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ON THE A_α SPECTRAL RADIUS AND A_α ENERGY OF NON–STRONGLY CONNECTED DIGRAPHS

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Abstract. Let $A_\alpha(G)$ be the A_α-matrix of a digraph G and $\lambda_{\alpha 1},\lambda_{\alpha 2},\ldots,\lambda_{\alpha n}$ be the eigenvalues of $A_\alpha(G)$. Let $\rho_\alpha(G)$ be the A_α spectral radius of G and $E_\alpha(G)=\sum_{i=1}^{n}\lambda_{\alpha i}^2$ be the A_α energy of G by using second spectral moment. Let $\mathcal{G}_n^m$ be the set of non-strongly connected digraphs with n vertices containing a unique strong component with m vertices and some directed trees hanging on each vertex of the strong component. In this paper, we characterize the digraph which has the maximal A_α spectral radius and the maximal (or minimal) A_α energy in $\mathcal{G}_n^m$.

1. Introduction

Let $G=(\mathcal{V}(G),\mathcal{A}(G))$ be a digraph where $\mathcal{V}(G)=\{v_1,v_2,\ldots,v_n\}$ is the vertex set of G and $\mathcal{A}(G)$ is the arc set of G. For an arc from the vertex v_i to v_j, we denote by (v_i,v_j), and v_i is the tail of (v_i,v_j) and v_j is the head of (v_i,v_j). The outdegree $d_i^+=d_G^+(v_i)$ of G is the number of arcs whose tail is vertex v_i and the indegree $d_i^-=d_G^-(v_i)$ of G is the number of arcs whose head is vertex v_i. We denote the maximum outdegree of G by $\Delta^+(G)$. A walk π of length l from vertex u to vertex v is a sequence of vertices π: $u=v_0,v_1,\ldots,v_l=v$, where (v_{k-1},v_k) is an arc of G for any $1\leqslant k\leqslant l$. If $u=v$ then π is called a closed walk. Let c_2 denote the number of all closed walks of length 2. A directed path P_n with n vertices is a digraph which the vertex set is $\{v_i|i=1,2,\ldots,n\}$ and the arc set is $\{(v_i,v_{i+1})|i=1,2,\ldots,n-1\}$. A directed cycle C_n with $n\geqslant 2$ vertices is a digraph which the vertex set is $\{v_i|i=1,2,\ldots,n\}$ and the arc set is $\{(v_i,v_{i+1})|i=1,\ldots,n-1\}\cup\{(v_n,v_1)\}$. A digraph G is connected if its underlying graph is connected. A digraph G is strongly connected if for each pair of vertices $v_i,v_j\in\mathcal{V}(G)$, there is a directed path from v_i to v_j. A strong component of G is a maximal strongly connected subdigraph of G. A directed tree T with n vertices is a digraph for which its underlying graph is connected and does not contain any cycles. A directed tree with n vertices will have $e=n-1$ arcs. Throughout this paper, we only consider a connected digraph G containing neither loops nor multiple arcs.

For a digraph G with n vertices, the adjacency matrix $A(G)=(a_{ij})_{n\times n}$ of G is a $(0,1)$-square matrix whose (i,j)-entry equals 1 if (v_i,v_j) is an arc of G, and

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equals 0 otherwise. The Laplacian matrix $L(G)$ and the signless Laplacian matrix $Q(G)$ of G are $L(G) = D^+(G) - A(G)$ and $Q(G) = D^+(G) + A(G)$, respectively, where $D^+(G) = diag(d_1^+, d_2^+, \ldots, d_n^+)$ is a diagonal outdegree matrix of G. In 2019, Liu et al. [12] defined the A_α-matrix of G as

$$A_\alpha(G) = \alpha D^+(G) + (1-\alpha)A(G),$$

where $\alpha \in [0,1]$. It is clear that if $\alpha = 0$, then $A_0(G) = A(G)$; if $\alpha = \frac{1}{2}$, then $A_{\frac{1}{2}}(G) = \frac{1}{2}Q(G)$; if $\alpha = 1$, then $A_1(G) = D^+(G)$. Since $D^+(G)$ is not interesting, we only consider $\alpha \in [0,1)$. The eigenvalue of $A_\alpha(G)$ with largest modulus is called the A_α spectral radius of G, denoted by $\rho_\alpha(G)$.

Actually, in 2017, Nikiforov [15] first proposed the A_α-matrix of a graph H of order n as

$$A_\alpha(H) = \alpha D(H) + (1-\alpha)A(H),$$

where $D(H) = diag(d_1, d_2, \ldots, d_n)$ is a diagonal degree matrix of H and $\alpha \in [0,1]$. After that, many scholars began to study the A_α-matrices of graphs. Nikiforov et al. [16] gave several results about the A_α-matrices of trees and gave the upper and lower bounds for the spectral radius of the A_α-matrices of arbitrary graphs. Let $\lambda_1(A_\alpha(H)) \geqslant \lambda_2(A_\alpha(H)) \geqslant \cdots \geqslant \lambda_n(A_\alpha(H))$ be the eigenvalues of $A_\alpha(H)$. Lin et al. [11] characterized the graph H with $\lambda_k(A_\alpha(H)) = \alpha n - 1$ for $2 \leqslant k \leqslant n$ and showed that $\lambda_n(A_\alpha(H)) \geqslant 2\alpha - 1$ if H contains no isolated vertices. Liu et al. [13] presented several upper and lower bounds on the k-th largest eigenvalue of A_α-matrix and characterized the extremal graphs corresponding to some of these obtained bounds. More results about A_α-matrix of a graph can be found in [8, 9, 10, 14, 17, 20]. Recently, Liu et al. [12] characterized the digraph which has the maximal A_α spectral radius in $\mathcal{G}_{n,r}$, where $\mathcal{G}_{n,r}$ is the set of digraphs of order n with dichromatic number r. Xi et al. [22] determined the digraphs which attain the maximum (or minimum) A_α spectral radius among all strongly connected digraphs with given parameters such as girth, clique number, vertex connectivity or arc connectivity. Xi and Wang [23] established some lower bounds on $\Delta^+ - \rho_\alpha(G)$ for strongly connected irregular digraphs with given maximum outdegree and some other parameters. Ganie and Baghipur [4] obtained some lower bounds for the spectral radius of $A_\alpha(G)$ in terms of the number of vertices, the number of arcs and the number of closed walks of the digraph G.

It is well-known that the energy of the adjacency matrix of a graph H first defined by Gutman [5] as $E_A(H) = \sum_{i=1}^n \nu_i$, where ν_i is an eigenvalue of the adjacency matrix of H. Peña and Rada [19] defined the energy of the adjacency matrix of a digraph G as $E_A(G) = \sum_{i=1}^n |\mathrm{Re}(z_i)|$, where z_i is an eigenvalue of the adjacency matrix of G and $\mathrm{Re}(z_i)$ is the real part of eigenvalue z_i. Some results about the energy of the adjacency matrices of graphs and digraphs have been obtained in [2, 3, 6]. Lazić [7] defined the Laplacian energy of a graph H as $LE(H) = \sum_{i=1}^n \mu_i^2$ by using second spectral moment, where μ_i is an eigenvalue of $L(H)$. Perera and Mizoguchi [18] defined the Laplacian energy $LE(G)$ of a digraph G as $LE(G) = \sum_{i=1}^n \lambda_i^2$ by using second spectral moment, where λ_i is an eigenvalue of $L(G)$. Yang and Wang [24] defined the signless Laplacian energy as $E_{SL}(G) = \sum_{i=1}^n q_i^2$ of a digraph G by using second spectral moment, where q_i

is an eigenvalue of $Q(G)$. In this paper, we study the A_α energy as $E_\alpha(G)=\sum_{i=1}^n \lambda_{\alpha i}^2$ of a digraph G by using second spectral moment, where $\lambda_{\alpha i}$ is an eigenvalue of $A_\alpha(G)$.

The arrangement of this paper is as follows. In Section 2, we introduce some special digraphs. In Section 3, we characterize the digraph which has the maximal A_α spectral radius in $\mathcal{G}_n^m$. In Section 4, we characterize the digraph which has the maximal (or minimal) A_α energy in $\mathcal{G}_n^m$.

2. Preliminaries

In this section, we will introduce some special digraphs.

Complete digraph:

Let $\overleftrightarrow{K}_n$ denote the complete digraph with n vertices in which two arbitrary distinct vertices $v_i, v_j \in \mathcal{V}(\overleftrightarrow{K}_n)$, there are arcs $(v_i, v_j) \in \mathcal{A}(\overleftrightarrow{K}_n)$ and $(v_j, v_i) \in \mathcal{A}(\overleftrightarrow{K}_n)$.

Out-star, in-star and star:

Let $\overrightarrow{K}_{1,n-1}$ be an out-star with n vertices which has one vertex with outdegree $n-1$ and other vertices with outdegree 0 (see $\overrightarrow{K}_{1,n-1}$ in Figure 1). Let $\overleftarrow{K}_{1,n-1}$ be an in-star with n vertices which has one vertex with indegree $n-1$ and other vertices with indegree 0 (see $\overleftarrow{K}_{1,n-1}$ in Figure 1). Let $\overleftrightarrow{K}_{1,n-1}$ be a star with n vertices which has one vertex with outdegree and indegree $n-1$ and other vertices with outdegree and indegree 1 (see $\overleftrightarrow{K}_{1,n-1}$ in Figure 1). The vertex with outdegree or indegree $n-1$ is called the centre of $\overrightarrow{K}_{1,n-1}$, $\overleftarrow{K}_{1,n-1}$ or $\overleftrightarrow{K}_{1,n-1}$.

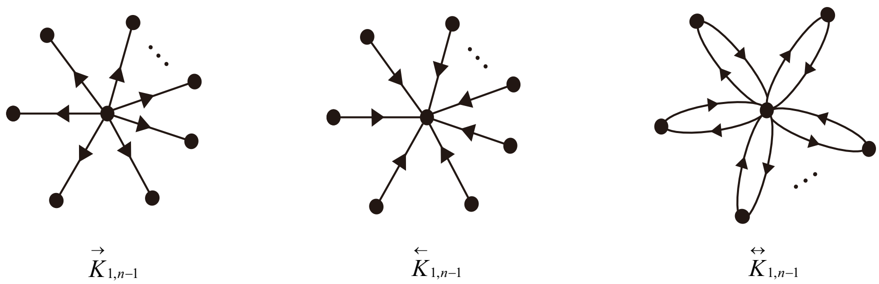

Figure 1: *An out-star* $\overrightarrow{K}_{1,n-1}$, *an in-star* $\overleftarrow{K}_{1,n-1}$ *and a star* $\overleftrightarrow{K}_{1,n-1}$.

In-tree:

Let in-tree be a directed tree with n vertices which the outdegree of each vertex of the directed tree is at most one. Then the in-tree has exactly one vertex with outdegree 0 and such vertex is called the root of the in-tree (see Figure 2).

Generalized ∞-digraph:

Let $\infty[m_1, m_2, \ldots, m_t]$ be a generalized ∞-digraph with $n=\sum_{i=1}^t m_i - t + 1$ $(m_i \geqslant 2)$ vertices which has t directed cycles C_{m_i} with exactly one common vertex (see $\infty[m_1, m_2, \ldots, m_t]$ in Figure 3).

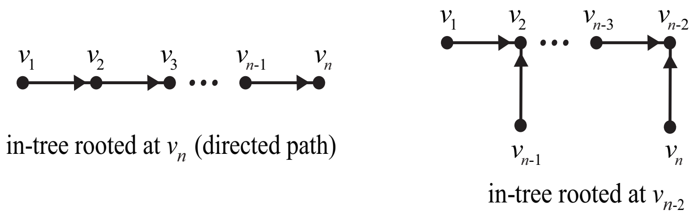

Figure 2: *Two different in-trees.*

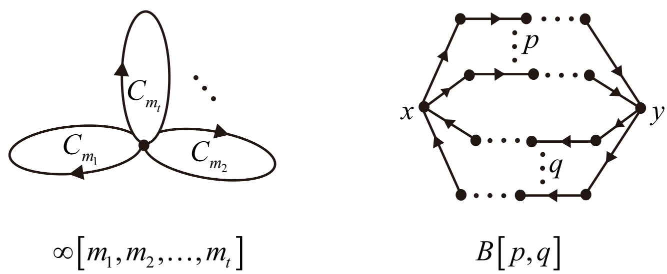

Figure 3: *A generalized ∞-digraph and a $(p+q)$-bispindle.*

p-spindle and $(p+q)$-bispindle:

A p-spindle with n vertices is the union of p internally disjoint (x,y)-directed paths for some vertices x and y. The vertex x is said to be the initial vertex of spindle and y its terminal vertex. A $(p+q)$-bispindle with n vertices is the internally disjoint union of a p-spindle with initial vertex x and terminal vertex y and a q-spindle with initial vertex y and terminal vertex x. Actually, it is the union of p (x,y)-directed paths and q (y,x)-directed paths. We denote the $(p+q)$-bispindle by $B[p,q]$ (see $B[p,q]$ in Figure 3).

The set of non-strongly connected digraphs $\mathcal{G}_n^m$:

Let $\mathcal{G}_n^m$ be the set of non-strongly connected digraphs with n vertices containing a unique strong component with m vertices and some directed trees hanging on each vertex of the strong component.

DEFINITION 2.1. Let G^* be a strong connected digraph with m vertices which $d^+_{G^*}(v_1) \geqslant d^+_{G^*}(v_2) \geqslant \cdots \geqslant d^+_{G^*}(v_m)$ is the outdegrees of vertices of G^*. Let T^i be the directed tree with n_i vertices, where $i = 1, 2, \ldots, m$ and $n = \sum_{i=1}^m n_i$. We give the digraphs G, G', G'' and G''' obtained by G^* and T^i as follow. (We take an example in Figure 4.)

(i) Let $G \in \mathcal{G}_n^m$ be a non-strongly connected digraphs with n vertices containing the unique strong component G^* with m vertices and some directed trees T^i hanging on each vertex of G^*, where $i = 1, 2, \ldots, m$ and $n = \sum_{i=1}^m n_i$. Then the vertex set of

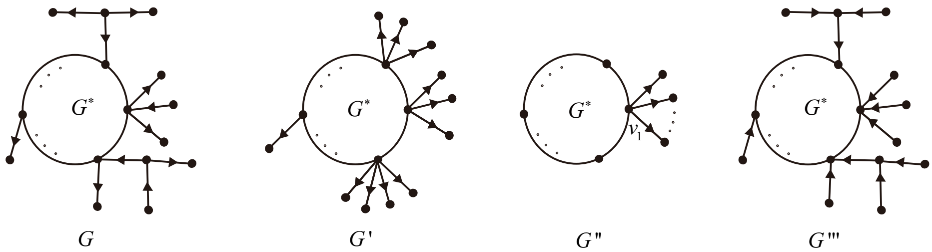

Figure 4: *The digraphs* $G, G', G'', G''' \in \mathcal{G}_n^m$.

G is $\mathcal{V}(G) = \bigcup_{i=1}^m \mathcal{V}(T^i)$, where $\mathcal{V}(T^i) = \{u_1^i, u_2^i, \ldots, u_{n_i}^i\}$, $\mathcal{V}(G^*) = \{v_1, v_2, \ldots, v_m\}$ and $v_i = u_1^i$, $i = 1, 2, \ldots, m$. Let $d_G^+(u_j^i)$ be the outdegree of vertex u_j^i of G, where $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n_i$.

(ii) Let

$$G' = G - \sum_{i=1}^{m}\sum_{s,t=1}^{n_i}(u_s^i, u_t^i) + \sum_{i=1}^{m}\sum_{j=2}^{n_i}(u_1^i, u_j^i),$$

where $(u_s^i, u_t^i) \in \mathcal{A}(G)$, $i = 1, 2, \ldots, m$ and $s, t, j = 1, 2, \ldots, n_i$. Then $G' \in \mathcal{G}_n^m$ is a non-strongly connected digraph which each directed tree T^i is an out-star $\overrightarrow{K}_{1,n_i-1}$ whose centre is v_i of G^*, where $i = 1, 2, \ldots, m$.

(iii) Let

$$\begin{aligned} G'' &= G - \sum_{i=1}^{m}\sum_{s,t=1}^{n_i}(u_s^i, u_t^i) + \sum_{i=1}^{m}\sum_{j=2}^{n_i}(u_1^1, u_j^i) \\ &= G' - \sum_{i=2}^{m}\sum_{j=2}^{n_i}(u_1^i, u_j^i) + \sum_{i=2}^{m}\sum_{j=2}^{n_i}(u_1^1, u_j^i), \end{aligned}$$

where $(u_s^i, u_t^i) \in \mathcal{A}(G)$, $i = 1, 2, \ldots, m$ and $s, t, j = 1, 2, \ldots, n_i$. Then $G'' \in \mathcal{G}_n^m$ is a non-strongly connected digraph which only has an out-star $\overrightarrow{K}_{1,n-m}$ whose centre is v_1 of G^*, where v_1 is the maximal outdegree vertex of G^*. Since the maximum outdegree vertex of G^* may not unique, the digraph G'' may not unique, too.

(iv) Let $G''' \in \mathcal{G}_n^m$ be a non-strongly connected digraph by changing each directed tree T^i of G to an in-tree whose root is v_i of G^*, where $i = 1, 2, \ldots, m$.

Digraphs K_n^m and C_n^m:

Let K_n^m be a non-strongly connected digraph with n vertices containing a complete digraph $\overleftrightarrow{K}_m$ and an out-star $\overrightarrow{K}_{1,n-m}$ with centre at any vertex of $\overleftrightarrow{K}_m$. Let C_n^m be a non-strongly connected digraph with n vertices containing a directed cycle C_m and some in-trees with roots at each vertex of C_m.

3. The maximal A_α spectral radius of non-strongly connected digraphs

In this section, we will consider the maximal A_α spectral radius of non-strongly connected digraphs in $\mathcal{G}_n^m$. First, we list some known results used for later.

DEFINITION 3.1. ([1]) Let $A=(a_{ij})$, $B=(b_{ij})$ be two $n\times n$ matrices. If $a_{ij}\leqslant b_{ij}$ for all i and j, then $A\leqslant B$. If $A\leqslant B$ and $A\neq B$, then $A<B$. If $a_{ij}<b_{ij}$ for all i and j, then $A\ll B$.

LEMMA 3.2. ([1]) *Let $A=(a_{ij})$, $B=(b_{ij})$ be two $n\times n$ matrices with the spectral radii $\rho(A)$ and $\rho(B)$, respectively. If $0\leqslant A\leqslant B$, then $\rho(A)\leqslant\rho(B)$. Furthermore, If $0\leqslant A<B$ and B is irreducible, then $\rho(A)<\rho(B)$.*

LEMMA 3.3. ([12]) *Let G be a digraph with the A_α spectral radius $\rho_\alpha(G)$ and maximal outdegree $\Delta^+(G)$. If H is a subdigraph of G, then $\rho_\alpha(H)\leqslant\rho_\alpha(G)$, especially, $\rho_\alpha(G)\geqslant\alpha\Delta^+(G)$. If G is strongly connected and H is a proper subdigraph of G, then $\rho_\alpha(H)<\rho_\alpha(G)$.*

Second, we give some lemmas to prove our main results.

LEMMA 3.4. *Let $G\in\mathcal{G}_n^m$ be a non-strongly connected digraph with $\mathcal{V}(G)=\{v_1,v_2,\ldots,v_n\}$. Let G^* be a unique strong component of G with $\mathcal{V}(G^*)=\{v_1,v_2,\ldots,v_m\}$. Let $\lambda_{\alpha 1},\lambda_{\alpha 2},\ldots,\lambda_{\alpha n}$ be the eigenvalues of $A_\alpha(G)$ and $d_1^+,d_2^+,\ldots,d_n^+$ be the outdegrees of vertices of G. Then*

$$\lambda_{\alpha i}=\alpha d_i^+,$$

for $i=m+1,m+2,\ldots,n$.

Proof. Let $A_\alpha(G)=\alpha D^+(G)+(1-\alpha)A(G)$ be the A_α-matrix of G. Let $\mathcal{V}(G)=\mathcal{V}_1\bigcup\mathcal{V}_2$ be the vertex set of G, where $\mathcal{V}_1=\mathcal{V}(G^*)=\{v_1,v_2,\ldots,v_m\}$ and $\mathcal{V}_2=\mathcal{V}(G-G^*)=\{v_{m+1},v_{m+2},\ldots,v_n\}$. According to the partition of vertex set of G, we partition $A_\alpha(G)$ into

$$A_\alpha(G)=\left(\begin{array}{c|c}A_{11}&A_{12}\\ \hline A_{21}&A_{22}\end{array}\right).$$

The characteristic polynomial $\phi_{A_\alpha(G)}(x)$ of G is $\phi_{A_\alpha(G)}(x)=|xI_n-A_\alpha(G)|$. Since the vertices of $\mathcal{V}_2$ are not on the strong component, there must exist a vertex with indegree 0 or outdegree 0. Then the elements of column or row of $A_\alpha(G)$ corresponding to that vertex are all 0, except the diagonal element. So by the property of determinant, we have $\phi_{A_\alpha(G)}(x)=|xI_n-A_\alpha(G)|=|xI_n-A_{11}|\prod_{i=m+1}^{n}(x-\alpha d_i^+)$. Hence $\lambda_{\alpha i}=\alpha d_i^+$, for $i=m+1,m+2,\ldots,n$. $\square$

With the above lemma, we can get a more general result.

COROLLARY 3.5. *Let G be an arbitrary digraph with n vertices. Let $\lambda_{\alpha 1}$, $\lambda_{\alpha 2},\ldots,\lambda_{\alpha n}$ be the eigenvalues of $A_\alpha(G)$ and $d_1^+,d_2^+,\ldots,d_n^+$ be the outdegrees of vertices of G. For any vertex v_i which is not on the strong components of G, we have*

$$\lambda_{\alpha i}=\alpha d_i^+.$$

LEMMA 3.6. *Let $G, G' \in \mathcal{G}_n^m$ be two non-strongly connected digraphs as defined in Definition 2.1. Then $\rho_\alpha(G') \geqslant \rho_\alpha(G)$.*

Proof. By the definition of G', we know $G' \in \mathcal{G}_n^m$ is a non-strongly connected digraph, which each directed tree T^i is an out-star $\overrightarrow{K}_{1,n_i-1}$ whose centre is v_i of G^*, where $i = 1,2,\ldots,m$. Then $d^+_{G'}(v_i) = d^+_{G'}(u^i_1) = d^+_{G^*}(v_i) + n_i - 1$, $d^+_{G'}(u^i_j) = 0$, where $i = 1,2,\ldots,m$ and $j = 2,3,\ldots,n_i$.

First, we consider the A_α-eigenvalues of G'. From Lemma 3.4, for the vertex u^i_j which is not on the strong component G^*, we have

$$\lambda_{\alpha G'}(u^i_j) = \alpha d^+_{G'}(u^i_j) = 0,$$

where $i = 1,2,\ldots,m$ and $j = 2,3,\ldots,n_i$. For the vertex $v_i = u^i_1$ which is on the strong component G^*, the A_α-eigenvalues $\lambda_{\alpha G'}(u^i_1)$ are equal to the eigenvalues of A'_{11}, where

$$A'_{11} = \alpha diag\left(d^+_{G^*}(v_1) + n_1 - 1, d^+_{G^*}(v_2) + n_2 - 1, \ldots, d^+_{G^*}(v_m) + n_m - 1\right) + (1-\alpha)A(G^*).$$

Obviously, $\rho_\alpha(G') = \rho(A'_{11})$.

Next, we consider the A_α-eigenvalues of G. From Lemma 3.4, for the vertex u^i_j which is not on the strong component G^*, we have

$$\lambda_{\alpha G}(u^i_j) = \alpha d^+_G(u^i_j),$$

where $i = 1,2,\ldots,m$ and $j = 2,3,\ldots,n_i$. For the vertex $v_i = u^i_1$ which is on the strong component G^*, the A_α-eigenvalues $\lambda_{\alpha G}(u^i_1)$ are equal to the eigenvalues of A_{11}, where

$$A_{11} = \alpha diag\left(d^+_G(v_1), d^+_G(v_2), \ldots, d^+_G(v_m)\right) + (1-\alpha)A(G^*).$$

Hence, $\rho_\alpha(G) = \max\limits_{1 \leqslant i \leqslant m, 2 \leqslant j \leqslant n_i} \left\{\rho(A_{11}), \alpha d^+_G(u^i_j)\right\}$.

Finally, we prove

$$\rho_\alpha(G') = \rho(A'_{11}) \geqslant \rho_\alpha(G) = \max_{1 \leqslant i \leqslant m, 2 \leqslant j \leqslant n_i} \{\rho(A_{11}), \alpha d^+_G(u^i_j)\}.$$

From Lemma 3.2, since

$$d^+_{G^*}(v_i) + n_i - 1 \geqslant d^+_G(v_i),$$

we have $A'_{11} \geqslant A_{11}$. Then $\rho(A'_{11}) \geqslant \rho(A_{11})$. From Lemma 3.3, we have

$$\rho_\alpha(G') \geqslant \alpha\Delta^+(G') \geqslant \alpha\Delta^+(G) \geqslant \alpha d^+_G(u^i_j).$$

Therefore, we have $\rho_\alpha(G') \geqslant \rho_\alpha(G)$. □

Finally, we give our main result.

THEOREM 3.7. *Among all digraphs in $\mathcal{G}_n^m$, K_n^m is the unique digraph which has the maximal A_α spectral radius.*

Proof. From the proof of Lemma 3.6, we know that $\rho_\alpha(G') = \rho(A'_{11}) \geqslant \rho_\alpha(G)$, where

$$A'_{11} = \alpha diag\left(d^+_{G^*}(v_1)+n_1-1, d^+_{G^*}(v_2)+n_2-1, \ldots, d^+_{G^*}(v_m)+n_m-1\right)+(1-\alpha)A(G^*).$$

When $G^* = \overleftrightarrow{K}_m$,

$$\overleftrightarrow{A'}_{11} = \alpha diag\,(m+n_1-2, m+n_2-2, \ldots, m+n_m-2)+(1-\alpha)A(\overleftrightarrow{K}_m).$$

From Lemmas 3.2 and 3.3, for the strong component G^*, we know that adding the arcs will increase the A_α spectral radius. So when $G^* = \overleftrightarrow{K}_m$, we have $\rho(\overleftrightarrow{A'}_{11}) \geqslant \rho(A'_{11}) = \rho_\alpha(G')$. Next we prove $\rho_\alpha(K_n^m) \geqslant \rho(\overleftrightarrow{A'}_{11})$.

Suppose that $\mathbf{x} = (x_1, x_2, \ldots, x_m)^T$ is a Perron vector of $\overleftrightarrow{A'}_{11}$ corresponding to $\rho(\overleftrightarrow{A'}_{11})$. We assume $x_t = \max\{x_i : i = 1, 2, \ldots, m\}$. Let

$$\overleftrightarrow{A''}_{11} = \alpha diag\Big(m-1, \ldots, m-1, \underbrace{n-1}_{t-th}, m-1, \ldots, m-1\Big)+(1-\alpha)A(\overleftrightarrow{K}_m).$$

Then we have

$$\begin{aligned}
\mathbf{x}^T\left(\overleftrightarrow{A''}_{11} - \overleftrightarrow{A'}_{11}\right)\mathbf{x} &= -\alpha\sum_{i\neq t}(n_i-1)x_i^2 + \alpha(n-m-n_t+1)x_t^2 \\
&= -\alpha\sum_{i\neq t}(n_i-1)x_i^2 + \alpha\sum_{i\neq t}(n_i-1)x_t^2 \\
&= \alpha\sum_{i\neq t}(n_i-1)\left(x_t^2 - x_i^2\right) \\
&\geqslant 0.
\end{aligned}$$

So $\rho(\overleftrightarrow{A''}_{11}) \geqslant \rho(\overleftrightarrow{A'}_{11})$.

Since K_n^m is a non-strongly connected digraph with n vertices containing a complete digraph $\overleftrightarrow{K}_m$ and an out-star $\overrightarrow{K}_{1,n-m}$ with centre at any vertex of $\overleftrightarrow{K}_m$, without loss of generality, let such vertex be v_t. Then $d^+_{K_n^m}(v_t) = d^+_{\overleftrightarrow{K}_m}(v_t)+n-m = n-1$, $d^+_{K_n^m}(u^t_j) = 0$ and $d^+_{K_n^m}(v_i) = d^+_{\overleftrightarrow{K}_m}(v_i) = m-1$, where $i = 1, \ldots, t-1, t+1, \ldots, m$ and $j = 2, 3, \ldots, n-m+1$. So we have $\rho_\alpha(K_n^m) = \rho(\overleftrightarrow{A''}_{11}) \geqslant \rho(\overleftrightarrow{A'}_{11})$. Hence, K_n^m is the unique digraph which has the maximal A_α spectral radius among all digraphs in $\mathcal{G}_n^m$. $\square$

REMARK 3.8. Let $G', G'' \in \mathcal{G}_n^m$ be two non-strongly connected digraphs as defined in Definition 2.1. If $\alpha = 0$, then $\rho_\alpha(G'') = \rho_\alpha(G')$. Actually, if the strong component G^* of G and n_i for $i = 1, 2, \ldots, m$ are fixed, can we get $\rho_\alpha(G'') \geqslant \rho_\alpha(G')$ for any $\alpha \in [0, 1)$?

4. The maximal (or minimal) A_α energy of non-strongly connected digraphs

In this section, we will consider the maximal (or minimal) A_α energy of non-strongly connected digraphs in $\mathcal{G}_n^m$. Firstly, we will introduce some basic concepts of A_α energy of digraphs.

Let $E_\alpha(G)$ be the A_α energy of a digraph G. By using second spectral moment, Xi [21] defined the A_α energy as $E_\alpha(G) = \sum_{i=1}^{n} \lambda_{\alpha i}^2$, where $\lambda_{\alpha i}$ is an eigenvalue of $A_\alpha(G)$. She also obtained the following result.

LEMMA 4.1. ([21]) *Let G be a connected digraph with n vertices. Let $d_1^+, d_2^+, \ldots, d_n^+$ be the outdegrees of vertices of G and c_2 be the number of all closed walks of length 2. Then*

$$E_\alpha(G) = \sum_{i=1}^{n} \lambda_{\alpha i}^2 = \alpha^2 \sum_{i=1}^{n} (d_i^+)^2 + (1-\alpha)^2 c_2.$$

From Lemma 4.1, we take the Example 4.2.

EXAMPLE 4.2. We give A_α energies of some special digraphs as follow:
(1) $E_\alpha(P_n) = \alpha^2(n-1)$;
(2) $E_\alpha(C_n) = \begin{cases} \alpha^2 n, & \text{if } n \geqslant 3, \\ 2\alpha^2 + 2(1-\alpha)^2, & \text{if } n = 2; \end{cases}$
(3) $E_\alpha(\overrightarrow{K}_{1,n-1}) = \alpha^2(n-1)^2$;
(4) $E_\alpha(\overleftarrow{K}_{1,n-1}) = \alpha^2(n-1)$;
(5) $E_\alpha(\overleftrightarrow{K}_{1,n-1}) = \alpha^2 n(n-1) + 2(1-\alpha)^2(n-1)$;
(6) $E_\alpha(\overleftrightarrow{K}_n) = \alpha^2 n(n-1)^2 + (1-\alpha)^2 n(n-1)$;
(7) $E_\alpha(\infty[m_1, m_2, \ldots, m_t]) = \alpha^2(t^2 + n - 1) + 2s(1-\alpha)^2$,
where $2 = m_1 \cdots = m_s < m_{s+1} \leqslant \cdots \leqslant m_t$;
(8) $E_\alpha(B[p,q]) = \begin{cases} \alpha^2(p^2+q^2+n-2) + 2(1-\alpha)^2, & \text{if } (x,y),(y,x) \in \mathcal{A}(B[p,q]), \\ \alpha^2(p^2+q^2+n-2), & \text{otherwise}; \end{cases}$
(9) $E_\alpha(K_n^m) = \alpha^2(n-1)^2 + \alpha^2(m-1)^3 + (1-\alpha)^2 m(m-1)$;
(10) $E_\alpha(C_n^m) = \begin{cases} \alpha^2 n, & \text{if } m \geqslant 3, \\ \alpha^2 n + 2(1-\alpha)^2, & \text{if } m = 2. \end{cases}$

LEMMA 4.3. ([21]) *Let T be a directed tree with n vertices. Then*

$$\alpha^2(n-1) \leqslant E_\alpha(T) \leqslant \alpha^2(n-1)^2.$$

Moreover, $E_\alpha(T) = \alpha^2(n-1)$ if and only if T is an in-tree with n vertices; $E_\alpha(T) = \alpha^2(n-1)^2$ if and only if T is an out-star $\overrightarrow{K}_{1,n-1}$ with n vertices.

Next, we give some lemmas to prove our main results.

LEMMA 4.4. *Let $G, G' \in \mathcal{G}_n^m$ be two non-strongly connected digraphs as defined in Definition 2.1. Then $E_\alpha(G') \geqslant E_\alpha(G)$ with equality holding if and only if $G \cong G'$.*

Proof. By the definition of G, we know $G \in \mathcal{G}_n^m$ is a non-strongly connected digraph with n vertices containing a unique strong component with m vertices and some directed trees hanging on each vertex of the strong component. From Lemma 4.3, we know the maximal A_α energy of T^i is

$$\big(E_\alpha(T^i)\big)_{\max} = \alpha^2 (n_i - 1)^2,$$

where $i = 1, 2, \ldots, m$. Then we have

$$\begin{aligned}
E_\alpha(G) &= \alpha^2 \sum_{i=1}^{m} \sum_{j=1}^{n_i} \big(d_G^+(u_j^i)\big)^2 + (1-\alpha)^2 c_2(G^*) \\
&= \alpha^2 \sum_{i=1}^{m} \big(d_{G^*}^+(u_1^i) + d_{T^i}^+(u_1^i)\big)^2 + \alpha^2 \sum_{i=1}^{m} \sum_{j=2}^{n_i} \big(d_G^+(u_j^i)\big)^2 + (1-\alpha)^2 c_2(G^*) \\
&= \alpha^2 \sum_{i=1}^{m} \Big(\big(d_{G^*}^+(v_i)\big)^2 + \big(d_{T^i}^+(u_1^i)\big)^2 + 2 d_{G^*}^+(v_i) d_{T^i}^+(u_1^i)\Big) \\
&\quad + \alpha^2 \sum_{i=1}^{m} \sum_{j=2}^{n_i} \big(d_{T^i}^+(u_j^i)\big)^2 + (1-\alpha)^2 c_2(G^*) \\
&= \alpha^2 \sum_{i=1}^{m} \big(d_{G^*}^+(v_i)\big)^2 + \alpha^2 \sum_{i=1}^{m} \sum_{j=1}^{n_i} \big(d_{T^i}^+(u_j^i)\big)^2 \\
&\quad + 2\alpha^2 \sum_{i=1}^{m} d_{G^*}^+(v_i) d_{T^i}^+(v_i) + (1-\alpha)^2 c_2(G^*) \\
&\leqslant \alpha^2 \sum_{i=1}^{m} \big(d_{G^*}^+(v_i)\big)^2 + \alpha^2 \sum_{i=1}^{m} (n_i - 1)^2 \\
&\quad + 2\alpha^2 \sum_{i=1}^{m} d_{G^*}^+(v_i)(n_i - 1) + (1-\alpha)^2 c_2(G^*) \\
&= \alpha^2 \sum_{i=1}^{m} \big(d_{G^*}^+(v_i) + (n_i - 1)\big)^2 + (1-\alpha)^2 c_2(G^*) \\
&= E_\alpha(G').
\end{aligned}$$

The equality holds if and only if

$$\sum_{j=1}^{n_i} \big(d_{T^i}^+(u_j^i)\big)^2 + 2 d_{G^*}^+(v_i) d_{T^i}^+(v_i) = (n_i - 1)^2 + 2 d_{G^*}^+(v_i)(n_i - 1),$$

for all $i = 1, 2, \ldots, m$. Anyway, the strong component G^* does not change, so $d_{G^*}^+(v_i)$ does not change. That is, $d_G^+(u_1^i) = d_{T^i}^+(v_i) = n_i - 1$, and $d_G^+(u_j^i) = 0$, where $i = 1, 2, \ldots, m$ and $j = 2, 3, \ldots, n_i$. Then each directed tree T^i is an out-star $\overrightarrow{K}_{1, n_i - 1}$.

Hence, we have $E_\alpha(G') \geqslant E_\alpha(G)$ with equality holding if and only if $G \cong G'$. $\square$

LEMMA 4.5. *Let $G', G'' \in \mathcal{G}_n^m$ be two non-strongly connected digraphs as defined in Definition 2.1. Then $E_\alpha(G'') \geqslant E_\alpha(G')$ with equality holding if and only if $G' \cong G''$.*

Proof. By the definition of G'', we know $G'' \in \mathcal{G}_n^m$ is a non-strongly connected digraph which only has an out-star $\overrightarrow{K}_{1,n-m}$ whose centre is v_1 of G^*, where v_1 is the maximal outdegree vertex of G^*. Then we have

$$E_\alpha(G'') = \alpha^2 \left(d^+_{G^*}(v_1) + n - m\right)^2 + \alpha^2 \sum_{i=2}^{m} (d^+_{G^*}(v_i))^2 + (1-\alpha)^2 c_2(G^*).$$

Since

$$\begin{aligned}
E_\alpha(G') &= \alpha^2 \sum_{i=1}^{m} \left(d^+_{G^*}(v_i) + (n_i - 1)\right)^2 + (1-\alpha)^2 c_2(G^*) \\
&= \alpha^2 \left(\sum_{i=1}^{m} (d^+_{G^*}(v_i))^2 + \sum_{i=1}^{m} (n_i-1)^2 + 2\sum_{i=1}^{m} d^+_{G^*}(v_i)(n_i-1) \right) + (1-\alpha)^2 c_2(G^*) \\
&\leqslant \alpha^2 \left(\sum_{i=1}^{m} (d^+_{G^*}(v_i))^2 + \left(\sum_{i=1}^{m} (n_i-1) \right)^2 + 2\sum_{i=1}^{m} d^+_{G^*}(v_1)(n_i-1) \right) \\
&\quad + (1-\alpha)^2 c_2(G^*) \\
&= \alpha^2 \left(\sum_{i=1}^{m} (d^+_{G^*}(v_i))^2 + (n-m)^2 + 2 d^+_{G^*}(v_1)(n-m) \right) + (1-\alpha)^2 c_2(G^*) \\
&= \alpha^2 \left(d^+_{G^*}(v_1) + n - m\right)^2 + \alpha^2 \sum_{i=2}^{m} (d^+_{G^*}(v_i))^2 + (1-\alpha)^2 c_2(G^*) \\
&= E_\alpha(G'').
\end{aligned}$$

The equality holds if and only if

$$\sum_{i=1}^{m} (n_i-1)^2 + 2\sum_{i=1}^{m} d^+_{G^*}(v_i)(n_i-1) = \left(\sum_{i=1}^{m} (n_i-1) \right)^2 + 2\sum_{i=1}^{m} d^+_{G^*}(v_1)(n_i-1).$$

Anyway, the strong component G^* does not change, so $d^+_{G^*}(v_i)$ does not change. That is, $n_i - 1 = 0$ for all $i = 2, 3, \ldots, m$ and $n_1 = n - m + 1$. Then the directed tree T^1 is an out-star $\overrightarrow{K}_{1,n-m}$, and each other directed tree is a vertex v_i, where $i = 2, 3, \ldots, m$.

Hence, we have $E_\alpha(G'') \geqslant E_\alpha(G')$ with equality holding if and only if $G' \cong G''$. □

Actually, since the maximum outdegree vertex of G^* may not unique, the digraph G'' may not unique, too. But by the property of A_α energy, it does not affect the value of A_α energy, we also have $E_\alpha(G'') \geqslant E_\alpha(G')$.

LEMMA 4.6. *Let $G, G''' \in \mathcal{G}_n^m$ be two non-strongly connected digraphs as defined in Definition 2.1. Then $E_\alpha(G) \geqslant E_\alpha(G''')$ with equality holding if and only if $G \cong G'''$.*

Proof. From Lemma 4.3, we know the minimal A_α energy of T^i is

$$\left(E_\alpha(T^i)\right)_{\min} = \alpha^2(n_i - 1),$$

where $i = 1, 2, \ldots, m$. Similar to the proof of Lemma 4.4, we can get the result easily. And

$$E_\alpha(G''') = \alpha^2 \sum_{i=1}^{m} (d_{G^*}^{+}(v_i))^2 + \alpha^2(n-m) + (1-\alpha)^2 c_2(G^*). \quad \square$$

From Lemmas 4.4–4.6, we have the following result.

COROLLARY 4.7. *Let $G, G'', G''' \in \mathcal{G}_n^m$ be non-strongly connected digraphs as defined in Definition 2.1. Then*

$$\alpha^2 \sum_{i=1}^{m} (d_{G^*}^{+}(v_i))^2 + \alpha^2(n-m) + (1-\alpha)^2 c_2(G^*) \leqslant E_\alpha(G)$$
$$\leqslant \alpha^2 \left(d_{G^*}^{+}(v_1) + n - m\right)^2 + \alpha^2 \sum_{i=2}^{m} (d_{G^*}^{+}(v_i))^2 + (1-\alpha)^2 c_2(G^*).$$

Moreover, the first equality holds if and only if $G \cong G'''$ and the second equality holds if and only if $G \cong G''$.

From Corollary 4.7, we can get bounds of A_α energies of some special non-strongly connected digraphs.

EXAMPLE 4.8. The bounds of A_α energies of special non-strongly connected digraphs $\widehat{U}_n^m$, $\widehat{\infty}[m_1, m_2, \ldots, m_t]$ and $\widehat{B}[p,q]$.

(i) Let $\widehat{U}_n^m \in \mathcal{G}_n^m$ be a unicyclic digraph with n vertices containing a unique directed cycle C_m and some directed trees hanging on each vertex of C_m, where $m \geqslant 2$. Then

$$2\alpha^2 + \alpha^2(n-2) + 2(1-\alpha)^2 \leqslant E_\alpha(\widehat{U}_n^2) \leqslant \alpha^2(n-1)^2 + \alpha^2 + 2(1-\alpha)^2,$$

and

$$\alpha^2 m + \alpha^2(n-m) \leqslant E_\alpha(\widehat{U}_n^m) \leqslant \alpha^2(n-m+1)^2 + \alpha^2(m-1) \ (m \geqslant 3).$$

Moreover, the first equality holds if and only if $\widehat{U}_n^m \cong C_n^m$; the second equality holds if and only if $\widehat{U}_n^m \in \mathcal{G}_n^m$ only has an out-star $\overrightarrow{K}_{1,n-m}$ whose centre is an any vertex of C_m.

(ii) Let $\widehat{\infty}[m_1, m_2, \ldots, m_t] \in \mathcal{G}_n^m$ be a generalized $\widehat{\infty}$-digraph with n vertices containing $\infty[m_1, m_2, \ldots, m_t]$ and some directed trees hanging on each vertex of $\infty[m_1,$

$m_2,\dots,m_t]$, where $2=m_1\cdots=m_s<m_{s+1}\leqslant\cdots\leqslant m_t$, $m=\sum_{i=1}^t m_i-t+1$ and the common vertex of t directed cycles C_{m_i} is v. Then

$$\alpha^2(m-1+t^2)+\alpha^2(n-m)+2s(1-\alpha)^2\leqslant E_\alpha(\widehat{\infty}[m_1,m_2,\dots,m_t])$$
$$\leqslant\alpha^2(n-m+t)^2+\alpha^2(m-1)+2s(1-\alpha)^2.$$

Moreover, the first equality holds if and only if each directed tree is an in-tree with root at each vertex of $\infty[m_1,m_2,\dots,m_t]$; the second equality holds if and only if $\widehat{\infty}[m_1,m_2,\dots,m_t]\in\mathcal{G}_n^m$ only has an out-star $\overrightarrow{K}_{1,n-m}$ whose centre is v.

(iii) Let $\widehat{B}[p,q]\in\mathcal{G}_n^m$ be a digraph with n vertices containing $B[p,q]$ and some directed trees hanging on each vertex of $B[p,q]$, where $\mathcal{V}(B[p,q])=m$ and $p\geqslant q$. If both (x,y) and (y,x) are arcs in $\widehat{B}[p,q]$, then

$$\alpha^2(m-2+p^2+q^2)+\alpha^2(n-m)+2(1-\alpha)^2\leqslant E_\alpha(\widehat{B}[p,q])$$
$$\leqslant\alpha^2(n-m+p)^2+\alpha^2(m-2+q^2)+2(1-\alpha)^2.$$

Otherwise,

$$\alpha^2(m-2+p^2+q^2)+\alpha^2(n-m)\leqslant E_\alpha(\widehat{B}[p,q])\leqslant\alpha^2(n-m+p)^2+\alpha^2(m-2+q^2).$$

Moreover, the first equality holds if and only if each directed tree is an in-tree with root at each vertex of $B[p,q]$; the second equality holds if and only if $\widehat{B}[p,q]\in\mathcal{G}_n^m$ only has an out-star $\overrightarrow{K}_{1,n-m}$ whose centre is x.

Finally, we give our main result.

THEOREM 4.9. *Among all digraphs in $\mathcal{G}_n^m$, K_n^m is the unique digraph which has the maximal A_α energy and C_n^m is the digraph which has the minimal A_α energy.*

Proof. From Lemmas 4.4–4.6, we have known $E_\alpha(G'')\geqslant E_\alpha(G')\geqslant E_\alpha(G)\geqslant E_\alpha(G''')$, if the strong component G^* of G and n_i for $i=1,2,\dots,m$ are fixed. By Corollary 4.7, we know $E_\alpha(G'')=\alpha^2\left(d^+_{G^*}(v_1)+n-m\right)^2+\alpha^2\sum_{i=2}^m(d^+_{G^*}(v_i))^2+(1-\alpha)^2c_2(G^*)$ and $E_\alpha(G''')=\alpha^2\sum_{i=1}^m(d^+_{G^*}(v_i))^2+\alpha^2(n-m)+(1-\alpha)^2c_2(G^*)$. Obviously, $m-1\geqslant d^+_{G^*}(v_i)\geqslant 1$ for all $i=1,2,\dots,m$ and $c_2(\overleftrightarrow{K}_m)\geqslant c_2(G^*)\geqslant c_2(C_m)$. So we get $E_\alpha(K_n^m)\geqslant E_\alpha(G'')$ and $E_\alpha(G''')\geqslant E_\alpha(C_n^m)$.

Hence among all digraphs in $\mathcal{G}_n^m$, K_n^m is the unique digraph which has the maximal A_α energy and C_n^m is the digraph which has the minimal A_α energy. □

REMARK 4.10. Since the in-trees of C_n^m is not unique, the minimal digraph of the lower bound of any $G\in\mathcal{G}_n^m$ is not unique, too. But by the property of A_α energy, we know the lower bound is unique. And

$$\alpha^2(n-1)^2+\alpha^2(m-1)^3+(1-\alpha)^2m(m-1)$$
$$\geqslant E_\alpha(G)\geqslant\begin{cases}\alpha^2n, & \text{if } m>2,\\ \alpha^2n+2(1-\alpha)^2, & \text{if } m=2.\end{cases}$$

5. Concluding

In this paper, we characterized the digraph which has the maximal A_α spectral radius and the maximal (or minimal) A_α energy in $\mathcal{G}_n^m$, where $\mathcal{G}_n^m$ is a special class of non-strongly connected digraphs with n vertices which contains a unique strong component with m vertices and some directed trees hanging on each vertex of the strong component. We want to further study the influence of the non-strongly connected part of the non-strongly connected digraph on the A_α spectral radius or A_α energy. Not just a directed tree, but an arbitrary acyclic digraph. We leave this as an open problem.

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