Operators
and
Matrices
Volume 18, Number 2 (2024), 361–374 doi:10.7153/oam-2024-18-22

POSITIVELY LIMITED p–CONVERGENT AND WEAK* POSITIVELY p–CONVERGENT OPERATORS

H. ARDAKANI* AND KH. AMJADI

(*Communicated by G. Misra*)

Abstract. The purpose of this article is to study two classes of operators, which we call positively limited p-convergent operators, and weak* positively p-convergent operators. We discuss the relationship between these two classes of operators, and other known classes of operators such as p-convergent operators, limited p-convergent operators, disjoint p-convergent operators, etc. Moreover, the positive DP* property of order p is studied, and the behavior of these two classes of operators on Banach lattices with this property (with focus on Banach lattices with the positively limited p-Schur property) is investigated. In addition, the domination properties of positively limited p-convergent operators, and weak* positively p-convergent operators on Banach lattices are considered.

1. Introduction and preliminaries

Throughout this paper E,F denote Banach lattices, X,Y denote Banach spaces. If A is a subset of a Banach space X, and for each weak*-null sequence (x_n^*) in X^*, $\lim_{n\to\infty}\sup_{a\in A}|\langle a,x_n^*\rangle| = 0$, then we say that A is *limited*. Each relatively compact set is limited [9, 12].

A subset A of a Banach lattice E is said to be *almost limited* if every disjoint weak*-null sequence (x_n^*) in E^* converges uniformly to zero on A. Each limited set is almost limited. B_{ℓ_∞} is an almost limited set which is not limited [6, 15].

A bounded set $A \subset E$ is *positively limited* if each positive weak*-null sequence (x_n^*) in E^* converges uniformly to zero on A. Each almost limited set is a positively limited set. Also, each order interval in a Banach lattice is positively limited [2].

A sequence $(x_n) \subset X$ is called *weakly p-summable*, where $1 \leqslant p < \infty$, if for each $x^* \in X^*$, $(x^*(x_n)) \in \ell_p$. Also $(x_n) \subset X$ is called *weakly p-convergent* to $x \in X$ if $(x_n - x) \in \ell_p^w(X)$, where $\ell_p^w(X)$ is the space of weakly p-summable sequences of X. For $p=\infty$, weakly p-convergent sequences are exactly weakly convergent sequences. A bounded set $A \subset X$ is called *relatively weakly p-compact* if each sequence in A has a weakly p-convergent subsequence. $A \subset X$ is *weakly p-compact* if the limit point is in A [13].

Mathematics subject classification (2020): 46B42, 46B40, 46A40, 47B65.
Keywords and phrases: Positively limited set, weakly p-compact set, p-convergent operator, p-Schur property, DP* property.

* Corresponding author.

Paper OaM-18-22

A Banach space X has the *limited p-Schur property* $(1 \leqslant p \leqslant \infty)$ if all limited weakly p-compact subsets of X are relatively compact or equivalently, every limited sequence $(x_n) \in \ell_p^w(X)$ is norm null [8].

Later, the concept of strong limited p-Schur property in Banach lattices was introduced. A Banach lattice E has the *strong limited p-Schur property* if each almost limited weakly p-compact subset of E is relatively compact. Each Banach lattice with the strong limited p-Schur property has the limited p-Schur property too [3].

Recently the concept of positively limited sets was defined and classes of Banach lattices with the positively limited p-Schur property were studied. A Banach lattice E has the *positively limited p-Schur property* if each positively limited weakly p-compact subset of E is relatively compact. Every Banach lattice with the positively limited p-Schur property has the strong limited p-Schur property too [4].

In this paper, at first the class of positively limited p-convergent operators is introduced, and some results of them are obtained. As an application, some characterizations of the positively limited p-Schur property, and the positive DP* property of order p of E are considered in terms of these operators.

Next, the weak* positively p-convergent operators are studied, and the relationships between them with the positively limited p-convergent operators, and the positive DP* property of order p of E are derived.

We also investigate the domination problem of positively limited p-convergent operators, and weak* almost p-convergent operators.

We recall some definitions, and notations. For a Banach lattice E, $E^+ = \{x \in E : x \geqslant 0\}$ refers to the positive cone of E. A subset A of E is called solid if $|x| \leqslant |y|$ for some $y \in A$ implies that $x \in A$. The solid hull of A is the set $Sol(A) = \{y \in E : |y| \leqslant |x|,$ for some $x \in A\}$. A norm bounded subset A of E is solid if $|x| \leqslant |y|$ for some $y \in A$ implies that $x \in A$. If for every weakly null sequence (x_n) in E, $|x_n| \xrightarrow{w} 0$, then the lattice operations are called weakly sequentially continuous. Also, if for every weak*-null sequence (x_n^*) in E^*, $|x_n^*| \xrightarrow{w^*} 0$, the lattice operations are called weak* sequentially continuous [1, 16].

All the concepts mentioned above and needed in this paper are collected:

1. A Banach lattice E has the:

   - *Schur property*, if each relatively weakly compact set in E is relatively compact.
   - *Gelfand-Phillips property (GP property)*, if each limited set in E is relatively compact.
   - *DP* property* if each relatively weakly compact set in E is limited [9].
   - *weak DP* property* if each relatively weakly compact set in E is almost limited [6].
   - *positive DP* property* if each relatively weakly compact set in X is positively limited [2].
   - *p-Schur property*, if every sequence $(y_n) \in \ell_p^w(E)$ is norm null [18].

- *property (d)*, if $|x_n^*| \xrightarrow{w^*} 0$ for every weak*-null disjoint sequence (x_n^*) in E^* [11].

2. An operator $T : E \to X$ is called

   (a) *completely continuous* if it carries weakly null sequences to norm null ones [1].

   (a) *p-convergent* if it carries weakly p-summable sequences to norm null ones [5].

   (d) *disjoint p-convergent* if it carries disjoint weakly p-summable sequences to norm null ones [18].

   (b) *limited p-convergent* (*abbr.* lpc) if it carries limited weakly p-summable sequences to norm null ones [8, 18].

   (c) *almost limited p-convergent* (*abbr.* alpc) if it carries almost limited weakly p-summable sequences to norm null ones [3].

3. An operator $T : X \to E$ is called *positively limited* if $T(B_X)$ is a positively limited set or equivalently, $\|T^*x_n^*\| \to 0$ for each positive weak*-null sequence (x_n^*) in E^* [2].

Throughout this article we assume that $1 \leqslant p < \infty$, unless otherwise stated.

2. Positively limited p-convergent operators

In this section, we consider a class of operators related to positively limited sets: positively limited p-convergent operators.

DEFINITION 2.1. A bounded linear operator $T : E \to X$ is positively limited p-convergent (*abbr.* plpc) if T carries weakly p-summable positively limited sequences of E to norm null sequences of X.

The set of all plpc operators from E into X is denoted by $\mathcal{L}_{plpc}(E,X)$. Clearly, $\mathcal{L}_{plpc}(E,X)$ is a linear subspace of $\mathcal{L}(E,X)$, where $\mathcal{L}(E,X)$ is the class of all bounded linear operators from E to X. Since each almost limited set is positively limited, a plpc operator is lpc and alpc. For the converse, we have the following which are the immediate consequences of [2, Theorem 2.5 & 2.7]:

1. If $T : E \to X$ is an lpc operator and E^* has the weak*-sequentially continuous lattice operations, then T is a plpc operator.

2. If $T : E \to X$ is an alpc operator and E has the property (d), then T is a plpc operator.

EXAMPLE 2.2. (a) The identity operator on each Banach lattice with the positively limited p-Schur property, and without the p-Schur property, such as c_0 is plpc, but it is not p-convergent.

(b) The identity operator on each Banach lattice with the limited p-Schur property, and without the positively limited p-Schur property, such as $L^1[0,1]$ is lpc, but it is not plpc for all $p \geqslant 2$.

(c) The identity operator on each Banach lattice with the strong limited p-Schur property, and without the positively limited p-Schur property, such as c is alpc, but it is not plpc.

It is trivial that TS is plpc if $F \xrightarrow{S} X \xrightarrow{T} Y$ where S is plpc and T is a bounded linear operator. It should be noted that order bounded operators between Banach lattices preserve positively limited sets, see [2, Theorem 2.11]. As a result, if $E \xrightarrow{T} F \xrightarrow{S} X$ where S is plpc and T is order bounded, then ST is also plpc. However, the following example shows that ST need not be plpc if T is not order bounded.

EXAMPLE 2.3. Let $T : L^1[0,1] \to c_0$ be defined as

$$Tf = \Big(\int_0^1 f(t) r_n(t)dt\Big)_{n=1}^{\infty}, \qquad \text{for all } f \in L^1[0,1],$$

where $r_n(t)$ is the n'th Rademacher function on $[0,1]$. The operator T is not order bounded. Indeed, $(r_n(t))_{n=1}^{\infty}$ is weakly p-summable, for all $2 \leqslant p \leqslant \infty$, and order bounded ($-\mathbf{1} \leqslant r_n \leqslant \mathbf{1}$, $n \in \mathbb{N}$), hence positively limited in $L^1[0,1]$, but $(Tr_n(t))_{n=1}^{\infty}$ is not order bounded in c_0. On the other hand, $\|Tr_n\| = 1$, $n \in \mathbb{N}$. Therefore T is not plpc. Note that the identity operator Id_{c_0} is plpc [2, Theorem 3.6]. However, $Id_{c_0}T = T$ is not plpc.

THEOREM 2.4. *An operator T on E is plpc if and only if for each positively limited weakly p-compact set $A \subseteq E$, the set $T(A)$ is relatively compact.*

Proof. To show that an operator T is plpc, we assume that (x_n) is a weakly p-summable positively limited sequence in E. For each subsequence of (x_n), which is denoted again by (x_n), the sequence (Tx_n) is relatively compact and so it has a convergent, and also weakly p-summable subsequence (Tx_{n_k}). Then $\|Tx_{n_k}\| \to 0$ as $k \to \infty$ which implies that T is plpc.

For the converse, let $A \subset E$ be a positively limited weakly p-compact set, and $T : E \to Y$ be a plpc operator. Then every sequence (x_n) in A has a weakly p-convergent subsequence, denoted again by (x_n). On the other hands, the difference set $A - A$ is positively limited. Hence the sequence $(x_n - x_m)$ is positively limited weakly p-summable, and by hypothesis (Tx_n) is Cauchy and so is norm convergent in Y. Thus $T(A)$ is relatively compact. $\square$

Note that, a plpc operator does not necessarily take positively limited sets to relatively compact sets.

EXAMPLE 2.5. An operator $R : C[0,1] \to c_0$ defined by

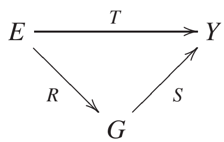
$$Rf = \Big(\int_0^1 f(t) r_n(t)dt\Big)_{n=1}^{\infty}, \qquad \text{for all } f \in L^1[0,1],$$

where r_n is the n'th Rademacher function on $[0,1]$. The operator R is weakly compact. By the Dunford-Pettis property of $C[0,1]$, R is completely continuous, and so it is plpc. However, R is not compact. Hence $R(B_{C[0,1]})$ is not a relatively compact set in c_0, while the closed unit ball $B_{C[0,1]}$ is a positively limited set.

Indeed, every plpc operator takes positively limited sets to relatively weakly compact sets.

THEOREM 2.6. *Every plpc operator $T : E \to Y$ carries positively limited sets to relatively weakly compact sets. In particular, every plpc operator is order weakly compact.*

Proof. Let (x_n) be an arbitrary order bounded disjoint sequence in E. Then (x_n) is weakly p-summable [3] and positively limited. Thus, $\|Tx_n\| \to 0$, since T is plpc. Hence T is order weakly compact. From (cf. [1, Theorem 5.58]), T admits a factorization through a Banach lattice G with order continuous norm



such that $R : E \to G$ is a lattice homomorphism. From [2, Theorem 2.11] for each positively limited set $B \subset E$, $R(B)$ is also positively limited in G, and so it is relatively weakly compact, see [2, Theorem 3.4]. Therefore, $T(B) = SR(B)$ is relatively weakly compact. $\square$

We can prove the following lemma. The proof is similar to the proof of Lemma 3.23 of [14].

LEMMA 2.7. *Let (x_n) be a positively limited weakly p-summable sequence in E. Then the operator $T : \ell_{p^*} \to E$, $T(b) = \sum_{n=1}^{\infty} b_n x_n$, $b = (b_n) \in \ell_{p^*}$ is positively limited $(1 < p < \infty)$.*

We then have the following composition result:

THEOREM 2.8. *Let $1 < p < \infty$, and $T : E \to Y$ be an operator. The following are equivalent:*

(a) T is a plpc operator,

(b) for every Banach space Z, and any positively limited, weakly p-compact operator $S : Z \to E$, the operator $TS : Z \to Y$ is compact,

(c) for every positively limited operator $S : \ell_{p^*} \to E$, the operator $TS : \ell_{p^*} \to Y$ is compact.

Proof. $(a) \Rightarrow (b)$ If $S : Z \to E$ is a positively limited, weakly p-compact operator, then $S(B_Z)$ is a positively limited weakly p-compact subset of E. Using the plpc-ness of T, and following the Theorem 2.4, we can prove that $TS(B_Z)$ is relatively compact, and so the operator TS is compact.

$(b) \Rightarrow (c)$ It follows easily from the fact that $Id(\ell_{p^*}) \in W_p$ (the class of weakly p-compact operators).

$(c) \Rightarrow (a)$ Let $(x_n) \in \ell_p^w(E)$ be a positively limited sequence. By Lemma 2.7 the operator $S : \ell_{p^*} \to E$ defined by $S(b) = \sum_{n=1}^{\infty} b_n x_n$, $b = (b_n) \in \ell_{p^*}$ such that $S(e_n) = x_n$ for all n is positively limited (note that, $\ell_p^w(E) = L(\ell_{p^*}, E)$). Hence the operator TS is compact and so $\|Tx_n\| = \|TS(e_n)\| \to 0$. □

As a consequence of Theorem 2.8, we obtain the following characterization:

COROLLARY 2.9. *Let $1 < p < \infty$. The following are equivalent:*

(a) E has the positively limited p-Schur property,

(b) for every Banach space Z, any positively limited weakly p-compact operator $S : Z \to E$ is compact,

(c) each positively limited operator $S : \ell_{p^*} \to E$ is compact.

The following theorem provides a characterization of the positively limited p-Schur property with respect to plpc operators.

THEOREM 2.10. *For a Banach lattice E, the following are equivalent:*

(a) E has the positively limited p-Schur property,

(b) for each Banach space Y, $\mathcal{L}_{plpc}(E,Y) = \mathcal{L}(E,Y)$,

(c) $\mathcal{L}_{plpc}(E,\ell_\infty) = \mathcal{L}(E,\ell_\infty)$.

Proof. $(a) \Rightarrow (b)$ Let $T : E \to Y$ be an operator and A be a positively limited weakly p-compact subset of E. By the positively limited p-Schur property of E, A and so $T(A)$ are relatively compact sets in Y. Then by Theorem 2.4, T is plpc.

$(b) \Rightarrow (c)$ It is obvious.

$(c) \Rightarrow (a)$ Assume by way of contradiction that E does not have the positively limited p-Schur property. Then, there is a weakly p-summable positively limited sequence (x_n) in E such that $\|x_n\| = 1$ for all n. Choose a normalized sequence (x_n^*) in E^* such that $|\langle x_n, x_n^* \rangle| = 1$ for all n. Then the operator $T : E \to \ell_\infty$ defined by

$$Tx = (\langle x, x_n^* \rangle), \qquad x \in E$$

is not plpc (since the sequence (x_n) is weakly p-summable and positively limited, and $\|Tx_n\| \geqslant 1$ for all n). This leads to a contradiction. □

Note that in Theorem 2.10 one cannot replace the positively limited p-Schur property of the domain E of related operators by their images. The Banach lattice c_0 has the positively limited p-Schur property, but the operator $T : L^1[0,1] \to c_0$ defined by

$$Tf = \left(\int_0^1 f(t)r_n(t)dt\right)_{n=1}^{\infty}, \quad \text{for all } f \in L^1[0,1],$$

is not plpc for all $p \geqslant 2$ (since the Rademacher sequence $f_n(t) = r_n(t)$ is weakly 2-summable and positively limited, but $\|Tf_n\| = 1$).

We can show that in order for the positively limited p-Schur property of the domain E of related operators to be replaced by their images in Theorem 2.10, it is sufficient for the operators to be positive.

PROPOSITION 2.11. *If $T : E \to F$ is a positive operator, and F has the positively limited p-Schur property, then T is plpc.*

Proof. Note that from [2], for each positively limited weakly p-compact set $A \subset E$, $T(A)$ is positively limited (and also weakly p-compact) in F. It follows from the positively limited p-Schur property of F that $T(A)$ is relatively compact. This proves that T is plpc. □

If $R : E \to F$ and $S : F \to X$ are two operators such that R is plpc, then SR is likewise plpc. In fact, if $(x_n) \in \ell_p^w(E)$ is a positively limited sequence, and R is an plpc operator, then $\|Rx_n\| \to 0$, and so $\|S(Rx_n)\| \to 0$. Consequently $SR : E \to X$ is plpc. However, if S is plpc, then SR is not necessarily plpc.

EXAMPLE 2.12. An operator $R : L^1[0,1] \to c_0$ defined by

$$Rf = \left(\int_0^1 f(t)r_n(t)dt\right)_{n=1}^{\infty}, \quad \text{for all } f \in L^1[0,1],$$

where r_n is the n'th Rademacher function on $[0,1]$, is not plpc. Consider an operator $S = Id_{c_0}$. Then S is plpc, but $SR = R$ is not plpc.

However, one easily verifies that if S is plpc and R is positive, then SR is plpc. It is enough to note that in this case, for each positively limited sequence $(x_n) \in \ell_p^w(E)$, (Rx_n) is a weakly p-summable and positively limited sequence. Hence plpc-ness of S ensures that $\|S(Rx_n)\| \to 0$. This proves that SR is plpc.

THEOREM 2.13. *Let E and F be two Banach lattices with F σ-Dedekind complete. If every bounded linear operator T from E into F is plpc, then either E or F has order continuous norm.*

Proof. If neither E nor F has order continuous norm, then there exists an order bounded disjoint sequence $(x_n)_n^\infty \subset E$ such that $\|x_n\| = 1$ for all $n \in \mathbb{N}$. Clearly, (x_n)

is positively limited and weakly p-summable. We choose a normalized sequence (x_n^*) in E^* such that $|\langle x_n, x_n^* \rangle| = 1$ for all $n \in \mathbb{N}$, and define the operator $T : E \to \ell_\infty$ by

$$Tx = (\langle x, x_n^* \rangle), \qquad x \in E.$$

Since F is a σ-Dedekind complete Banach lattice without order continuous norm, ℓ_∞ lattice embeds in F. Let $j : \ell_\infty \to F$ be the lattice embedding. Then it is easily verified $joT : E \to F$ is not plpc since $\|Tx_n\| \geqslant 1$ for all $n \in \mathbb{N}$. $\square$

REMARK 2.14. Example 2.12 implies that the converse of Proposition 2.13 does not necessarily hold. However, if E^* has the weak*-sequentially continuous lattice operations and either E or F has order continuous norm, then each operator T from E into F is plpc. Indeed, if either E or F has order continuous norm, then either E or F has the limited p-Schur property, and hence every operator $T : E \to F$ is lpc. On the other hand, since E^* has the weak*-sequentially continuous lattice operations, T is plpc.

Also, note that the σ-Dedekind completeness of a Banach lattice F cannot be removed. Each operator $T : \ell_\infty \to c$ is plpc, while neither ℓ_∞ nor c has order continuous norm.

Theorem 2.13 has the two following consequences.

COROLLARY 2.15. *If E^* has the weak* sequentially continuous lattice operations, Then the following are equivalent:*

(a) Each operator $T : E \to \ell_\infty$ is plpc.

(b) E has order continuous norm.

The order continuity of the norm is also characterized.

COROLLARY 2.16. *Let F be a σ-Dedekind complete Banach lattice. Then the following are equivalent:*

(a) Each operator $T : c \to F$ is plpc.

(b) F has order continuous norm.

Recently, the authors introduced the p-positive DP* property in [4]. A Banach lattice E has the *p-positive DP* property* if every relatively weakly p-compact subset of E is positively limited; alternatively, $x_n^*(x_n) \to 0$ for every weakly p-summable sequence (x_n), and every weak*-null sequence $(x_n^*) \subset (E^*)^+$. For our convenience, let $\mathcal{L}_{pc}(X,Y)$ denote the set of all p-convergent operators from X into Y. $\mathcal{L}_{pc}(E,X) \subset \mathcal{L}_{plpc}(E,X)$ holds.

THEOREM 2.17. *For a Banach lattice E the following assertions are equivalent.*

(a) E has the p-positive DP* property.

(b) for every Banach space X, $\mathcal{L}_{plpc}(E,X) = \mathcal{L}_{pc}(E,X)$.

(c) $\mathcal{L}_{plpc}(E,c_0) = \mathcal{L}_{pc}(E,c_0)$.

Proof. $(a) \Rightarrow (b)$ Let $T \in \mathcal{L}_{plpc}(E,X)$, and $(x_n)_{n=1}^{\infty}$ be a weakly p-summable sequence in E. Since E has the p-positive DP* property, (x_n) is positively limited in E and by hypothesis, $\|T(x_n)\| \to 0$. This implies that $T \in \mathcal{L}_{pc}(E,X)$

$(b) \Rightarrow (c)$ Obvious.

$(c) \Rightarrow (a)$ Let $(x_n)_{n=1}^{\infty}$ be a weakly p-summable sequence in E, and $(x_n^*)_{n=1}^{\infty}$ be a sequence in $(E^*)^+$ satisfying $x_n^* \xrightarrow{w^*} 0$. We define the positive operator $T : E \to c_0$ by

$$Tx = (x_n^*(x)), \qquad x \in E.$$

Since c_0 has the positively limited p-Schur property, and the operator T is positive, by Proposition 2.11, T is plpc. By hypothesis, $T \in \mathcal{L}_{pc}(E,c_0)$ and hence $\|Tx_n\| \to 0$. As a result, $x_n^*(x_n) \to 0$ since the inequality $|x_n^*(x_n)| \leqslant \|Tx_n\|$ holds for all $n \in \mathbb{N}$. This implies that E has the p-positive DP* property. $\square$

The p-positive DP* property is then also characterized by plpc operators as follows. For this, we need the following lemma of [4]. See more information of type and cotype in [10, Chapter 16].

LEMMA 2.18. *Suppose that E is a Banach lattice with the type q (with $1 < q \leqslant 2$), and $p \geqslant q'$. Then each disjoint sequence (z_n) in the solid hull of a bounded set $W \subset E$ is weakly p-summable.*

THEOREM 2.19. *Assume that the type of a Banach lattice E is q $(1 < q \leqslant 2)$, and $p \geqslant q'$. Then the following assertions are equivalent:*

(a) E has the p-positive DP* property,

(b) every plpc operator $T : E \to Y$ is disjoint p-convergent for every Banach space Y,

(c) every plpc operator $T : E \to c_0$ is disjoint p-convergent.

Proof. $(a) \Rightarrow (b)$ In this case, every plpc operator $T : E \to Y$ is p-convergent, for every Banach space Y.

$(b) \Rightarrow (c)$ It is clear.

$(c) \Rightarrow (a)$ Let $(x_n) \in \ell_p^w(E)$ be a disjoint sequence, and (x_n^*) be a positive weak*-null sequence in E^*. It is enough to show that $x_n^*(x_n) \to 0$, see [4, Theorem 3.6]. Consider the positive operator $T : E \to c_0$ defined by $Tx = (\langle x, x_n^* \rangle)$ for all $x \in E$. According to Proposition 2.11, T is plpc, and so it is disjoint p-convergent. Thus, $\|Tx_n\| \to 0$, and hence $x_n^*(x_n) \to 0$, as desired. $\square$

PROPOSITION 2.20. *Assume that E is a weak p-consistent Banach lattice, F is a Banach lattice and $S,T : E \to F$ are two positive operators satisfying $0 \leqslant S \leqslant T$. If T is a plpc operator, then S is a plpc operator.*

Proof. Let $(x_n) \in \ell_p^w(E)$ be a positively limited sequence. Since E is weak p-consistent, $(|x_n|) \in \ell_p^w(E)$ also holds in E. Also, $(|x_n|)$ is positively limited [2] and so $\|T|x_n|\| \to 0$. Our conclusion follows from the inequalities $|S(x_n)| \leqslant S|x_n| \leqslant T|x_n|$. □

If $0 \leqslant S \leqslant T : E \to F$ are two positive operators between Banach lattices and T is plpc , is then S necessarily plpc? We pose a question, and have to leave it open.

3. Weak* positively p-convergent operators

This section focuses on the so-called weak* positively p-convergent operators, and discuss some properties of them related to plpc operators, and the positively limited p-Schur property.

DEFINITION 3.1. An operator $T : X \to E$ is called *weak* positively p-convergent* if for each sequence $(x_n) \in \ell_p^w(X)$, and positive weak*-null sequence (x_n^*) in E^*, $x_n^*(Tx_n) \to 0$.

THEOREM 3.2. *Let $T : X \to E$ be an operator. The following are equivalent:*

(a) T is a weak* positively p-convergent operator,

(b) T maps each weakly p-compact set in X to a positively limited set in E,

(c) for each Banach space Z, and each weakly p-compact operator $S : Z \to X$, the operator TS is positively limited,

(d) for each operator $S : \ell_{p^*} \to X$, TS is positively limited.

Proof. $(a) \Rightarrow (b)$ Let W be a weakly p-compact set in X. We show that $T(W)$ is a positively limited set in E; that is, for each positive weak*-null sequence (x_n^*) in E^*, and each sequence (x_n) in W, $x_n^*(Tx_n) \to 0$. Since W is weakly p-compact, (x_n) has a weakly p-convergent subsequence to $x \in X$ which is denoted by (x_n) again. Then $(x_n - x)$, and so $(Tx_n - Tx)$ are weakly p-summable, and by the definition of weak* positively p-convergent operators, $x_n^*(Tx_n) = x_n^*(Tx_n - Tx) + x_n^*(Tx) \to 0$.

$(b) \Rightarrow (c)$ Obvious.

$(c) \Rightarrow (d)$ It is clear (since $B_{\ell_{p^*}}$ is a weakly p-compact set).

$(d) \Rightarrow (a)$. We show that for each positive weak*-null sequence (x_n^*) in E^*, and every sequence $(x_n) \in \ell_p^w(X)$, $x_n^*(Tx_n) \to 0$. An operator $S : \ell_{p^*} \to E$ defined by $S(b) = \sum_{n=1}^{\infty} b_n x_n$, $b = (b_n) \in \ell_{p^*}$ is weakly p-compact, and so TS is positively limited. Then $TS(B_{\ell_{p^*}})$ is a positively limited set which implies that $x_n^*(Tx_n) \to 0$. □

The following result characterizes the class of Banach lattices with the p-positive DP* property by weak* positively p-convergent operators.

COROLLARY 3.3. *Let E be a Banach lattice. Then the following are equivalent:*

(a) E has the p-positive DP* property,

(b) the identity operator on E is weak* positively p-convergent.

(c) every weakly p-compact operator T from an arbitrary Banach space Z to E is a positively limited operator.

(d) every bounded linear operator $S : \ell_{p^*} \to E$ is positively limited.

THEOREM 3.4. *Let E be a Banach lattice and $T : X \to E$ be an operator. Then the following are equivalent:*

(a) T is a weak* positively p-convergent operator,

(b) for each positive operator S from E to a Banach lattice F with the positively limited p-Schur property, the operator ST is p-convergent,

(c) for each positive operator $S : E \to c_0$ the operator ST is p-convergent.

Proof. $(a) \Rightarrow (b)$ Let W be a weakly p-compact set in X. By hypothesis (a), $T(W)$ is a positively limited and weakly p-compact set in E. Since $S : E \to F$ is positive, $ST(W)$ is a positively limited and weakly p-compact set in F. Hence by the positively limited p-Schur property of F, $ST(W)$ is relatively compact. Thus ST is p-convergent.

$(b) \Rightarrow (c)$ It is clear.

$(c) \Rightarrow (a)$ We show that for each positive weak* null sequence (x_n^*) in E^* and every weakly p-summable sequence (x_n) in X, $x_n^*(Tx_n) \to 0$. Consider the operator $S : E \to c_0$ defined by $Sx = (x_n^*(x))$, $x \in E$. Then S is a positive operator, and by hypothesis, ST is p-convergent. Thus, $\|S(T(x_n))\| \to 0$ which implies that $x_n^*(Tx_n) \to 0$. $\square$

We have immediate consequences of Theorem 3.4.

1. If $T : E \to F$ is weak* positively p-convergent and F has the positively limited p-Schur property, then T is p-convergent.
2. If $T : E \to F$ is weak* positively p-convergent, then for each Banach space Y and each plpc operator $S : F \to Y$, the operator $ST : E \to Y$ is p-convergent.

THEOREM 3.5. *Let $T : E \to F$ be an operator. The following are equivalent:*

(a) T is a weak* positively p-convergent operator,

(b) for each positive operator $S : F \to c_0$, the operator ST is p-convergent.

Proof. $(a) \Rightarrow (b)$ It is clear, since the operator $S : F \to c_0$ is plpc (Proposition 2.11).

$(b) \Rightarrow (a)$ We show that for each positive weak* null sequence (x_n^*) in F^* and every weakly p-summable sequence (x_n) in E, $x_n^*(Tx_n) \to 0$. Consider the operator $S : F \to c_0$ defined by $Sx = (x_n^*(x))$, $x \in F$. S is a positive operator. By hypothesis, ST is p-convergent, and so $\|S(T(x_n))\| \to 0$ which implies that $x_n^*(Tx_n) \to 0$. □

THEOREM 3.6. *Suppose that F is a σ-Dedekind complete Banach lattice with the weakly sequentially continuous lattice operations, and E is a Banach lattice. Then the following are equivalent:*

(a) Each weak* positively p-convergent operator $T : E \to F$ is plpc.

(b) E has the positively limited p-Schur property or F has order continuous norm.

Proof. $(a) \Rightarrow (b)$ Suppose that E does not have the positively limited p-Schur property and F does not have order continuous norm. Then there is a weakly p-summable and positively limited sequence (x_n) in E with $\|x_n\| > \varepsilon$, for some $\varepsilon > 0$. So, there is a sequence (x_n^*) in E^* such that $\|x_n^*\| = 1$ and $x_n^*(x_n) = \|x_n\|$. Consider the operator $S : E \to \ell_\infty$ defined by

$$Sx = (\langle x, x_n^* \rangle), \qquad x \in E.$$

Also, F is σ-Dedekind complete without order continuous norm and so there is a lattice embedding $i : \ell_\infty \to F$. Thus an operator $T := ioS : E \to F$ defined by $Tx = (\langle x, x_n^* \rangle)$ for all $x \in E$ is weak* (positively) p-convergent, but it is not plpc.

$(b) \Rightarrow (a)$ If E has the positively limited p-Schur property, then clearly each operator $T : E \to F$ is plpc. If F has order continuous norm and $T : E \to F$ is a weak* positively p-convergent operator, then we show that T is p-convergent. Let (x_n) be a weakly p-summable sequence in E. Then (Tx_n) is a weakly p-summable positively limited sequence in F, because T is weak* positively p-convergent. Since E has the weakly sequentially continuous lattice operations, and order continuous norm, it is discrete [7, Corollary 2.3]. By [2, Theorem 2.5], (Tx_n) is a weakly p-summable limited sequence in F. Since F has the GP property [17], $\|Tx_n\| \to 0$, and so the operator T is p-convergent. □

As some consequences of Theorem 3.6, we obtain two following characterizations. Note that ℓ_∞ is a σ-Dedekind complete Banach lattice with the weakly sequentially continuous lattice operations.

COROLLARY 3.7. *For a Banach lattice E, the following are equivalent:*

(a) each weak* positively p-convergent operator $T : E \to \ell_\infty$ is plpc.

(b) E has the positively limited p-Schur property.

COROLLARY 3.8. *Let F be a σ-Dedekind complete Banach lattice with the weakly sequentially continuous lattice operations. Then the following are equivalent:*

(a) each weak* positively p-convergent operator $T:\ell_\infty \to F$ is plpc,

(b) F has order continuous norm.

Following the same arguments as in the proof of Theorem 3.6, we conclude that if F is a σ-Dedekind complete Banach lattice with the weakly sequentially continuous lattice operations, then each weak* positively p-convergent operator $T:E\to F$ is p-convergent if and only if E has the p-Schur property or F has order continuous norm.

An operator $T:X\to Y$ is called *weak* p-convergent* if for each weakly p-summable sequence (x_n) in X and weak* null sequence (x_n^*) in Y^*, $x_n^*(Tx_n)\to 0$, see [18, Definition 4.1.1]. It is clear that each weak* p-convergent operator carries weakly p-compact sets into limited sets. If $T:X\to E$ is an operator and E has the positively limited p-Schur property, then it is easily verified that T is p-convergent if and only if T is weak* p-convergent if and only if T is weak* positively p-convergent.

PROPOSITION 3.9. *Suppose that $R:E\to F$ is a positive operator. If $S:X\to E$ is a weak* positively p-convergent operator, and $T:Y\to X$ is an operator. Then the composition operators $RST:Y\to F$ is a weak* positively p-convergent operator.*

Proof. Let W be a weakly p-compact set in Y. Then $T(W)$ is a weakly p-compact set in X, and so $S(TW)$ is a positively limited set in E (since S is a weak* positively p-convergent operator). Also, $R:E\to F$ is a positive operator and so $R(STW)$ is a positively limited set in F. Hence RST is a weak* positively p-convergent operator. □

We consider a result of the domination property of the weak* positively p-convergent operators.

THEOREM 3.10. *Let $T:E\to F$ be a weak* positively p-convergent operator. If E is weak p-consistent, and $0\leqslant S\leqslant T$. Then S is a weak* positively p-convergent operator.*

Proof. Let (x_n) be a weakly p-summable sequence in E and (x_n^*) be a positive weak* null sequence in F^*. By assumption, the sequence $(|x_n|)$ is weakly p-summable in E. Then

$$x_n^*(S(x_n))\leqslant x_n^*(S(|x_n|))\leqslant x_n^*(T(|x_n|))\to 0.$$

This proves that S is a weak* positively p-convergent operator. □

REFERENCES

[1] C. D. Aliprantis and O. Burkishaw, *Positive Operators*, Academic Press, New York, London, 1978.

[2] H. Ardakani and J. X. Chen, *Positively limited sets in Banach lattices*, J. Math. Anal. Appl. **526** (1) (2023), 127220, https://doi.org/10.1016/j.jmaa.2023.127220.

[3] H. Ardakani and Kh. Taghavinejad, *The strong limited p-Schur property in Banach lattices*, OaM. **16** (2022), 811–825.

[4] H. Ardakani and Kh. Amjadi, *The positively limited p-Schur property in Banach lattices*, Ann. Funct. Anal. **16** (2023), https://doi.org/10.1007/s43034-023-00292-y.

[5] J. Castillo and F. Sanchez, *Dunford-Pettis-like Properties of Continuous Vector Function Spaces*, Revista Mathematica, **6** (1993), 43–59.

[6] J. X. Chen, Z. L. Chen and G. X. Ji, *Almost limited sets in Banach lattices*, J. Math. Anal. Appl. **412** (2014), 547–563.

[7] Z. L. Chen and A. W. Wickstead, *Relative weak compactness of solid hulls in Banach lattices*, Indag. Mathem. N.S. **9** (1995), 187–196.

[8] M. B. Dehghani, S. M. Moshtaghioun and M. Dehghani, *On the limited p-Schur property of soe operator spaces*, IJAA. **16** (2018), 50–61.

[9] J. Diestel, *Sequences and Series in Banach Spaces*, Graduate Texts in Math. **92**, Springer–Verlag, Berlin, 1984.

[10] J. Diestel, *Absolutely Summing Operators*, Cambridge University Press, 1995.

[11] A. Elbour, *Some characterizations of almost limited operators*, Positivity **21** (2017), 865–874.

[12] G. Emmanuele, *On Banach spaces with the Gelfand–Phillips property, III*, J. Math. Pures Appl. **72** (1993), 327–333.

[13] J. H. Fourie and E. D. Zeekoei, *DP^*-properties of order p on Banach spaces*, Quaest. Math. **37** (2014), 349–358.

[14] I. Ghenciu, *Some classes of Banach spaces and complemented subspaces of operators*, Adv. Oper. Theory **4** (2019), 369–387.

[15] N. Machrafi, A. Elbour and M. Moussa, *Some characterizations of almost limited sets and applications* (2013), arXiv:1312.2770v1.

[16] P. Meyer-Nieberg, *Banach Lattices*, Universitext, Springer-Verlag, Berlin, 1991.

[17] W. Wnuk, *Banach Lattices with Order Continuous Norms*, Polish Scientific Publishers PWN, Warsaw, 1999.

[18] E. Zeekoei, *Classes of Dunford-Pettis-type operators with applications to Banach spaces and Banach lattices*, Ph.D. Thesis, 2017.

(Received October 5, 2023)

H. Ardakani
Department of Mathematics
Payame Noor University Tehran
Iran
e-mail: ardakani@pnu.ac.ir
halimeh_ardakani@yahoo.com

Kh. Amjadi
Department of Mathematics
Payame Noor University Tehran
Iran
e-mail: amjadi55@student.pnu.ac.ir