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PERTURBATION OF CONTINUOUS FRAMES ON QUATERNIONIC HILBERT SPACES

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Abstract. In this note, we explore the theory of continuous frame perturbations in the quaternion settings by inspiration of the well-developed theory of perturbing discrete frames in complex Hilbert spaces. We examine the perturbation of continuous frames of rank n, including Bessel and Riesz families, within right quaternionic Hilbert spaces. We also investigate some results on the bounds of perturbing continuous frames under some conditions.

1. Introduction

Frame is a spanning set of vectors introduced by Duffin and Schaeffer in 1952 in the study of non-harmonic Fourier series [12]. It has been widely studied since 1986, following a landmark development by Daubechies [10]. The study of frames has attracted interest in recent years due to their applications in various areas of Engineering, Applied Mathematics, and Mathematical Physics. Many applications of frames have emerged in recent years, including Internet coding [25], sampling [14], filter bank theory [2], system modeling [13], digital signal processing [6, 16] and many more.

Perturbation theory plays a significant role in several areas of mathematics. Frame perturbations were first explicitly introduced by Chris Heil in his Ph.D. thesis [17], and then widely studied by other authors [3,5,7,9,15]. As far as we know, these perturbation results have not been extended even to complex continuous frames. In this paper, we investigate certain perturbations of rank n continuous frames in a right quaternionic Hilbert space, which was introduced in [18], following the arguments given in [5, 8], where frame perturbations were studied for complex discrete frames.

Hilbert spaces may be rigorously formulated over various mathematical fields, including the field of real numbers ($\mathbb{R}$), the field of complex numbers ($\mathbb{C}$), and exclusively over the field of quaternions ($\mathbb{H}$) [1]. The fields $\mathbb{R}$ and $\mathbb{C}$ exhibit the properties of associativity and commutativity. Consequently, the theory of functional analysis is systematically developed and well-established when applied to both real and complex Hilbert spaces. However, the quaternions constitute a non-commutative associative algebra, thereby imposing significant limitations on mathematicians seeking to formulate a comprehensive theory of functional analysis within quaternionic Hilbert spaces.

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Moreover, the inherent non-commutativity gives rise to distinct categories of Hilbert spaces on quaternions, denoted as right quaternion Hilbert space ($V_{\mathbb{H}}^{R}$) and left quaternion Hilbert space ($V_{\mathbb{H}}^{L}$).

The theory of quaternionic operators finds applications in diverse fields such as quantum mechanics and quaternionic schur analysis [4]. Notably the difference between complex and quaternionic operator theory lies in the definition of S-spectrum. In quaternionic operator theory, the conventional notions of resolvent operator and spectrum necessitate replacement with the two S-resolvent operators an the S-spectrum due to the inherent non-commutatitivity of quaternionic setting.

This article is organized as follows. In Section 2, we provide some basic notations and preliminary results about quaternions and frames necessary for the development of the results presented in this article. In Section 3, we present the main results of this article, namely, perturbations of rank n continuous frames, rank n continuous Bessel families, and rank n continuous Riesz families in right quaternionic Hilbert spaces. These perturbations follow their discrete counterparts studied in complex Hilbert spaces.

2. Mathematical preliminaries

We recall few facts about quaternions, quaternionic Hilbert spaces and quaternionic functional calculus which may not be very familiar to the reader. For quaternions and quaternionic Hilbert spaces we refer the reader to [1].

2.1. Quaternions

Let $\mathbb{H}$ denote the field of quaternions. Its elements are of the form $q = x_0 + x_1 i + x_2 j + x_3 k$, where x_0, x_1, x_2 and x_3 are real numbers, and i, j, k are imaginary units such that $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$ and $ki = -ik = j$. The quaternionic conjugate of q is defined to be $\overline{q} = x_0 - x_1 i - x_2 j - x_3 k$. Quaternions do not commute in general. However q and $\overline{q}$ commute, and quaternions commute with real numbers. $|q|^2 = q\overline{q} = \overline{q}q$ and $\overline{qp} = \overline{p}\,\overline{q}$. Quaternion can also be represented by 2×2 complex matrices.

$$q = x_0 \sigma_0 + i\underline{x} \cdot \underline{\sigma} \tag{2.1}$$

with $x_0 \in \mathbb{R}$, $\underline{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$, $\sigma_0 = \mathbb{I}_2$, the 2×2 identity matrix, and $\underline{\sigma} = (\sigma_1, -\sigma_2, \sigma_3)$, where the σ_ℓ, $\ell = 1, 2, 3$ are the usual Pauli matrices. The quaternionic imaginary units are identified as, $i = \sqrt{-1}\sigma_1$, $j = -\sqrt{-1}\sigma_2$, $k = \sqrt{-1}\sigma_3$. Thus

$$q = \begin{pmatrix} x_0 + ix_3 & -x_2 + ix_1 \\ x_2 + ix_1 & x_0 - ix_3 \end{pmatrix}. \tag{2.2}$$

and $\overline{q} = q^\dagger$ (matrix adjoint). Introducing the polar coordinates,

$$\begin{aligned} x_0 &= r\cos\vartheta \\ x_1 &= r\sin\vartheta\sin\varphi\cos\Psi \\ x_2 &= r\sin\vartheta\sin\varphi\sin\Psi \\ x_3 &= r\sin\vartheta\cos\varphi, \end{aligned}$$

where $(r,\varphi,\vartheta,\Psi)\in[0,\infty)\times[0,\pi]\times[0,2\pi)^2$, we may write

$$q=A(r)e^{i\vartheta\sigma(\tilde{n})}, \tag{2.3}$$

where

$$A(r)=r\sigma_0 \text{ and } \sigma(\tilde{n})=\begin{pmatrix}\cos\varphi & \sin\varphi e^{i\Psi}\\ \sin\varphi e^{-i\Psi} & -\cos\varphi\end{pmatrix}. \tag{2.4}$$

The matrices $A(r)$ and $\sigma(\tilde{n})$ satisfy the conditions,

$$A(r)=A(r)^{\dagger},\ \ \sigma(\tilde{n})^2=\sigma_0, \tag{2.5}$$

with

$$\sigma(\tilde{n})^{\dagger}=\sigma(\tilde{n}),\ \ [A(r),\sigma(\tilde{n})]=0. \tag{2.6}$$

Note that a real norm on $\mathbb{H}$ is defined by

$$|q|^2:=\overline{q}q=r^2\sigma_0=(x_0^2+x_1^2+x_2^2+x_3^2)\sigma_0.$$

A typical measure on $\mathbb{H}$ may take the form

$$d\mu(r,\vartheta,\varphi,\Psi)=d\tau(r)d\vartheta d\Omega(\varphi,\Psi), \tag{2.7}$$

with $d\Omega(\varphi,\Psi)=\dfrac{1}{4\pi}\sin\varphi d\varphi d\Psi$ and $d\mu(r,\vartheta,\varphi,\Psi)=\dfrac{r}{2\pi}e^{-r^2}\sin\varphi drd\vartheta d\varphi d\Psi$. For details, we refer the reader to [20, 22, 26, 27]. So that $d\mu(q)=d\mu(r,\vartheta,\varphi,\Psi)=\dfrac{r}{2\pi}e^{-r^2}\sin\varphi drd\vartheta d\varphi d\Psi=\dfrac{1}{4\pi}e^{-r^2}\sin\varphi d(r^2)d\vartheta d\varphi d\Psi$. One may obtain the integral

$$\int_{\mathbb{H}}d\mu(q)=\int_{\mathbb{H}}d\mu(r,\vartheta,\varphi,\Psi)=\frac{1}{4\pi}\int_0^{\infty}\int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi}e^{-r^2}\sin\varphi d(r^2)d\vartheta d\varphi d\Psi=2\pi. \tag{2.8}$$

2.2. Right quaternionic Hilbert space

Let $V_{\mathbb{H}}^R$ be a linear vector space under right multiplication by quaternionic scalars (again $\mathbb{H}$ standing for the field of quaternions). For $\phi,\psi,\omega\in V_{\mathbb{H}}^R$ and $q\in\mathbb{H}$, the inner product

$$\langle\cdot\mid\cdot\rangle:V_{\mathbb{H}}^R\times V_{\mathbb{H}}^R\longrightarrow\mathbb{H}$$

satisfies the following properties

(i) $\overline{\langle\phi\mid\psi\rangle}=\langle\psi\mid\phi\rangle$

(ii) $\|\phi\|^2=\langle\phi\mid\phi\rangle>0$ unless $\phi=0$, a real norm

(iii) $\langle\phi\mid\psi+\omega\rangle=\langle\phi\mid\psi\rangle+\langle\phi\mid\omega\rangle$

(iv) $\langle\phi\mid\psi q\rangle=\langle\phi\mid\psi\rangle q$

(v) $\langle\phi q\mid\psi\rangle=\overline{q}\langle\phi\mid\psi\rangle$

where $\overline{q}$ stands for the quaternionic conjugate. We assume that the space $V_{\mathbb{H}}^{R}$ is complete under the norm given above. This, together with the inner product $\langle\cdot\mid\cdot\rangle$, defines a right quaternionic Hilbert space.We shall assume this space to be separable. Quaternionic Hilbert spaces share most of the standard properties of complex Hilbert spaces. In particular, the Cauchy-Schwarz inequality and the Riesz representation theorem for their duals hold in quaternionic Hilbert spaces. Consequently, the Dirac bra-ket notation can be adapted to quaternionic Hilbert spaces:

$$\mid\phi q\rangle=\mid\phi\rangle q, \qquad \langle\phi q\mid=\overline{q}\langle\phi\mid,$$

for a right quaternionic Hilbert space, with $|\phi\rangle$ denoting the vector ϕ and $\langle\phi|$ its dual vector. Let A be an operator on a right quaternionic Hilbert space.The scalar multiple of A should be written as qA and the action must take the form [23, 26].

$$(qA)\mid\phi\rangle=(A\mid\phi\rangle)\overline{q}. \tag{2.9}$$

The adjoint $A^{\dagger}$ of A is defined as

$$\langle\psi\mid A\phi\rangle=\langle A^{\dagger}\psi\mid\phi\rangle; \quad \text{for all} \quad \phi,\psi\in V_{H}^{R}. \tag{2.10}$$

An operator A is said to be self-adjoint if $A=A^{\dagger}$. If $\phi\in V_{H}^{R}\setminus\{0\}$, then $|\phi\rangle\langle\phi|$ is a rank one projection operator. For operators A,B, by convention, we have

$$|A\phi\rangle\langle B\phi|=A|\phi\rangle\langle\phi|B^{\dagger}. \tag{2.11}$$

Let $\mathcal{D}(A)$ denote the domain of A. A is said to be right linear if

$$A(\phi q+\psi p)=(A\phi)q+(A\psi)p; \quad \forall\phi,\psi\in\mathcal{D}(A),\ q,p\in\mathbb{H}.$$

The set of all right linear operators will be denoted by $\mathcal{L}(V_{\mathbb{H}}^{R})$. We call an operator $A\in\mathcal{L}(V_{\mathbb{H}}^{R})$ bounded if

$$\|A\|=\sup_{\|\phi\|=1}\|A\phi\|<\infty.$$

or equivalently, there exists $K\geqslant 0$ such that $\|A\phi\|\leqslant K\|\phi\|$ for $\phi\in\mathcal{D}(A)$. The set of all bounded right linear operators will be denoted by $\mathcal{B}(V_{\mathbb{H}}^{R})$.

PROPOSITION 2.1. [21] Let $A\in\mathcal{B}(V_{\mathbb{H}}^{R})$ and suppose that $\|A\|<1$. Then $(I_{V_{\mathbb{H}}^{R}}-A)^{-1}$ exists.

DEFINITION 2.2. [19] (*Discrete Frames*) A countable family of elements $\{f_k\}_{k=1}^{m}$ in $V_{\mathbb{H}}^{R}$ is a frame for $V_{\mathbb{H}}^{R}$ if there exist constants $A,B>0$ such that

$$A\|f\|^{2}\leqslant\sum_{k=1}^{m}|\langle f|f_k\rangle|^{2}\leqslant B\|f\|^{2}, \tag{2.12}$$

for all $f\in V_{\mathbb{H}}^{R}$.

Let $\{f_k\}_{k=1}^m$ be a frame in $V_{\mathbb{H}}^R$ and define a linear mapping $T:\mathbb{H}^m \longrightarrow V_{\mathbb{H}}^R$, by

$$T\{c_k\}_{k=1}^m = \sum_{k=1}^m f_k c_k,\ c_k \in \mathbb{H}. \tag{2.13}$$

T is usually called the *pre-frame operator*, or the *synthesis operator*. The adjoint operator $T^\dagger : V_{\mathbb{H}}^R \longrightarrow \mathbb{H}^m$, given by

$$T^\dagger f = \{\langle f|f_k\rangle\}_{k=1}^m \tag{2.14}$$

is called the *analysis operator*. By composing T with its adjoint we obtain the *frame operator* $S: V_{\mathbb{H}}^R \longrightarrow V_{\mathbb{H}}^R$, by

$$Sf = TT^\dagger f = \sum_{k=1}^m f_k \langle f|f_k\rangle. \tag{2.15}$$

THEOREM 2.3. [18] *For each $q\in\mathbb{H}$, let the set $\{\eta_q^i : i=1,2,\cdots,n\}$ be linearly independent in $V_{\mathbb{H}}^R$. We define an operator A by using Dirac bra-ket notation,*

$$\sum_{i=1}^n \int_{\mathbb{H}} |\eta_q^i\rangle\langle\eta_q^i|\, d\mu(q) = A \tag{2.16}$$

and we always assume that $A\in GL(V_{\mathbb{H}}^R)$, where

$$GL(V_{\mathbb{H}}^R) = \left\{A : V_{\mathbb{H}}^R \longrightarrow V_{\mathbb{H}}^R : A \text{ bounded and } A^{-1} \text{ bounded}\right\}.$$

Then the operator A is positive and self adjoint.

DEFINITION 2.4. [18] (*Continuous frame*) A set of vectors $\{\eta_q^i \in V_{\mathbb{H}}^R \mid i = 1,2,\cdots,n,\ q\in\mathbb{H}\}$ constitute a rank n right quaternionic continuous frame, denoted by $F(\eta_q^i, A, n)$, if

(i) for each $q\in\mathbb{H}$, the set of vectors $\{\eta_q^i \in V_{\mathbb{H}}^R \mid i=1,2,\cdots,n\}$ is a linearly independent set.

(ii) there exists a positive operator $A\in GL(V_{\mathbb{H}}^R)$ such that

$$\sum_{i=1}^n \int_{\mathbb{H}} |\eta_q^i\rangle\langle\eta_q^i| d\mu(q) = A.$$

THEOREM 2.5. [18] *For $\phi\in V_{\mathbb{H}}^R$, we have*

$$m(A)\|\phi\|^2 \leqslant \sum_{i=1}^n \int_{\mathbb{H}} |\langle\eta_q^i|\phi\rangle|^2 d\mu(q) \leqslant M(A)\|\phi\|^2, \tag{2.17}$$

where $M(A) = \sup_{\|\phi\|=1} \langle\phi|A\phi\rangle$ and $m(A) = \inf_{\|\phi\|=1} \langle\phi|A\phi\rangle$.

The inequality (2.17) presents the frame condition for the set of vectors

$$\{\eta_q^i \in V_{\mathbb{H}}^R \mid i=1,2,\cdots,n,\ q\in\mathbb{H}\}$$

with frame bounds $m(A)$ and $M(A)$.

THEOREM 2.6. [18] (Frame decomposition) *Let* $\{\eta_q^i \in V_{\mathbb{H}}^R \mid i=1,2,\cdots,n,\ q\in\mathbb{H}\}$ *be a rank n continuous frame with bounds $m(A)$ and $M(A)$. Then for any $\phi\in V_{\mathbb{H}}^R$, we have*

$$\begin{aligned}\phi &= \sum_{i=1}^{n}\int_{\mathbb{H}} \eta_q^i \langle \phi | A^{-1}\eta_q^i\rangle \, d\mu(q)\\ &= \sum_{i=1}^{n}\int_{\mathbb{H}} A^{-1}\eta_q^i \langle \phi | \eta_q^i\rangle \, d\mu(q),.\end{aligned}$$

where A is the frame operator of the frame $\{\eta_q^i \in V_{\mathbb{H}}^R \mid i=1,2,\cdots,n,\ q\in\mathbb{H}\}$.

THEOREM 2.7. [18] *Let* $\{\eta_q^i \in V_{\mathbb{H}}^R \mid i=1,2,\cdots,n,\ q\in\mathbb{H}\}$ *be a rank n continuous frame with bounds $m(A)$ and $M(A)$. Then* $\{A^{-1}\eta_q^i \in V_{\mathbb{H}}^R \mid i=1,2,\cdots,n,\ q\in\mathbb{H}\}$ *is a rank n continuous frame with bounds* $\dfrac{1}{M(A)}$ *and* $\dfrac{1}{m(A)}$.

DEFINITION 2.8. We call a family $\{\xi_q^i \in V_{\mathbb{H}}^R \mid i=1,2,\cdots,n,\ q\in\mathbb{H}\}$ of elements in $V_{\mathbb{H}}^R$ a rank n continuous Bessel family if there exists $D>0$ such that

$$\sum_{i=1}^{n}\int_{\mathbb{H}} \left|\langle \xi_q^i | \phi\rangle\right|^2 d\mu(q) \leqslant D\,\|\phi\|^2, \tag{2.18}$$

for all $\phi\in V_{\mathbb{H}}^R$.

A rank n continuous Bessel family $\{\xi_q^i \in V_{\mathbb{H}}^R \mid i=1,2,\cdots,n,\ q\in\mathbb{H}\}$ will be called a rank n continuous frame if there exists $C>0$ such that

$$C\,\|\phi\|^2 \leqslant \sum_{i=1}^{n}\int_{\mathbb{H}} \left|\langle \xi_q^i | \phi\rangle\right|^2 d\mu(q), \tag{2.19}$$

for all $\phi\in V_{\mathbb{H}}^R$. The following result is an adaptation of the discrete case considered in [24].

THEOREM 2.9. *Let* $\{\xi_q^i \in V_{\mathbb{H}}^R \mid i=1,2,\cdots,n,\ q\in\mathbb{H}\}$ *be a rank n continuous Bessel family of $V_{\mathbb{H}}^R$ with bound D. Then the mapping T from $\mathbb{H}^n$ to $V_{\mathbb{H}}^R$ defined by*

$$T(\{c_i\}_{i=1}^n) := \sum_{i=1}^{n}\int_{\mathbb{H}} \zeta_q^i c_i \, d\mu(q) \tag{2.20}$$

is a right linear and bounded operator with $\|T\| \leqslant \sqrt{D}$.

Proof. It is not difficult to see that T is right linear. Now for $\phi \in V_{\mathbb{H}}^{R}$,

$$\begin{aligned}
\|T\{c_i\}\| &= \sup_{\|\phi\|=1} |\langle T\{c_i\}|\phi\rangle| \\
&= \sup_{\|\phi\|=1} \left|\left\langle \sum_{i=1}^{n}\int_{\mathbb{H}} \zeta_q^i c_i d\mu(q)|\phi \right\rangle\right| \\
&= \sup_{\|\phi\|=1} \left|\sum_{i=1}^{n}\int_{\mathbb{H}} \overline{c_i}\left\langle \zeta_q^i|\phi\right\rangle d\mu(q)\right| \\
&\leqslant \sup_{\|\phi\|=1} \sum_{i=1}^{n}\int_{\mathbb{H}} \left|\overline{c_i}\left\langle \zeta_q^i|\phi\right\rangle\right| d\mu(q) \\
&\leqslant \sup_{\|\phi\|=1} \left(\sum_{i=1}^{n}\int_{\mathbb{H}} \left|\left\langle \zeta_q^i|\phi\right\rangle\right|^2 d\mu(q)\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n}|c_i|^2\right)^{\frac{1}{2}} \\
&\leqslant \sup_{\|\phi\|=1} \left(B\|\phi\|^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n}|c_i|^2\right)^{\frac{1}{2}} \\
&= \sqrt{B}\left(\sum_{i=1}^{n}|c_i|^2\right)^{\frac{1}{2}}.
\end{aligned}$$

Hence $\|T\| \leqslant \sqrt{B}$. □

By composing the operator T in (2.20) with its adjoint operator $T^{\dagger}$ we get the frame operator.

PROPOSITION 2.10. Let $A = \sum_{i=1}^{n}\int_{\mathbb{H}} |\eta_q^i\rangle\langle\zeta_q^i| d\mu(q)$. Then

$$A^{\dagger} = \sum_{i=1}^{n}\int_{\mathbb{H}} |\zeta_q^i\rangle\langle\eta_q^i| d\mu(q).$$

Proof. For $\psi, \phi \in V_{\mathbb{H}}^{R}$, $A\psi = \sum_{i=1}^{n}\int_{\mathbb{H}} |\eta_q^i\rangle\langle\zeta_q^i|\psi\rangle d\mu(q)$, we have

$$\langle\phi|A\psi\rangle = \sum_{i=1}^{n}\int_{\mathbb{H}} \langle\phi|\eta_q^i\rangle\langle\zeta_q^i|\psi\rangle d\mu(q). \tag{2.21}$$

If we take $T = \sum_{i=1}^{n}\int_{\mathbb{H}} |\zeta_q^i\rangle\langle\eta_q^i| d\mu(q)$ then $T\phi = \sum_{i=1}^{n}\int_{\mathbb{H}} |\zeta_q^i\rangle\langle\eta_q^i|\phi\rangle d\mu(q)$. Hence,

$$\langle\psi|T\phi\rangle = \sum_{i=1}^{n}\int_{\mathbb{H}} \langle\psi|\zeta_q^i\rangle\langle\eta_q^i|\phi\rangle d\mu(q).$$

Now

$$\begin{aligned}\langle T\phi|\psi\rangle &= \overline{\langle\psi|T\phi\rangle}\\ &= \sum_{i=1}^{n}\int_{\mathbb{H}}\overline{\langle\psi|\zeta_q^i\rangle\langle\eta_q^i|\phi\rangle}d\mu(q).\\ &= \sum_{i=1}^{n}\int_{\mathbb{H}}\overline{\langle\eta_q^i|\phi\rangle}\;\overline{\langle\psi|\zeta_q^i\rangle}d\mu(q).\\ &= \sum_{i=1}^{n}\int_{\mathbb{H}}\langle\phi|\eta_q^i\rangle\langle\zeta_q^i|\psi\rangle d\mu(q)\\ &= \langle\phi|A\psi\rangle.\end{aligned}$$

Therefore, $\langle\phi|A\psi\rangle = \langle T\phi|\psi\rangle$. That is, $T = A^\dagger$. □

3. Frame perturbation

In this section, we present perturbations of rank n continuous frames in $V_{\mathbb{H}}^R$ following the frame perturbation theory presented for complex discrete frames in [5, 8].

THEOREM 3.1. *Let $\{\eta_q^i \in V_{\mathbb{H}}^R \mid i = 1,2,\cdots,n,\ q\in\mathbb{H}\}$ be a rank n right quaternionic continuous frame with bounds $m(A)$ and $M(A)$ and frame operator A. Then any family $\{\zeta_q^i \in V_{\mathbb{H}}^R \mid i = 1,2,\cdots,n,\ q\in\mathbb{H}\}$ satisfying*

$$\kappa := \sum_{i=1}^{n}\int_{\mathbb{H}}\left\|\eta_q^i - \zeta_q^i\right\|^2 d\mu(q) < m(A) \tag{3.1}$$

is a rank n continuous frame for $V_{\mathbb{H}}^R$ with bounds $m(A)\left(1-\sqrt{\frac{\kappa}{m(A)}}\right)^2$ and

$$M(A)\left(1+\sqrt{\frac{\kappa}{M(A)}}\right)^2.$$

Proof. Suppose that $\{\eta_q^i \in V_{\mathbb{H}}^R \mid i = 1,2,\cdots,n,\ q\in\mathbb{H}\}$ is a rank n right quaternionic continuous frame with bounds $m(A)$ and $M(A)$. Then

$$m(A)\|\phi\|^2 \leqslant \sum_{i=1}^{n}\int_{\mathbb{H}}|\langle\eta_q^i|\phi\rangle|^2 d\mu(q) \leqslant M(A)\|\phi\|^2.$$

From 3.1, $\{\zeta_q^i \in V_{\mathbb{H}}^R \mid i = 1,2,\cdots,n,\ q\in\mathbb{H}\}$ is a continuous Bessel family in $V_{\mathbb{H}}^R$. Thus, we can define an operator $\mathfrak{U} : V_{\mathbb{H}}^R \longrightarrow V_{\mathbb{H}}^R$ by

$$\mathfrak{U}\phi = \sum_{i=1}^{n}\int_{\mathbb{H}}\zeta_q^i\langle\phi|A^{-1}\eta_q^i\rangle d\mu(q). \tag{3.2}$$

The operator $\mathfrak{U}$ is bounded. To see it, let $\phi \in V_{\mathbb{H}}^{R}$,

$$\begin{aligned}
\|\mathfrak{U}\phi\|^2 &= \langle \mathfrak{U}\phi | \mathfrak{U}\phi \rangle \\
&= \left\langle \sum_{i=1}^{n} \int_{\mathbb{H}} \zeta_q^i \langle \phi | A^{-1} \eta_q^i \rangle d\mu(q) | \sum_{j=1}^{n} \int_{\mathbb{H}} \zeta_p^j \langle \phi | A^{-1} \eta_p^j \rangle d\mu(p) \right\rangle \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \int_{\mathbb{H}} \int_{\mathbb{H}} \overline{\langle \phi | A^{-1} \eta_q^i \rangle} \langle \zeta_q^i | \zeta_p^j \rangle \langle \phi | A^{-1} \eta_p^j \rangle d\mu(q) d\mu(p) \\
&\leqslant \left| \sum_{i=1}^{n} \sum_{j=1}^{n} \int_{\mathbb{H}} \int_{\mathbb{H}} \overline{\langle \phi | A^{-1} \eta_q^i \rangle} \langle \zeta_q^i | \zeta_p^j \rangle \langle \phi | A^{-1} \eta_p^j \rangle d\mu(q) d\mu(p) \right| \\
&\leqslant \sum_{i=1}^{n} \sum_{j=1}^{n} \int_{\mathbb{H}} \int_{\mathbb{H}} \left| \overline{\langle \phi | A^{-1} \eta_q^i \rangle} \right| \left| \langle \zeta_q^i | \zeta_p^j \rangle \right| \left| \langle \phi | A^{-1} \eta_p^j \rangle \right| d\mu(q) d\mu(p) \\
&\leqslant \sum_{i=1}^{n} \sum_{j=1}^{n} \int_{\mathbb{H}} \int_{\mathbb{H}} \left| \overline{\langle \phi | A^{-1} \eta_q^i \rangle} \right| \|\zeta_q^i\| \, \|\zeta_p^j\| \left| \langle \phi | A^{-1} \eta_p^j \rangle \right| d\mu(q) d\mu(p) \\
&\leqslant \sum_{i=1}^{n} \sum_{j=1}^{n} \int_{\mathbb{H}} \int_{\mathbb{H}} \left| \overline{\langle \phi | A^{-1} \eta_q^i \rangle} \right| \alpha \left| \langle \phi | A^{-1} \eta_p^j \rangle \right| d\mu(q) d\mu(p), \\
&\qquad \text{where } \alpha = (\max_i \sup_{q \in \mathbb{H}} \|\zeta_q^i\|)(\max_j \sup_{p \in \mathbb{H}} \|\zeta_p^j\|) \\
&= \alpha \sum_{i=1}^{n} \sum_{j=1}^{n} \int_{\mathbb{H}} \int_{\mathbb{H}} \left| \overline{\langle \phi | A^{-1} \eta_q^i \rangle} \right| \left| \langle \phi | A^{-1} \eta_p^j \rangle \right| d\mu(q) d\mu(p) \\
&\leqslant \frac{\alpha}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \int_{\mathbb{H}} \int_{\mathbb{H}} \left(\left| \overline{\langle \phi | A^{-1} \eta_q^i \rangle} \right|^2 + \left| \langle \phi | A^{-1} \eta_p^j \rangle \right|^2 \right) d\mu(q) d\mu(p) \\
&= \frac{\alpha}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \int_{\mathbb{H}} \int_{\mathbb{H}} \left| \overline{\langle \phi | A^{-1} \eta_q^i \rangle} \right|^2 d\mu(q) d\mu(p) \\
&\quad + \frac{\alpha}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \int_{\mathbb{H}} \int_{\mathbb{H}} \left| \langle \phi | A^{-1} \eta_p^j \rangle \right|^2 d\mu(q) d\mu(p) \\
&= \frac{n\alpha}{2} \sum_{i=1}^{n} \int_{\mathbb{H}} \int_{\mathbb{H}} \left| \langle \phi | A^{-1} \eta_q^i \rangle \right|^2 d\mu(q) d\mu(p) \\
&\quad + \frac{n\alpha}{2} \sum_{j=1}^{n} \int_{\mathbb{H}} \int_{\mathbb{H}} \left| \langle \phi | A^{-1} \eta_p^j \rangle \right|^2 d\mu(q) d\mu(p) \\
&= \frac{2\pi n\alpha}{2} \sum_{i=1}^{n} \int_{\mathbb{H}} \left| \langle \phi | A^{-1} \eta_q^i \rangle \right|^2 d\mu(q) \\
&\quad + \frac{2\pi n\alpha}{2} \sum_{j=1}^{n} \int_{\mathbb{H}} \left| \langle \phi | A^{-1} \eta_p^j \rangle \right|^2 d\mu(p), \quad \text{by 2.8} \\
&\leqslant \pi n\alpha \frac{1}{m(A)} \|\phi\|^2 + \pi n\alpha \frac{1}{m'(A)} \|\phi\|^2, \quad \text{by theorem 2.7} \\
&= \pi n\alpha \left(\frac{1}{m(A)} + \frac{1}{m'(A)} \right) \|\phi\|^2
\end{aligned}$$

It follows that there exists $K>0$ such that $\|\mathfrak{U}\phi\| \leqslant K\|\phi\|$, where

$$K=\sqrt{\pi n\alpha\left(\frac{1}{m(A)}+\frac{1}{m'(A)}\right)}.$$

Hence $\mathfrak{U}$ is bounded. Now, from Theorem 2.6,

$$\begin{aligned}
\|\phi-\mathfrak{U}\phi\|^2 &= \left\|\sum_{i=1}^{n}\int_{\mathbb{H}}\eta_q^i\langle\phi|A^{-1}\eta_q^i\rangle\,d\mu(q)-\sum_{i=1}^{n}\int_{\mathbb{H}}\zeta_q^i\langle\phi|A^{-1}\eta_q^i\rangle\,d\mu(q)\right\|^2\\
&=\left\|\sum_{i=1}^{n}\int_{\mathbb{H}}(\eta_q^i-\zeta_q^i)\langle\phi|A^{-1}\eta_q^i\rangle\,d\mu(q)\right\|^2\\
&\leqslant\sum_{i=1}^{n}\int_{\mathbb{H}}\left|\langle\phi|A^{-1}\eta_q^i\rangle\right|^2\left\|\eta_q^i-\zeta_q^i\right\|^2 d\mu(q)\\
&\leqslant\sum_{i=1}^{n}\int_{\mathbb{H}}\left\|\eta_q^i-\zeta_q^i\right\|^2 d\mu(q)\sum_{i=1}^{n}\int_{\mathbb{H}}\left|\langle\phi|A^{-1}\eta_q^i\rangle\right|^2 d\mu(q)\\
&\leqslant\kappa\frac{1}{m(A)}\|\phi\|^2.
\end{aligned}$$

That is,

$$\|\phi-\mathfrak{U}\phi\|^2=\left\|\mathcal{I}_{V_{\mathbb{H}}^R}\phi-\mathfrak{U}\phi\right\|^2=\left\|(\mathcal{I}_{V_{\mathbb{H}}^R}-\mathfrak{U})\phi\right\|^2\leqslant\frac{\kappa}{m(A)}\|\phi\|^2.$$

Therefore, $\left\|(\mathcal{I}_{V_{\mathbb{H}}^R}-\mathfrak{U})\phi\right\|\leqslant\sqrt{\frac{\kappa}{m(A)}}\|\phi\|$. It follows that $\left\|\mathcal{I}_{V_{\mathbb{H}}^R}-\mathfrak{U}\right\|\leqslant\sqrt{\frac{\kappa}{m(A)}}$. Thus $\left\|\mathcal{I}_{V_{\mathbb{H}}^R}-\mathfrak{U}\right\|\leqslant\sqrt{\frac{\kappa}{m(A)}}<1$ and $\mathfrak{U}$ is invertible. Also we have

$$\left|\|\mathfrak{U}\|-\left\|\mathcal{I}_{V_{\mathbb{H}}^R}\right\|\right|\leqslant\left\|\mathcal{I}_{V_{\mathbb{H}}^R}-\mathfrak{U}\right\|\leqslant\sqrt{\frac{\kappa}{m(A)}}.$$

Hence, $\|\mathfrak{U}\|\leqslant 1+\sqrt{\frac{\kappa}{m(A)}}$ and $\left\|\mathfrak{U}^{-1}\right\|\leqslant\frac{1}{1-\sqrt{\frac{\kappa}{m(A)}}}$.

For $\phi\in V_{\mathbb{H}}^R$,

$$\phi=\mathfrak{U}\mathfrak{U}^{-1}\phi=\sum_{i=1}^{n}\int_{\mathbb{H}}\zeta_q^i\langle\mathfrak{U}^{-1}\phi|A^{-1}\eta_q^i\rangle\,d\mu(q).$$

Therefore

$$\begin{aligned}
\|\phi\|^4 &= \left|\left\langle \sum_{i=1}^{n}\int_{\mathbb{H}} \zeta_q^i \langle \mathfrak{U}^{-1}\phi|A^{-1}\eta_q^i\rangle\, d\mu(q)|\phi\right\rangle\right|^2 \\
&= \left|\sum_{i=1}^{n}\int_{\mathbb{H}} \overline{\langle \mathfrak{U}^{-1}\phi|A^{-1}\eta_q^i\rangle}\,\langle \zeta_q^i|\phi\rangle\, d\mu(q)\right|^2 \\
&\leqslant \sum_{i=1}^{n}\int_{\mathbb{H}} \left|\overline{\langle \mathfrak{U}^{-1}\phi|A^{-1}\eta_q^i\rangle}\right|^2 d\mu(q)\cdot \sum_{i=1}^{n}\int_{\mathbb{H}} \left|\langle \zeta_q^i|\phi\rangle\right|^2 d\mu(q) \\
&\leqslant \frac{1}{m(A)}\left\|\mathfrak{U}^{-1}\phi\right\|^2\cdot \sum_{i=1}^{n}\int_{\mathbb{H}} \left|\langle \zeta_q^i|\phi\rangle\right|^2 d\mu(q) \\
&\leqslant \frac{1}{m(A)}\left\|\mathfrak{U}^{-1}\right\|^2\|\phi\|^2\cdot \sum_{i=1}^{n}\int_{\mathbb{H}} \left|\langle \zeta_q^i|\phi\rangle\right|^2 d\mu(q) \\
&\leqslant \frac{\|\phi\|^2}{m(A)\left(1-\sqrt{\frac{\kappa}{m(A)}}\right)^2}\cdot \sum_{i=1}^{n}\int_{\mathbb{H}} \left|\langle \zeta_q^i|\phi\rangle\right|^2 d\mu(q).
\end{aligned}$$

Hence,

$$\sum_{i=1}^{n}\int_{\mathbb{H}} \left|\langle \zeta_q^i|\phi\rangle\right|^2 d\mu(q) \geqslant m(A)\left(1-\sqrt{\frac{\kappa}{m(A)}}\right)^2\|\phi\|^2, \tag{3.3}$$

for all $\phi \in V_{\mathbb{H}}^R$.

On the other hand define a right linear operator $T:\mathbb{H}^n \longrightarrow V_{\mathbb{H}}^R$ by

$$T\{c_i\} := \sum_{i=1}^{n}\int_{\mathbb{H}} \zeta_q^i c_i d\mu(q) \tag{3.4}$$

The frame operator for $\{\zeta_q^i \in V_{\mathbb{H}}^R \mid i=1,2,\cdots,n,\ q\in\mathbb{H}\}$ is $TT^\dagger$, so the optimal upper frame bound for $\{\zeta_q^i \in V_{\mathbb{H}}^R \mid i=1,2,\cdots,n,\ q\in\mathbb{H}\}$ is $\|T\|^2$.

For $\{c_i\}\in\mathbb{H}^n$,

$$\begin{aligned}
\|T\{c_i\}\| &= \left\|\sum_{i=1}^{n}\int_{\mathbb{H}} \zeta_q^i c_i d\mu(q)\right\| \\
&\leqslant \left\|\sum_{i=1}^{n}\int_{\mathbb{H}} (\zeta_q^i-\eta_q^i) c_i d\mu(q)\right\| + \left\|\sum_{i=1}^{n}\int_{\mathbb{H}} \eta_q^i c_i d\mu(q)\right\| \\
&\leqslant (\sqrt{M(A)}+\sqrt{\kappa})\,\|\{c_i\}\|.
\end{aligned}$$

Hence $\|T\{c_i\}\| \leqslant (\sqrt{M(A)}+\sqrt{\kappa})\,\|\{c_i\}\|$ and $\|T\|^2 \leqslant (\sqrt{M(A)}+\sqrt{\kappa})^2$.

Therefore $\|T\|^2 \leqslant M(A)\left(1+\sqrt{\dfrac{\kappa}{M(A)}}\right)^2$.

Since $\|T\|^2$ is the optimal upper frame bound for $\{\zeta_q^i \in V_{\mathbb{H}}^R \mid i = 1,2,\cdots,n,\ q \in \mathbb{H}\}$, we have

$$\sum_{i=1}^{n}\int_{\mathbb{H}}\left|\langle\zeta_q^i|\phi\rangle\right|^2 d\mu(q) \leqslant M(A)\left(1+\sqrt{\frac{\kappa}{M(A)}}\right)^2\|\phi\|^2, \tag{3.5}$$

for all $\phi \in V_{\mathbb{H}}^R$.

From 3.3 and 3.5, we get

$$m(A)\left(1-\sqrt{\frac{\kappa}{m(A)}}\right)^2\|\phi\|^2 \leqslant \sum_{i=1}^{n}\int_{\mathbb{H}}\left|\langle\zeta_q^i|\phi\rangle\right|^2 d\mu(q) \leqslant M(A)\left(1+\sqrt{\frac{\kappa}{M(A)}}\right)^2\|\phi\|^2, \tag{3.6}$$

for all $\phi \in V_{\mathbb{H}}^R$.

Hence $\{\zeta_q^i \in V_{\mathbb{H}}^R \mid i = 1,2,\cdots,n,\ q \in \mathbb{H}\}$ is a rank n continuous frame with bounds $m(A)\left(1-\sqrt{\frac{\kappa}{m(A)}}\right)^2$ and $M(A)\left(1+\sqrt{\frac{\kappa}{M(A)}}\right)^2$. $\square$

THEOREM 3.2. *Let $\{\eta_q^i \in V_{\mathbb{H}}^R \mid i = 1,2,\cdots,n,\ q \in \mathbb{H}\}$ be a rank n right quaternionic continuous frame with bounds $m(A)$ and $M(A)$. Let $\{\zeta_q^i \in V_{\mathbb{H}}^R \mid i = 1,2,\cdots,n,\ q \in \mathbb{H}\}$ be any family defined in (3.1). Then $\{\eta_q^i + \zeta_q^i \in V_{\mathbb{H}}^R \mid i = 1,2,\cdots,n,\ q \in \mathbb{H}\}$ is a rank n continuous frame in $V_{\mathbb{H}}^R$.*

Proof. For $\phi \in V_{\mathbb{H}}^R$, we have

$$\begin{aligned}
&\sum_{i=1}^{n}\int_{\mathbb{H}}\left|\langle\eta_q^i+\zeta_q^i|\phi\rangle\right|^2 d\mu(q)\\
&= \sum_{i=1}^{n}\int_{\mathbb{H}}\left|\langle\phi|\eta_q^i+\zeta_q^i\rangle\right|^2 d\mu(q)\\
&= \sum_{i=1}^{n}\int_{\mathbb{H}}\left|\langle\phi|\eta_q^i\rangle+\langle\phi|\zeta_q^i\rangle\right|^2 d\mu(q)\\
&\leqslant \sum_{i=1}^{n}\int_{\mathbb{H}}\{\left|\langle\phi|\eta_q^i\rangle\right|+\left|\langle\phi|\zeta_q^i\rangle\right|\}^2 d\mu(q)\\
&= \sum_{i=1}^{n}\int_{\mathbb{H}}\{\left|\langle\phi|\eta_q^i\rangle\right|^2+2\left|\langle\phi|\eta_q^i\rangle\right|\left|\langle\phi|\zeta_q^i\rangle\right|+\left|\langle\phi|\zeta_q^i\rangle\right|^2\} d\mu(q)\\
&\leqslant \sum_{i=1}^{n}\int_{\mathbb{H}}\{\left|\langle\phi|\eta_q^i\rangle\right|^2+\left|\langle\phi|\eta_q^i\rangle\right|^2+\left|\langle\phi|\zeta_q^i\rangle\right|^2+\left|\langle\phi|\zeta_q^i\rangle\right|^2\} d\mu(q)\\
&= 2\sum_{i=1}^{n}\int_{\mathbb{H}}\left|\langle\eta_q^i|\phi\rangle\right|^2 d\mu(q)+2\sum_{i=1}^{n}\int_{\mathbb{H}}\left|\langle\zeta_q^i|\phi\rangle\right|^2 d\mu(q).
\end{aligned}$$

Since $\{\eta_q^i \in V_{\mathbb{H}}^R \mid i = 1,2,\cdots,n,\ q \in \mathbb{H}\}$ and $\{\zeta_q^i \in V_{\mathbb{H}}^R \mid i = 1,2,\cdots,n,\ q \in \mathbb{H}\}$ are rank n continuous frames in $V_{\mathbb{H}}^R$, for $\phi \in V_{\mathbb{H}}^R$,

$$m(A)\|\phi\|^2 \leqslant \sum_{i=1}^{n}\int_{\mathbb{H}}|\langle\eta_q^i|\phi\rangle|^2 d\mu(q) \leqslant M(A)\|\phi\|^2$$

and

$$m(A)\left(1-\sqrt{\frac{\kappa}{m(A)}}\right)^2\|\phi\|^2 \leqslant \sum_{i=1}^n \int_{\mathbb{H}} \left|\langle \zeta_q^i|\phi\rangle\right|^2 d\mu(q) \leqslant M(A)\left(1+\sqrt{\frac{\kappa}{M(A)}}\right)^2\|\phi\|^2.$$

Therefore

$$\sum_{i=1}^n \int_{\mathbb{H}} \left|\langle \eta_q^i+\zeta_q^i|\phi\rangle\right|^2 d\mu(q) \leqslant 2M(A)\left[1+\left(1+\sqrt{\frac{\kappa}{M(A)}}\right)^2\right]\|\phi\|^2.$$

Similarly one can obtain

$$2m(A)\left[1+\left(1-\sqrt{\frac{\kappa}{m(A)}}\right)^2\right]\|\phi\|^2 \leqslant \sum_{i=1}^n \int_{\mathbb{H}} \left|\langle \eta_q^i+\zeta_q^i|\phi\rangle\right|^2 d\mu(q).$$

Hence

$$\begin{aligned}2m(A)\left[1+\left(1-\sqrt{\frac{\kappa}{m(A)}}\right)^2\right]\|\phi\|^2 &\leqslant \sum_{i=1}^n \int_{\mathbb{H}} \left|\langle \eta_q^i+\zeta_q^i|\phi\rangle\right|^2 d\mu(q)\\ &\leqslant 2M(A)\left[1+\left(1+\sqrt{\frac{\kappa}{M(A)}}\right)^2\right]\|\phi\|^2,\end{aligned}$$

for all $\phi \in V_{\mathbb{H}}^R$. Therefore, $\{\eta_q^i+\zeta_q^i \in V_{\mathbb{H}}^R \mid i=1,2,\cdots,n,\ q\in\mathbb{H}\}$ is a rank n continuous frame in $V_{\mathbb{H}}^R$ with bounds

$$2m(A)\left[1+\left(1-\sqrt{\frac{\kappa}{m(A)}}\right)^2\right] \quad \text{and} \quad 2M(A)\left[1+\left(1+\sqrt{\frac{\kappa}{M(A)}}\right)^2\right]. \quad \square$$

PROPOSITION 3.3. Let $\{\eta_q^i+\zeta_q^i \in V_{\mathbb{H}}^R \mid i=1,2,\cdots,n,\ q\in\mathbb{H}\}$ be a rank n continuous frame in $V_{\mathbb{H}}^R$ with frame operator

$$A' = \sum_{i=1}^n \int_{\mathbb{H}} |\eta_q^i+\zeta_q^i\rangle\langle\eta_q^i+\zeta_q^i| d\mu(q). \tag{3.7}$$

Then A' is self adjoint and positive.

Proof. For $\phi \in V_{\mathbb{H}}^R, A'|\phi\rangle = \sum_{i=1}^n \int_{\mathbb{H}} |\eta_q^i+\zeta_q^i\rangle\langle\eta_q^i+\zeta_q^i|\phi\rangle d\mu(q)$.
Since $\langle A'\phi|\psi\rangle = (A'|\phi\rangle)^\dagger|\psi\rangle$,

$$\begin{aligned}\langle A'\phi|\psi\rangle &= \left(\sum_{i=1}^n \int_{\mathbb{H}} |\eta_q^i+\zeta_q^i\rangle\langle\eta_q^i+\zeta_q^i|\phi\rangle d\mu(q)\right)^\dagger (|\psi\rangle)\\ &= \sum_{i=1}^n \int_{\mathbb{H}} \langle\phi|\eta_q^i+\zeta_q^i\rangle\langle\eta_q^i+\zeta_q^i|\psi\rangle d\mu(q)\\ &= \langle\phi|A'\psi\rangle.\end{aligned}$$

Hence $\langle A'\phi|\psi\rangle = \langle\phi|A'\psi\rangle$. It follows that A' is self adjoint.

Now

$$\begin{aligned}\langle A'\phi|\phi\rangle &= \sum_{i=1}^{n}\int_{\mathbb{H}}\langle\phi|\eta_q^i+\zeta_q^i\rangle\langle\eta_q^i+\zeta_q^i|\phi\rangle d\mu(q)\\ &= \sum_{i=1}^{n}\int_{\mathbb{H}}\left|\langle\phi|\eta_q^i+\zeta_q^i\rangle\right|^2 d\mu(q)\\ &\geqslant 0.\end{aligned}$$

Thereby A' is positive. □

The following results are the quaternionic continuous counterparts of certain perturbations considered for complex discrete frames in [5].

THEOREM 3.4. *Let* $\{\eta_q^i\in V_{\mathbb{H}}^R \mid i=1,2,\cdots,n,\ q\in\mathbb{H}\}$ *be a rank* n *right quaternionic continuous frame with bounds* $m(A),M(A)$ *and* $\{\overline{\eta}_q^i\in V_{\mathbb{H}}^R \mid i=1,2,\cdots,n,\ q\in\mathbb{H}\}$ *be the dual frame of* $\{\eta_q^i\in V_{\mathbb{H}}^R \mid i=1,2,\cdots,n,\ q\in\mathbb{H}\}$ *with bounds* C,D*. Assume that the family* $\{\Psi_q^i\in V_{\mathbb{H}}^R \mid i=1,2,\cdots,n,\ q\in\mathbb{H}\}$ *satisfies the following two conditions:*

1. $\lambda := \sum_{i=1}^{n}\int_{\mathbb{H}}\left\|\eta_q^i-\Psi_q^i\right\|^2 d\mu(q)<\infty;$

2. $\gamma := \sum_{i=1}^{n}\int_{\mathbb{H}}\|\eta_q^i-\Psi_q^i\|\,\|\overline{\eta}_q^i\|\,d\mu(q)<1.$

Then $\{\Psi_q^i\in V_{\mathbb{H}}^R \mid i=1,2,\cdots,n,\ q\in\mathbb{H}\}$ *is a rank* n *continuous frame for* $V_{\mathbb{H}}^R$ *with bounds* $\dfrac{(1-\gamma)^2}{D}$ *and* $M(A)\left(1+\sqrt{\dfrac{\lambda}{M(A)}}\right)^2$.

Proof. Let $T:\mathbb{H}^n\longrightarrow V_{\mathbb{H}}^R$ defined by

$$T(\{c_i\}_{i=1}^n)=\sum_{i=1}^{n}\int_{\mathbb{H}}\eta_q^i c_i\,d\mu(q),\ \text{where } \{c_i\}_{i=1}^n\in\mathbb{H}^n$$

be the pre-frame operator of the frame $\{\eta_q^i\in V_{\mathbb{H}}^R \mid i=1,2,\cdots,n,\ q\in\mathbb{H}\}$. From Theorem 2.9, $\|T\|\leqslant\sqrt{M(A)}$. Now define $U:\mathbb{H}^n\longrightarrow V_{\mathbb{H}}^R$ by

$$U(\{c_i\}_{i=1}^n)=\sum_{i=1}^{n}\int_{\mathbb{H}}\Psi_q^i c_i\,d\mu(q),\ \text{where } \{c_i\}_{i=1}^n\in\mathbb{H}^n.$$

We have

$$\begin{aligned}
\|U\| &= \sup_{\|\phi\|=1} \|U\phi\| \\
&= \sup_{\|\phi\|=1} \left\| \sum_{i=1}^{n} \int_{\mathbb{H}} \Psi_q^i c_i d\mu(q) \right\|, \text{ where } \phi = \{c_i\}_{i=1}^{n} \in \mathbb{H}^n \\
&= \sup_{\|\phi\|=1} \left\| \sum_{i=1}^{n} \int_{\mathbb{H}} (\Psi_q^i - \eta_q^i + \eta_q^i) c_i d\mu(q) \right\| \\
&\leqslant \sup_{\|\phi\|=1} \left\| \sum_{i=1}^{n} \int_{\mathbb{H}} (\Psi_q^i - \eta_q^i) c_i d\mu(q) \right\| + \sup_{\|\phi\|=1} \left\| \sum_{i=1}^{n} \int_{\mathbb{H}} \eta_q^i c_i d\mu(q) \right\| \\
&\leqslant \sup_{\|\phi\|=1} \sum_{i=1}^{n} |c_i| \int_{\mathbb{H}} \|\Psi_q^i - \eta_q^i\| \, d\mu(q) + \|T\| \\
&\leqslant \sqrt{\lambda} + \|T\|, \text{ by } (1) \\
&\leqslant \sqrt{\lambda} + \sqrt{M(A)}.
\end{aligned}$$

Hence U is well defined and $\|U\| \leqslant \sqrt{\lambda} + \sqrt{M(A)}$. Now the adjoint $U^\dagger$ of U can be defined by

$$U^\dagger : V_{\mathbb{H}}^R \longrightarrow \mathbb{H}^n \text{ by } U^\dagger(\phi) = \{\langle \phi | \Psi_q^i \rangle\}_{i=1}^{n}, \ \forall \phi \in V_{\mathbb{H}}^R.$$

We have

$$\begin{aligned}
\sum_{i=1}^{n} \int_{\mathbb{H}} \left| \langle \phi | \Psi_q^i \rangle \right|^2 d\mu(q) &= \left\| U^\dagger \phi \right\|^2 \\
&\leqslant \left\| U^\dagger \|^2 \, \|\phi \right\|^2 \\
&= \{\|U\| \, \|\phi\|\}^2 \\
&\leqslant \{\sqrt{\lambda} + \sqrt{M(A)}\}^2 \, \|\phi\|^2 \\
&= M(A) \left(1 + \sqrt{\frac{\lambda}{M(A)}} \right)^2 \|\phi\|^2 .
\end{aligned}$$

Therefore

$$\sum_{i=1}^{n} \int_{\mathbb{H}} \left| \langle \phi | \Psi_q^i \rangle \right|^2 d\mu(q) \leqslant M(A) \left(1 + \sqrt{\frac{\lambda}{M(A)}} \right)^2 \|\phi\|^2 . \tag{3.8}$$

Now define $L : V_{\mathbb{H}}^R \longrightarrow V_{\mathbb{H}}^R$ by

$$L(\phi) = \sum_{i=1}^{n} \int_{\mathbb{H}} \Psi_q^i \langle \phi | \overline{\eta}_q^i \rangle \, d\mu(q), \ \forall \phi \in V_{\mathbb{H}}^R.$$

For $\phi \in V_{\mathbb{H}}^{R}$,

$$\begin{aligned}
\|\phi - L(\phi)\| &= \left\| \sum_{i=1}^{n} \int_{\mathbb{H}} \eta_q^i \langle \phi | \overline{\eta}_q^i \rangle \, d\mu(q) - \sum_{i=1}^{n} \int_{\mathbb{H}} \Psi_q^i \langle \phi | \overline{\eta}_q^i \rangle \, d\mu(q) \right\| \\
&= \left\| \sum_{i=1}^{n} \int_{\mathbb{H}} (\eta_q^i - \Psi_q^i) \langle \phi | \overline{\eta}_q^i \rangle \, d\mu(q) \right\| \\
&\leqslant \sum_{i=1}^{n} \int_{\mathbb{H}} \left\| (\eta_q^i - \Psi_q^i) \langle \phi | \overline{\eta}_q^i \rangle \right\| d\mu(q) \\
&= \sum_{i=1}^{n} \int_{\mathbb{H}} \left\| \eta_q^i - \Psi_q^i \right\| \left| \langle \phi | \overline{\eta}_q^i \rangle \right| d\mu(q) \\
&\leqslant \sum_{i=1}^{n} \int_{\mathbb{H}} \left\| \eta_q^i - \Psi_q^i \right\| \|\phi\| \left\| \overline{\eta}_q^i \right\| d\mu(q) \\
&= \gamma \|\phi\| .
\end{aligned}$$

That is $\|\phi - L(\phi)\| \leqslant \gamma \|\phi\|$, for all $\phi \in V_{\mathbb{H}}^{R}$. It follows that $\left\| I_{V_{\mathbb{H}}^{R}} - L \right\| \leqslant \gamma$ and $\left\| I_{V_{\mathbb{H}}^{R}} - L \right\| \leqslant 1$. So that $\|L\| \leqslant 1 + \gamma$ and $\left\| L^{-1} \right\| \leqslant \dfrac{1}{1-\gamma}$. Each $\phi \in V_{\mathbb{H}}^{R}$ can be written as

$$\begin{aligned}
\phi &= L L^{-1} \phi \\
&= \sum_{i=1}^{n} \int_{\mathbb{H}} \Psi_q^i \langle L^{-1}\phi | \overline{\eta}_q^i \rangle \, d\mu(q).
\end{aligned}$$

Now

$$\begin{aligned}
\|\phi\|^2 &= \langle \phi | \phi \rangle \\
&= \left\langle \phi | \sum_{i=1}^{n} \int_{\mathbb{H}} \Psi_q^i \langle L^{-1}\phi | \overline{\eta}_q^i \rangle \, d\mu(q) \right\rangle \\
&= \sum_{i=1}^{n} \int_{\mathbb{H}} \langle \phi | \Psi_q^i \rangle \langle L^{-1}\phi | \overline{\eta}_q^i \rangle \, d\mu(q) \\
&\leqslant \left(\sum_{i=1}^{n} \int_{\mathbb{H}} \left| \langle \phi | \Psi_q^i \rangle \right|^2 d\mu(q) \right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} \int_{\mathbb{H}} \left| \langle L^{-1}\phi | \overline{\eta}_q^i \rangle \right|^2 d\mu(q) \right)^{\frac{1}{2}} \\
&\leqslant \left(\sum_{i=1}^{n} \int_{\mathbb{H}} \left| \langle \phi | \Psi_q^i \rangle \right|^2 d\mu(q) \right)^{\frac{1}{2}} \left(D . \left\| L^{-1}\phi \right\|^2 \right)^{\frac{1}{2}} \\
&= \sqrt{D} \left\| L^{-1}\phi \right\| \left(\sum_{i=1}^{n} \int_{\mathbb{H}} \left| \langle \phi | \Psi_q^i \rangle \right|^2 d\mu(q) \right)^{\frac{1}{2}} \\
&\leqslant \sqrt{D} \frac{1}{1-\gamma} \|\phi\| \left(\sum_{i=1}^{n} \int_{\mathbb{H}} \left| \langle \phi | \Psi_q^i \rangle \right|^2 d\mu(q) \right)^{\frac{1}{2}} .
\end{aligned}$$

It follows that

$$\sum_{i=1}^{n}\int_{\mathbb{H}}\left|\langle\phi|\Psi_q^i\rangle\right|^2 d\mu(q) \geqslant \frac{(1-\gamma)^2}{D}\,\|\phi\|^2. \tag{3.9}$$

From 3.8 and 3.9, we get

$$\frac{(1-\gamma)^2}{D}\,\|\phi\|^2 \leqslant \sum_{i=1}^{n}\int_{\mathbb{H}}\left|\langle\phi|\Psi_q^i\rangle\right|^2 d\mu(q) \leqslant M(A)\left(1+\sqrt{\frac{\lambda}{M(A)}}\right)^2\|\phi\|^2, \tag{3.10}$$

for all $\phi\in V_{\mathbb{H}}^R$. Hence $\{\Psi_q^i\in V_{\mathbb{H}}^R \mid i=1,2,\cdots,n,\ q\in\mathbb{H}\}$ is a rank n continuous frame for $V_{\mathbb{H}}^R$ with bounds $\dfrac{(1-\gamma)^2}{D}$ and $M(A)\left(1+\sqrt{\dfrac{\lambda}{M(A)}}\right)^2$. $\square$

DEFINITION 3.5. Let K and L be subspaces of $V_{\mathbb{H}}^R$. When $K\neq\{0\}$, the gap from K to L is given by

$$\delta(K,L) := \sup_{\phi\in K,\|\phi\|=1}\ \inf_{\psi\in L}\|\phi-\psi\|.$$

Also when $K=\{0\}$, we define $\delta(K,L)=0$.

THEOREM 3.6. *Let $\{\eta_q^i\in V_{\mathbb{H}}^R \mid i=1,2,\cdots,n,\ q\in\mathbb{H}\}$ be a rank n continuous frame in $V_{\mathbb{H}}^R$ with bounds $m(A)$ and $M(A)$ and let $\{\overline{\eta}_q^i\in V_{\mathbb{H}}^R \mid i=1,2,\cdots,n,\ q\in\mathbb{H}\}$ be the dual frame of $\{\eta_q^i\in V_{\mathbb{H}}^R \mid i=1,2,\cdots,n,\ q\in\mathbb{H}\}$ with bounds C,D. Suppose that $\{\Psi_q^i\in V_{\mathbb{H}}^R \mid i=1,2,\cdots,n,\ q\in\mathbb{H}\}$ is a family in $V_{\mathbb{H}}^R$. Let $K=\overline{\text{rightspan}}\{\Psi_q^i\}_{i=1}^n, L=\overline{\text{rightspan}}\{\eta_q^i\}_{i=1}^n$, where $q\in\mathbb{H}$. and the right span is taken over $\mathbb{H}$. Assume that $\delta(K,L)<1$. If $\{\Psi_q^i\in V_{\mathbb{H}}^R \mid i=1,2,\cdots,n,\ q\in\mathbb{H}\}$ satisfies the following conditions:*

1. $\lambda := \sum_{i=1}^{n}\int_{\mathbb{H}}\left\|\eta_q^i-\Psi_q^i\right\|^2 d\mu(q)<\infty$;
2. $\gamma := \sum_{i=1}^{n}\int_{\mathbb{H}}\left\|\eta_q^i-\Psi_q^i\right\|\left\|\overline{\eta}_q^i\right\| d\mu(q)<1.$

Then $\{\Psi_q^i\in V_{\mathbb{H}}^R \mid i=1,2,\cdots,n,\ q\in\mathbb{H}\}$ is a rank n continuous frame with bounds $\dfrac{(1-\gamma)^2}{D}$ and $M(A)\left(1+\sqrt{\dfrac{\lambda}{M(A)}}\right)^2\dfrac{1}{(1-\delta(K,L))^2}$. Moreover, the restriction of the orthogonal projection P_L to K is an isomorphism from K onto L.

Proof. Let $h \in K$ then $h = h_L + h - h_L$, where $h_L \in L$ with $P_L h = h_L,\ h = P_L h + h - h_L$. Therefore

$$\begin{aligned}\|P_L h\| &\geqslant \|h\| - \|h - h_L\| \\ &= \|h\| - \|h\| \left\| \frac{h}{\|h\|} - \frac{h_L}{\|h\|} \right\| \\ &\geqslant \|h\| - \|h\| \sup_{\phi \in K, \|\phi\|=1} \inf_{\psi \in L} \|\phi - \psi\| \\ &= \|h\| - \|h\| \, \delta(K,L) \\ &= (1 - \delta(K,L)) \, \|h\| .\end{aligned}$$

Therefore $\|P_L h\| \geqslant (1-\delta(K,L)) \|h\|$, for all $h \in K$. Let $V_{\mathbb{H}}^R = L \oplus L^{\perp}$ and $P_L : V_{\mathbb{H}}^R \longrightarrow L$ be the orthogonal projection. Now for each $i, V_{\mathbb{H}}^R \ni \Psi_q^i = P_L(\Psi_q^i) + \Phi_q^i$, for some $\Phi_q^i \in L^{\perp}$. Now $\eta_q^i - \Psi_q^i = (\eta_q^i - P_L(\Psi_q^i)) - \Phi_q^i$, for all i. Note that $\eta_q^i - \Psi_q^i \in V_{\mathbb{H}}^R, \eta_q^i - P_L(\Psi_q^i) \in L$ and $-\Phi_q^i \in L^{\perp}$. Then for each i,

$$\begin{aligned}\left\|\eta_q^i - \Psi_q^i\right\|^2 &= \langle u+v | u+v \rangle, \text{ where } u = \eta_q^i - P_L(\Psi_q^i) \in L \text{ and } v = -\Phi_q^i \in L^{\perp} \\ &= \|u\|^2 + \langle u|v\rangle + \langle v|u\rangle + \|v\|^2 \\ &= \|u\|^2 + \|v\|^2 \text{ as } \langle u|v\rangle = \langle v|u\rangle = 0 \\ &\geqslant \|u\|^2 \\ &= \left\|\eta_q^i - P_L(\Psi_q^i)\right\|^2\end{aligned}$$

Therefore $\left\|\eta_q^i - P_L(\Psi_q^i)\right\| \leqslant \left\|\eta_q^i - \Psi_q^i\right\|$, for all i. Hence,

$$\sum_{i=1}^n \int_{\mathbb{H}} \left\|\eta_q^i - P_L(\Psi_q^i)\right\|^2 d\mu(q) \leqslant \sum_{i=1}^n \int_{\mathbb{H}} \left\|\eta_q^i - \Psi_q^i\right\|^2 d\mu(q)$$

and

$$\sum_{i=1}^n \int_{\mathbb{H}} \left\|\eta_q^i - P_L(\Psi_q^i)\right\| \left\|\overline{\eta}_q^i\right\| d\mu(q) \leqslant \sum_{i=1}^n \int_{\mathbb{H}} \left\|\eta_q^i - \Psi_q^i\right\| \left\|\overline{\eta}_q^i\right\| d\mu(q).$$

We apply Theorem 3.4 to the sequence $\{P_L(\Psi_q^i)\}_{i=1}^n$ in L and to the frame $\{\eta_q^i\}_{i=1}^n$ for L to obtain $\{P_L(\Psi_q^i)\}_{i=1}^n$ as a frame for L with bounds $\dfrac{(1-\gamma)^2}{D}$ and $M(A)\left(1+\sqrt{\dfrac{\lambda}{M(A)}}\right)^2$. We have $P_L(K) = L$ and hence the restriction $Q := P_L|_K$ is an isomorphism from K onto L. Now the claim is $\{\Psi_q^i \in V_{\mathbb{H}}^R \mid i = 1,2,\cdots,n,\ q \in \mathbb{H}\}$ is a frame for K. For $\Psi \in K$, we have

$$\begin{aligned}\sum_{i=1}^n \int_{\mathbb{H}} \left|\langle \Psi | \Psi_q^i \rangle\right|^2 d\mu(q) &= \sum_{i=1}^n \int_{\mathbb{H}} \left|\langle \Psi | Q^{-1} Q(\Psi_q^i) \rangle\right|^2 d\mu(q) \\ &= \sum_{i=1}^n \int_{\mathbb{H}} \left|\langle (Q^{-1})^{\dagger} \Psi | Q(\Psi_q^i) \rangle\right|^2 d\mu(q)\end{aligned}$$

$$\leqslant M(A)\left(1+\sqrt{\frac{\lambda}{M(A)}}\right)^2 \left\|(Q^{-1})^{\dagger}\Psi\right\|^2$$
$$\leqslant M(A)\left(1+\sqrt{\frac{\lambda}{M(A)}}\right)^2 \left\{\left\|(Q^{-1})^{\dagger}\right\| \|\Psi\|\right\}^2$$
$$= M(A)\left(1+\sqrt{\frac{\lambda}{M(A)}}\right)^2 \left\|Q^{-1}\right\|^2 \|\Psi\|^2$$
$$\leqslant M(A)\left(1+\sqrt{\frac{\lambda}{M(A)}}\right)^2 \frac{1}{(1-\delta(K,L))^2} \|\Psi\|^2$$

Hence

$$\sum_{i=1}^{n}\int_{\mathbb{H}}\left|\langle\Psi|\Psi_q^i\rangle\right|^2 d\mu(q) \leqslant M(A)\left(1+\sqrt{\frac{\lambda}{M(A)}}\right)^2 \frac{1}{(1-\delta(K,L))^2}\|\Psi\|^2, \tag{3.11}$$

for all $\Psi \in K$. Now

$$\begin{aligned}\sum_{i=1}^{n}\int_{\mathbb{H}}\left|\langle\Psi|\Psi_q^i\rangle\right|^2 d\mu(q) &= \sum_{i=1}^{n}\int_{\mathbb{H}}\left|\langle(Q^{-1})^{\dagger}\Psi|Q(\Psi_q^i)\rangle\right|^2 d\mu(q)\\ &\geqslant \frac{(1-\gamma)^2}{D}\left\|(Q^{-1})^{\dagger}\Psi\right\|^2\\ &= \frac{(1-\gamma)^2}{D}\left\|(Q^{\dagger})^{-1}\Psi\right\|^2\\ &\geqslant \frac{(1-\gamma)^2}{D}\|\Psi\|^2.\end{aligned}$$

Hence

$$\sum_{i=1}^{n}\int_{\mathbb{H}}\left|\langle\Psi|\Psi_q^i\rangle\right|^2 d\mu(q) \geqslant \frac{(1-\gamma)^2}{D}\|\Psi\|^2, \tag{3.12}$$

for all $\Psi \in K$. Therefore, $\{\Psi_q^i \in V_{\mathbb{H}}^R \mid i=1,2,\cdots,n,\ q\in\mathbb{H}\}$ is a rank n continuous frame for K with bounds $\frac{(1-\gamma)^2}{D}$ and $M(A)\left(1+\sqrt{\frac{\lambda}{M(A)}}\right)^2 \frac{1}{(1-\delta(K,L))^2}$. □

DEFINITION 3.7. We call a sequence $\{\eta_q^i \in V_{\mathbb{H}}^R \mid i=1,2,\cdots,n,\ q\in\mathbb{H}\}$ a quaternionic rank n continuous Riesz family if there exists two constants $A,B>0$ such that for every scalar sequence $\{c_i\}_{i=1}^n \subseteq \mathbb{H}^n$,

$$A\sum_{i=1}^{n}|c_i|^2 \leqslant \left\|\sum_{i=1}^{n}\int_{\mathbb{H}}\eta_q^i c_i d\mu(q)\right\|^2 \leqslant B\sum_{i=1}^{n}|c_i|^2,$$

where A,B are called Riesz bounds.

THEOREM 3.8. *Let* $\{\eta_q^i \in V_{\mathbb{H}}^R \mid i = 1,2,\cdots,n,\ q \in \mathbb{H}\}$ *be a quaternionic rank* n *continuous Riesz family in* $V_{\mathbb{H}}^R$ *with bounds* $m(A)$ *and* $M(A)$ *and let* $\{\xi_q^i \in V_{\mathbb{H}}^R \mid i = 1,2,\cdots,n,\ q \in \mathbb{H}\}$ *be a family in* $V_{\mathbb{H}}^R$ *which satisfies* $\gamma = \sum_{i=1}^{n}\int_{\mathbb{H}} \|\eta_q^i - \xi_q^i\|\,\|S^{-1}\eta_q^i\|\,d\mu(q) < 1$. *Then* $\{\xi_q^i \in V_{\mathbb{H}}^R \mid i = 1,2,\cdots,n,\ q \in \mathbb{H}\}$ *is a rank* n *continuous Riesz family with bounds* $m(A)(1-\gamma)^2$ *and* $M(A)\left(1+\sqrt{\dfrac{\lambda}{M(A)}}\right)^2$, *where* $\lambda := \sum_{i=1}^{n}\int_{\mathbb{H}} \|\eta_q^i - \xi_q^i\|^2\,d\mu(q)$ *and* S *is a frame operator of* $\{\eta_q^i \in V_{\mathbb{H}}^R \mid i = 1,2,\cdots,n,\ q \in \mathbb{H}\}$ *in* $L := \overline{\text{rightspan}}\{\eta_q^i \mid q \in \mathbb{H}\}_{i=1}^n$.

Proof. For $\{c_i\}_{i=1}^n \in \mathbb{H}^n$,

$$\begin{aligned}
\left\|\sum_{i=1}^{n}\int_{\mathbb{H}} \xi_q^i c_i d\mu(q)\right\| &\leqslant \sum_{i=1}^{n}\int_{\mathbb{H}} \|\xi_q^i c_i\|\,d\mu(q)\\
&= \sum_{i=1}^{n}\int_{\mathbb{H}} \|c_i(\xi_q^i - \eta_q^i + \eta_q^i)\|\,d\mu(q)\\
&\leqslant \sum_{i=1}^{n}\int_{\mathbb{H}} \|c_i(\eta_q^i - \xi_q^i)\|\,d\mu(q) + \sum_{i=1}^{n}\int_{\mathbb{H}} \|c_i\eta_q^i\|\,d\mu(q)\\
&= \sum_{i=1}^{n}|c_i|\int_{\mathbb{H}} \|(\eta_q^i - \xi_q^i)\|\,d\mu(q) + \sum_{i=1}^{n}\int_{\mathbb{H}} \|c_i\eta_q^i\|\,d\mu(q)\\
&\leqslant \left(\sum_{i=1}^{n}|c_i|^2\right)^{\frac12}\left(\sum_{i=1}^{n}\int_{\mathbb{H}} \|(\eta_q^i - \xi_q^i)\|^2\,d\mu(q)\right)^{\frac12}\\
&\quad + \sqrt{M(A)}\left(\sum_{i=1}^{n}|c_i|^2\right)^{\frac12}\\
&= \sqrt{\lambda}\left(\sum_{i=1}^{n}|c_i|^2\right)^{\frac12} + \sqrt{M(A)}\left(\sum_{i=1}^{n}|c_i|^2\right)^{\frac12}\\
&= M(A)\left(1+\sqrt{\frac{\lambda}{M(A)}}\right)^2\left(\sum_{i=1}^{n}|c_i|^2\right)^{\frac12}.
\end{aligned}$$

Define $U : V_{\mathbb{H}}^R \longrightarrow V_{\mathbb{H}}^R$, by $U\phi = \sum_{i=1}^{n}\int_{\mathbb{H}} \xi_q^i \langle\phi|S^{-1}\eta_q^i\rangle\,d\mu(q)$. For any $\phi \in V_{\mathbb{H}}^R$, we have

$$\begin{aligned}
\left\|\sum_{i=1}^{n}\int_{\mathbb{H}} \xi_q^i\langle\phi|S^{-1}\eta_q^i\rangle\,d\mu(q)\right\| &\leqslant \sum_{i=1}^{n}\int_{\mathbb{H}} \|\xi_q^i\langle\phi|S^{-1}\eta_q^i\rangle\|\,d\mu(q)\\
&\leqslant (\sqrt{\lambda}+\sqrt{M(A)})\left(\sum_{i=1}^{n}\int_{\mathbb{H}} |\langle\phi|S^{-1}\eta_q^i\rangle|^2\right)^{\frac12}
\end{aligned}$$

$$= (\sqrt{\lambda} + \sqrt{M(A)}) \left(\sum_{i=1}^{n} \int_{\mathbb{H}} \left| \langle P_L \phi | S^{-1} \eta_q^i \rangle \right|^2 \right)^{\frac{1}{2}}$$
$$\leqslant (\sqrt{\lambda} + \sqrt{M(A)}) \left(\frac{1}{m(A)} \|P_L \phi\|^2 \right)^{\frac{1}{2}}$$
$$\leqslant \frac{\sqrt{\lambda} + \sqrt{M(A)}}{\sqrt{m(A)}} \|\phi\|.$$

Since $\{\eta_q^i \in V_{\mathbb{H}}^R \mid i = 1,2,\cdots,n,\ q \in \mathbb{H}\}$ is a continuous frame for L, by the frame decomposition, $U(\eta_q^i) = \xi_q^i$, for all $i = 1,2,\ldots,n$ and $q \in \mathbb{H}$. For $\phi \in L$,

$$\begin{aligned}
\|\phi - U\phi\| &= \left\| \sum_{i=1}^{n} \int_{\mathbb{H}} \eta_q^i \langle \phi | S^{-1} \eta_q^i \rangle \, d\mu(q) - \sum_{i=1}^{n} \int_{\mathbb{H}} \xi_q^i \langle \phi | S^{-1} \eta_q^i \rangle \, d\mu(q) \right\| \\
&= \left\| \sum_{i=1}^{n} \int_{\mathbb{H}} (\eta_q^i - \xi_q^i) \langle \phi | S^{-1} \eta_q^i \rangle \, d\mu(q) \right\| \\
&\leqslant \sum_{i=1}^{n} \int_{\mathbb{H}} \left\| (\eta_q^i - \xi_q^i) \langle \phi | S^{-1} \eta_q^i \rangle \right\| d\mu(q) \\
&= \sum_{i=1}^{n} \int_{\mathbb{H}} \|\eta_q^i - \xi_q^i\| \left| \langle \phi | S^{-1} \eta_q^i \rangle \right| d\mu(q) \\
&\leqslant \sum_{i=1}^{n} \int_{\mathbb{H}} \|\eta_q^i - \xi_q^i\| \, \|S^{-1} \eta_q^i\| \, \|\phi\| \, d\mu(q) \\
&= \gamma \|\phi\|.
\end{aligned}$$

Hence $\|\phi - U\phi\| \leqslant \gamma \|\phi\|$. It follows that $\left| \|\phi\| - \|U\phi\| \right| \leqslant \|\phi - U\phi\| \leqslant \gamma \|\phi\|$ and $\|U\phi\| \geqslant (1-\gamma)\|\phi\|$. We have

$$\begin{aligned}
\left\| \sum_{i=1}^{n} \int_{\mathbb{H}} \xi_q^i c_i d\mu(q) \right\| &= \left\| U \left(\sum_{i=1}^{n} \int_{\mathbb{H}} \eta_q^i c_i d\mu(q) \right) \right\| \\
&\geqslant (1-\gamma) \left\| \sum_{i=1}^{n} \int_{\mathbb{H}} \eta_q^i c_i d\mu(q) \right\| \\
&\geqslant (1-\gamma) \sqrt{m(A)} \left(\sum_{i=1}^{n} |c_i|^2 \right)^{\frac{1}{2}}. \quad \square
\end{aligned}$$

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