) \quad \text{and} \quad |T| \leqslant \sqrt{\beta}(|T|\sharp|T^*|) \tag{3.7}$$

by utilizing the same approach as in the proof of inequality (2.2). Inequalities (3.6) and (3.7) say that

$$|T| \leqslant \min\left\{\frac{1}{\sqrt{\alpha}}, \sqrt{\beta}\right\}(|T|\sharp|T^*|) \quad \text{and} \quad |T^*| \leqslant \min\left\{\frac{1}{\sqrt{\alpha}}, \sqrt{\beta}\right\}(|T|\sharp|T^*|).$$

Adding the above two inequalities together implies the desired result. □

REMARK 3.3. Inequalities (3.6) and (3.7) can be shown in another way. Since $f(t) = \sqrt{t}$ is operator monotone on $(0,\infty)$, and since $\alpha^2|T|^2 \leqslant |T^*|^2$, we infer that

$$|T| \leqslant \frac{1}{\alpha}|T^*|.$$

This implies

$$|T| \leqslant \frac{1}{\sqrt{\alpha}}(|T|\sharp|T^*|),$$

where we have used the fact that if $A,B,C,D \geqslant O$ are such that $A \leqslant B$ and $C \leqslant D$, then $A\sharp C \leqslant B\sharp D$.

For the following result, we remind the reader of positive linear maps. A linear map $\Phi : \mathscr{M}_n \to \mathscr{M}_n$ is said to be positive if $\Phi(A) \geqslant O$ whenever $A \geqslant O$.

THEOREM 3.4. *Let $T \in \mathscr{M}_n$ be (α,β)-normal and let Φ be a positive linear map. If $\Phi(T) = U|\Phi(T)|$ is the polar decomposition of $\Phi(T)$, then*

$$|\Phi(T)| \leqslant \frac{1}{\sqrt{\alpha}}\left(\Phi(|T^*|)\sharp U^*\Phi(|T^*|)U\right)$$

and

$$|\Phi(T)| \leqslant \sqrt{\beta}\left(\Phi(|T|)\sharp U^*\Phi(|T|)U\right).$$

Proof. First notice that every positive linear map is adjoint-preserving; i.e., $\Phi^*(T) = \Phi(T^*)$ for all T [3, Lemma 2.3.1]. It follows from (3.3) that

$$\begin{bmatrix} \frac{1}{\sqrt{\alpha}}\Phi(|T^*|) & \Phi^*(T) \\ \Phi(T) & \frac{1}{\sqrt{\alpha}}\Phi(|T^*|) \end{bmatrix} \geqslant O \text{ and } \begin{bmatrix} \sqrt{\beta}\Phi(|T|) & \Phi^*(T) \\ \Phi(T) & \sqrt{\beta}\Phi(|T|) \end{bmatrix} \geqslant O$$

thanks to [3, Exercise 3.2.2]. We get the desired result by mimicking the technique of the proof of Theorem 2.1. □

The following result presents an interesting reverse of the well-known inequality $\|T^2\| \leqslant \|T\|^2$, for any T. We recall that a contraction K satisfies $KK^* \leqslant I$, the identity. We also recall that the spectral radius $r(X)$ coincides with the operator norm $\|X\|$ when $X \geqslant O$.

THEOREM 3.5. *Let $T \in \mathscr{M}_n$ be (α,β)-normal. Then*

$$\|T\|^2 \leqslant \frac{1}{\alpha}\left\|T^2\right\| \text{ and } \|T\|^2 \leqslant \beta\left\|T^2\right\|.$$

Indeed,

$$\|T\|^2 \leqslant \min\left\{\frac{1}{\alpha},\beta\right\}\left\|T^2\right\|.$$

Proof. Ando [2] proved that $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \geqslant O$ if and only if there exists a contraction K such that $X = A^{\frac{1}{2}}KB^{\frac{1}{2}}$. It has been revealed in the proof of Theorem 3.3 that $\begin{bmatrix} \frac{1}{\sqrt{\alpha}}|T^*| & |T| \\ |T| & \frac{1}{\sqrt{\alpha}}|T| \end{bmatrix} \geqslant O$. Therefore, there exists a contraction K such that $|T| = \frac{1}{\sqrt{\alpha}}|T^*|^{\frac{1}{2}}K|T|^{\frac{1}{2}} \geqslant O$. So, we have

$$\begin{aligned} \|\,|T|\,\| &= \|T\| \\ &= \frac{1}{\sqrt{\alpha}}\left\||T^*|^{\frac{1}{2}}K|T|^{\frac{1}{2}}\right\| \\ &= \frac{1}{\sqrt{\alpha}}r\left(|T^*|^{\frac{1}{2}}K|T|^{\frac{1}{2}}\right) \quad (\text{since } r(X) = \|X\| \text{ for positive } X) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{\alpha}} r\left(K|T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}}\right) \quad (\text{since } r(XY) = r(YX)) \\
&\leqslant \frac{1}{\sqrt{\alpha}} \left\|K|T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}}\right\| \quad (\text{since } r(X) \leqslant \|X\|) \\
&\leqslant \frac{1}{\sqrt{\alpha}} \|K\| \left\||T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}}\right\| \\
&\qquad\qquad (\text{by the submultiplicative property of usual operator norm}) \\
&\leqslant \frac{1}{\sqrt{\alpha}} \left\||T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}}\right\| \quad (\text{since } K \text{ is contraction}) \\
&\leqslant \frac{1}{\sqrt{\alpha}} \||T|\,|T^*|\|^{\frac{1}{2}} \quad (\text{by [3, Theorem IX.2.1]}) \\
&= \frac{1}{\sqrt{\alpha}} \left\|T^2\right\|^{\frac{1}{2}}.
\end{aligned}
$$

The second inequality comes from $\begin{bmatrix} \sqrt{\beta}|T| & |T^*| \\ |T^*| & \sqrt{\beta}|T^*| \end{bmatrix} \geqslant O$ and the same method as above. This completes the proof. □

REMARK 3.4. We will give another method to prove Theorem 3.5. The operator inequality

$$|T| \leqslant \frac{1}{\sqrt{\alpha}} (|T| \sharp |T^*|)$$

implies the following norm inequality

$$\|T\| \leqslant \frac{1}{\sqrt{\alpha}} \||T| \sharp |T^*|\|.$$

But, for any positive operators A, B, we know that

$$\|A \sharp B\| \leqslant \left\|A^{\frac{1}{2}} B^{\frac{1}{2}}\right\|.$$

That is,

$$\|T\| \leqslant \frac{1}{\sqrt{\alpha}} \left\||T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}}\right\| = \frac{1}{\sqrt{\alpha}} \left\|T^2\right\|^{\frac{1}{2}}.$$

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