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STABILITY ANALYSIS OF TIME—INVARIANT PERTURBED SINGULAR SYSTEMS

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Abstract. This paper considers a class of linear time–invariant perturbed singular systems. The main aim of this paper is to develop the practical exponential stability of this class of systems based on Lyapunov techniques. Finally, to illustrate our results more clearly, we introduce a numerical example.

1. Introduction

Singular systems are those dynamics of which are governed by a mixture of algebraic and differential equations. In that sense, singular systems represent the constraints to the solution of the differential part. These systems are also called degenerate systems, generalized systems, descriptor systems, semi–state systems, and differential–algebraic systems.

In [22], Rosenbrock proposed singular systems for the first time and handled the transformation of linear singular systems. Later on, singular systems representation has been used as a perfect tool to model a wide variety of problems, such as electrical engineering, aircraft dynamics, robotics, economics, optimization problems, chemical, biology, etc.

The stability theory of differential systems is an active research topic. In [18], Lyapunov was the first who developed the problem of stability for systems of ordinary differential equations. Later on, different authors investigate the problem of stability of differential equations, see [1].

Because of the existence of algebraic equations, the investigation of singular systems is more complicated than standard ordinary differential equations. The complex nature of descriptor systems causes many difficulties in the analysis.

Owing to the difficulty resulting in analysis, few results are concerned with the stability of this class of systems.

The stability theory of linear time–invariant singular systems is an active research topic. Various authors attacked the problem of stability and stabilization of these systems, the interested reader is referred to [4, 5, 6, 11, 16, 17, 19, 23].

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The main objective of our manuscript is to develop the problem stability of linear time–invariant singular systems under perturbation based on the explicit solution form via Lyapunov techniques.

Indeed, the qualitative behavior of the solutions of linear time–invariant perturbed singular systems is analyzed by regarding the Lyapunov function candidate for the nominal system as an appropriate Lyapunov function candidate for the perturbed system.

Systems can innately show a perturbed structure where the solutions of unperturbed equations are in general supposed to be stable, and some restrictions are imposed on the uncertainties or disturbances like special growth conditions to derive conclusions about the behavior of solutions of the perturbed state equation.

Different authors have introduced the concept of practical stability. In such a situation, all state trajectories are bounded and approach a sufficiently small neighborhood of the origin. One also desires that the state approaches the origin (or some sufficiently small neighborhood of it) in a sufficiently fast manner especially in presence of perturbations, we mention here [2, 7, 8].

Our results are related to the relation between a perturbed linear time–invariant singular system and the associated unperturbed one. Given two solutions to the perturbed singular system and the associated unperturbed one with initial conditions that are close at the same value of time, these solutions will remain close over the entire time interval and not just at the initial time.

In [9], Caraballo et al. developed the problem of stabilization of stochastic non linear affine systems via Gamidov's inequality and based on the explicit solution form. The novelty of our paper is to analyze the problem of stability of linear time–invariant perturbed singular systems through Gamidov's inequality and based on Lyapunov techniques.

This paper is structured in the following way: In Section 2, we introduce some notations, definitions, and preliminaries lemmas about linear time–invariant singular systems, which will be needed in the sequel. In Section 3, we introduce a class of linear time–invariant perturbed singular system and we derive some new results on the stability via Gamidov's type inequality. In Section 4, we provide a numerical example to validate the effectiveness of our main result. Finally, some conclusions are included in Section 5.

Notations

R : Real vector space.

R_+ : the set of all nonnegative real numbers, i.e., $\mathrm{R}_+ = [0,\infty)$

$\mathcal{S}^n$: the set of $n \times n$ symmetric matrices

$\mathcal{C}$: Complex vector space.

I : Identity matrix.

$\mathrm{B} := (b_{ij}) \in \mathrm{R}^{n\times n}$: Real matrix.

B^{T} : Transpose of the matrix B.

$B > 0$: Positive definite matrix.

B^D : Drazin inverse of the matrix B.

$\aleph(B)$: Kernel of the matrix B.

$\mathrm{im}(B)$: Image of the matrix B.

$\lambda(B)$: Eigenvalue of the matrix B.

$\lambda_{\max}(B)$ $(\lambda_{\min}(B))$: The maximum (minimum) eigenvalue of a symmetric matrix B.

$||B|| := \sqrt{\lambda_{\max}(B^T B)}$: Euclidean matrix norm of B.

$||x|| := \sqrt{x^T x}$: Euclidean norm of $x \in R^n$.

2. Preliminaries

In this section, some needed preliminaries about linear time–invariant singular systems are introduced.

Consider the linear time–invariant singular system:

$$E\dot{x}(t) = Ax(t), \quad x(t_0) = x_0, \tag{2.1}$$

where $x(t) \in R^n$ is the system state vector, $x(t_0) = x_0 \in R^n$ is the initial condition.

$E, A \in R^{n\times n}$ are constant matrices, with E is a singular matrix and $\mathrm{rank}(E) = r < n$.

DEFINITION 2.1. The linear time–invariant singular system (2.1) is said to be:

1. Regular, if $\det(sE - A)$ is different from zero for certain $s \in \mathcal{C}$.
2. Impulse–free, if $\deg(\det(sE - A)) = \mathrm{rank}(E)$.

LEMMA 2.1. [4] *If the linear time–invariant singular system* (2.1) *is regular and impulse–free, then the solution of the system* (2.1) *exists, impulse–free and unique on* R.

The singularity of the matrix E will ensure that solutions of equation (2.1) exist only for particular choices of x_0. We will say that an initial condition $x_0 \in R^n$ is consistent if there exists a differentiable, continuous solution of (2.1). The problem of consistent initial conditions is not characteristic for the systems in the classical form but is a fundamental one for the singular systems. The analysis and generation of the subspace of consistent initial conditions $\mathcal{W}_{k^*}$ have received very much attention in the literature, we refer the reader to [4, 20].

Campbell [4] characterized the subspace of consistent initial conditions by the following theorem.

THEOREM 2.2. *The initial condition x_0 is a consistent initial condition for equation (2.1), if and only if:*

$$(\mathrm{I}-\check{\mathrm{E}}\check{\mathrm{E}}^{\mathrm{D}})x_0 = 0.$$

That is,

$$\mathcal{W}_{k^*} = \aleph(\mathrm{I}-\check{\mathrm{E}}\check{\mathrm{E}}^{\mathrm{D}}),$$

where $\check{\mathrm{E}}^{\mathrm{D}}$ is the Drazin inverse of the matrix $\check{\mathrm{E}}$ with $\check{\mathrm{E}} = (\lambda\mathrm{E}+\mathrm{A})^{-1}_{|\lambda=0}.\mathrm{E}$.

Later on, Ownes & Debeljkovic [20] proved that the radical geometric tool in the characterization of the subspace of consistent initial conditions $\mathcal{W}_{k^*}$ is the subspace sequence, as the following:

$$\mathcal{W}_0 \in \mathrm{R}^n,$$
$$\vdots$$
$$\mathcal{W}_{i+1} = \mathrm{A}^{-1}(\mathrm{E}\mathcal{W}_i), \quad i \geqslant 0.$$

LEMMA 2.3. [20] *The subsequence $\{\mathcal{W}_0, \mathcal{W}_1, \mathcal{W}_2, \cdots\}$ is nested in the sense that*

$$\mathcal{W}_0 \supset \mathcal{W}_1 \supset \mathcal{W}_3 \supset \cdots$$

Furthermore,

$$\aleph(\mathrm{A}) \subset \mathcal{W}_i, \quad \forall i \geqslant 0,$$

and there exists an integer $k \geqslant 0$, such that

$$\mathcal{W}_{k+1} = \mathcal{W}_k.$$

Then it is obvious that

$$\mathcal{W}_{k+i} = \mathcal{W}_k, \quad \forall i \geqslant 1.$$

If k^* is the smallest integer with this property, then

$$\mathcal{W}_k \cap \aleph(\mathrm{E}) = \{0\}, \quad k \geqslant k^*, \tag{2.2}$$

provided that $(\lambda\mathrm{E}-\mathrm{A})$ is invertible for some $\lambda \in \mathrm{R}$.

3. Stability analysis

Assume that some parameters of the linear time–invariant singular system (2.1) are excited or perturbed, and the perturbed singular system defined as

$$\mathrm{E}\dot{x}(t) = \mathrm{A}x(t) + \Pi f(t,x(t)), \quad x(0) = x_0, \tag{3.1}$$

where $\Pi \in \mathrm{R}^{n\times n}$ is a constant matrix such that $\mathrm{im}\Pi = \mathcal{W}_{k^*}$, and $f : \mathrm{R}_+ \times \mathrm{R}^n \to \mathrm{R}^n$ is continuous in (t,x), Lipschitz in x, uniformly in t.

REMARK 3.1. Systems can naturally show a perturbed structure where the solutions of unperturbed equations are in general supposed to be stable and some restrictions are imposed on the uncertainties or disturbances like special growth conditions in order to derive conclusions about the behavior of solutions of the perturbed state equation.

REMARK 3.2. The term Πf is a structured perturbation. In fact, it is necessary to contain the set of allowable perturbations to warranty consistency with the perturbed singular system (3.1), Πf is a structured perturbation that ensures "consistency". Thus,

$$\Pi f(t,x(t)) \in \mathcal{W}_{k^*}, \quad \text{for all } t \geqslant 0.$$

REMARK 3.3. Different authors handled the problem of stability of linear time–invariant singular systems $(\mathrm{E},\mathrm{A}) \in \mathrm{R}^{n\times n} \times \mathrm{R}^{n\times n}$, see [12, 13, 20]. It is well known that we look for solutions $\mathrm{P},\mathrm{Q} \in \mathrm{R}^{n\times n}$ such that for all $\mathrm{Q} \in \mathrm{R}^{n\times n}$, there exists $\mathrm{P} \in \mathrm{R}^{n\times n}$ solves the following equation:

$$\mathrm{A}^{\mathrm{T}}\mathrm{PE} + \mathrm{E}^{\mathrm{T}}\mathrm{PA} = -\mathrm{Q}, \tag{3.2}$$

and the corresponding Lyapunov function candidate is the following:

$$\mathcal{V} : \mathcal{W}_{k^*} \setminus \{0\} \to \mathrm{R}, \quad x \mapsto (\mathrm{E}x)^{\mathrm{T}}\mathrm{P}(\mathrm{E}x).$$

DEFINITION 3.1. [14] The perturbed singular system (3.1) is said to be uniformly exponentially stable, if there exist two positive constants α_1 and α_2, such that for $t_0 \in \mathrm{R}_+$, and x_0 a consistent initial condition,

$$||x(t)|| \leqslant \alpha_1 ||x_0|| e^{-\alpha_2(t-t_0)}, \quad \forall t \geqslant t_0.$$

We suppose the following assumption which is required for stability purposes.

$(\mathcal{H}_\infty)$ There exists a continuous non–negative known function $\theta(t)$, such that

$$\|f(t,x)\| \leqslant \theta(t)||x||, \quad \forall (t,x) \in \mathrm{R}_+ \times \mathcal{W}_{k^*},$$

where $\lim\limits_{t\to\infty} \theta(t) = 0$.

THEOREM 3.4. *The perturbed singular system* (3.1) *is uniformly exponentially stable, if there exists a positive definite symmetric matrix* P, *being the solution of Lyapunov matrix equation* (3.2)*, where the matrix* $\mathrm{Q} = \mathrm{Q}^{\mathrm{T}} > 0$ *such that*

$$x^{\mathrm{T}}\mathrm{Q}x > 0, \quad \forall x \in \mathcal{W}_{k^*} \setminus \{0\}, \tag{3.3}$$

where $\mathcal{W}_{k^*}$ *is the subspace of consistent initial conditions. Moreover, the perturbation term satisfies assumption* $(\mathcal{H}_1)$.

To prove our theorem, we need to recall the following technical lemma.

LEMMA 3.5. [10] *Consider* $X \in \mathbb{R}^{n\times m}$, $\widetilde{Y} \in \mathbb{R}^{n\times m}$, $\mathrm{P} \in \mathcal{S}^n$, *and* $m > 0$, *where* $\mathrm{P} > 0$, *we have*

$$X^{\mathrm{T}}\mathrm{P}\widetilde{Y} + \widetilde{Y}^{\mathrm{T}}\mathrm{P}X \leqslant mX^{\mathrm{T}}\mathrm{P}X + \frac{1}{m}\widetilde{Y}^{\mathrm{T}}\mathrm{P}\widetilde{Y}.$$

Proof of Theorem 3.4.. To establish sufficiency, note that Eq. (2.2) indicates that

$$\mathcal{V}(x) = x^{\mathrm{T}}\mathrm{E}^{\mathrm{T}}\mathrm{PE}x, \tag{3.4}$$

is a positive quadratic form on $\mathcal{W}_{k^*}$. Furthermore, all smooth solutions $x(t)$ evolve in $\mathcal{W}_{k^*}$.

As a result, $\mathcal{V}(x)$ can be used as a Lyapunov function for the perturbed singular system (3.1).

That is, there exist $\rho_1, \rho_2 > 0$, such that we have

$$\rho_1 x^{\mathrm{T}}x \leqslant x^{\mathrm{T}}\mathrm{E}^{\mathrm{T}}\mathrm{PE}x \leqslant \rho_2 x^{\mathrm{T}}x, \quad \forall t \geqslant 0, \quad \forall x \in \mathcal{W}_{k^*}\backslash\{0\}. \tag{3.5}$$

The total derivative of $\mathcal{V}(x(\cdot))$ along the trajectory of the singular perturbed system (3.1), is provided by the following:

$$\begin{aligned}
\dot{\mathcal{V}}(x(t)) &= \dot{x}^{\mathrm{T}}(t)\mathrm{E}^{\mathrm{T}}\mathrm{PE}x(t) + x^{\mathrm{T}}(t)\mathrm{E}^{\mathrm{T}}\mathrm{PE}\dot{x}(t)\\
&= (\mathrm{E}\dot{x}(t))^{\mathrm{T}}\mathrm{PE}x(t) + x^{\mathrm{T}}(t)\mathrm{E}^{\mathrm{T}}\mathrm{P}(\mathrm{E}\dot{x}(t))\\
&= (\mathrm{A}x(t) + \Pi f(t,x(t)))^{\mathrm{T}}\,\mathrm{PE}x(t) + x^{\mathrm{T}}(t)\mathrm{E}^{\mathrm{T}}\mathrm{P}\left(\mathrm{A}x(t) + \Pi f(t,x(t))\right)\\
&= x^{\mathrm{T}}(t)\left(\mathrm{A}^{\mathrm{T}}\mathrm{PE} + \mathrm{E}^{\mathrm{T}}\mathrm{PA}\right)x(t) + 2x^{\mathrm{T}}(t)\mathrm{E}^{\mathrm{T}}\mathrm{P}\Pi f(t,x(t))\\
&= -x^{\mathrm{T}}(t)\mathrm{Q}x(t) + 2x^{\mathrm{T}}(t)\mathrm{E}^{\mathrm{T}}\mathrm{P}\Pi f(t,x(t)),\\
&= -x^{\mathrm{T}}(t)\mathrm{Q}x(t) + (\mathrm{E}x(t))^{\mathrm{T}}\,\mathrm{P}\left(\Pi f(t,x(t)\right) + \left(\Pi f(t,x(t))\right)^{\mathrm{T}}\mathrm{P}\left(\mathrm{E}x\right).
\end{aligned}$$

From Lemma 3.5, we arrive at

$$\dot{\mathcal{V}}(x(t)) \leqslant -x^{\mathrm{T}}(t)\mathrm{Q}x(t) + m\,(\mathrm{E}x(t))^{\mathrm{T}}\,\mathrm{PE}x(t) + \frac{1}{m}\left(\Pi f(t,x(t))\right)^{\mathrm{T}}\mathrm{P}\left(\Pi f(t,x(t))\right).$$

Condition (3.3) yields that there exist positive constants q_1 and q_2, such that

$$q_1 x^{\mathrm{T}}x \leqslant x^{\mathrm{T}}\mathrm{Q}x \leqslant q_2 x^{\mathrm{T}}x, \quad \forall t \geqslant 0, \forall x \in \mathcal{W}_{k^*}\backslash\{0\}. \tag{3.6}$$

Based on Eq. (3.5), Eq. (3.6), and assumption $(\mathcal{H}_1)$, it derives that

$$\begin{aligned}
\dot{\mathcal{V}}(x(t)) &\leqslant -q_1 x^{\mathrm{T}}(t)x(t) + m\rho_2 x^{\mathrm{T}}(t)x(t) + \frac{\lambda_{\max}(\mathrm{P})}{m}||\Pi||^2||f(t,x(t))||^2\\
&\leqslant -q_1 x^{\mathrm{T}}(t)x(t) + m\rho_2 x^{\mathrm{T}}(t)x(t) + \frac{\theta^2(t)}{m}\lambda_{\max}(\mathrm{P})\lambda_{max}(\Pi^{\mathrm{T}}\Pi)x^{\mathrm{T}}(t)x(t).
\end{aligned}$$

Since $\lim\limits_{t\to\infty}\theta(t) = 0$, there exists $\widetilde{\theta}$, such that

$$\theta(t) \leqslant \widetilde{\theta}, \quad \forall t \geqslant 0. \tag{3.7}$$

Accordingly, we see

$$\dot{\mathcal{V}}(x(t)) \leqslant -\left(q_1 - m\rho_2 - \frac{\widetilde{\theta}^2}{m}\lambda_{\max}(\mathrm{P})\lambda_{max}(\Pi^{\mathrm{T}}\Pi)\right)x^{\mathrm{T}}(t)x(t) = -\zeta x^{\mathrm{T}}(t)x(t),$$

where $\zeta = q_1 - m\rho_2 - \dfrac{\widetilde{\theta}^2}{m}\lambda_{\max}(\mathrm{P})\lambda_{max}(\Pi^{\mathrm{T}}\Pi)$.

Without loss of generality, we may choose m such as

$$q_1 > m\rho_2 + \frac{\widetilde{\theta}^2}{m}\lambda_{\max}(\mathrm{P})\lambda_{\max}(\Pi^{\mathrm{T}}\Pi).$$

By virtue of (3.5), it follows

$$\dot{\mathcal{V}}(x(t)) \leqslant -\frac{\zeta}{\rho_2}(\mathrm{E}x(t))^{\mathrm{T}}\mathrm{P}(\mathrm{E}x(t)) = -\frac{\zeta}{\rho_2}\mathcal{V}(x(t)).$$

Integration over s from 0 to t, gives

$$\mathcal{V}(x(t)) \leqslant \mathcal{V}(x(0))\exp\left(-\frac{\zeta}{\rho_2}t\right).$$

Now, we are at a point to determine an estimate for the norm of $x(\cdot)$.

$$\rho_1||x(t)||^2 \leqslant x^{\mathrm{T}}(t)\mathrm{E}^{\mathrm{T}}\mathrm{PE}x(t) \leqslant \mathcal{V}(x(0))\exp\left(-\frac{\zeta}{\rho_2}t\right) \leqslant \rho_2||x_0||^2\exp\left(-\frac{\zeta}{\rho_2}t\right).$$

Finally, for all $t \geqslant 0$, and all consistent initial conditions x_0, we arrive at

$$||x(t)|| \leqslant \sqrt{\frac{\rho_2}{\rho_1}}||x_0||\exp\left(-\frac{\zeta}{2\rho_2}t\right).$$

That is, the singular perturbed system (3.1) is uniformly exponentially stable. □

Assume that there exists t such that $f(t,0) \neq 0$, i.e., the linear time–invariant perturbed singular system (3.1) does not have the trivial solution $x \equiv 0$.

DEFINITION 3.2. [14] The perturbed singular system (3.1) is said to be practically uniformly exponentially stable, if there exist two positive constants α_1, α_2, and $r > 0$ such that for $t_0 \in \mathrm{R}_+$, and x_0 a consistent initial condition the following inequality is:

$$\| x(t) \| \leqslant \alpha_1 \| x_0 \| \exp(-\alpha_2 t) + r, \quad \forall t \geqslant 0. \tag{3.8}$$

REMARK 3.6. Eq. (3.8) implies that $x(t)$ will be bounded by a small bound $r > 0$, thus $||x(t)||$ will be small for sufficiently large t. That is to say, the solution provided in Eq. (3.8) will be uniformly ultimately bounded for sufficiently large t. The factor α_2 in Eq. (3.8) is called the convergence speed, whereas the factor α_1 is called the transient estimate.

It is even worth seeing that, in the earlier definition, if we take $r = 0$, then we recover the standard concept of the uniform exponential stability of the origin viewed as an equilibrium point.

Next our principal purpose is to state sufficient conditions to provide the practical uniform exponential stability of the linear time–invariant perturbed singular system (3.1), under different restrictions are imposed on the uncertainties or disturbances. In fact, if we suppose that the perturbation term is bounded, then the origin is not necessarily an equilibrium point. For that reason, we will analyze the convergence of the solutions toward a neighborhood of origin.

$(\mathcal{H}_{\in})$ The perturbation term $f(t,x)$ satisfies, the following condition:

$$\|f(t,x)\| \leqslant \theta(t)||x|| + \delta(t), \quad \forall (t,x) \in \mathrm{R}_{+} \times \mathcal{W}_{k^*},$$

where $\lim\limits_{t\to\infty} \theta(t) = 0$, and $\delta(\cdot)$ is a continuous non–negative bounded function.

THEOREM 3.7. *The perturbed singular system* (3.1) *is practically uniformly exponentially stable, if there exists a positive definite symmetric matrix* P*, being the solution of Lyapunov matrix equation* (3.2)*, where the matrix* $\mathrm{Q} = \mathrm{Q}^{\mathrm{T}} > 0$ *satisfies* (3.3)*, and the perturbation term satisfies assumption* $(\mathcal{H}_2)$*.*

In order to prove this Theorem, we need to recall an important Gronwall–lemma.

LEMMA 3.8. [21] *Let* $g : [0,\infty) \longrightarrow [0,\infty)$ *be a continuous function,* ε *is a positive real number and* m *is a strictly positive real number. Assume that for all* $t \in [0,\infty)$ *and* $0 \leqslant u \leqslant t$*, we have*

$$g(t) - g(u) \leqslant \int_u^t (-mg(s) + \varepsilon) ds.$$

Then we obtain

$$g(t) \leqslant \frac{\varepsilon}{m} + g(0)\exp(-mt).$$

Proof of Theorem 3.7. We reconsider the Lyapunov function (3.4), and the total derivative of $\mathcal{V}(x(\cdot))$, where $x(\cdot)$ is a trajectory of the perturbed system (3.1) is given by the following:

$$\begin{aligned}
\dot{\mathcal{V}}(x(t)) &= \dot{x}^{\mathrm{T}}(t)\mathrm{E}^{\mathrm{T}}\mathrm{PE}x(t) + x^{\mathrm{T}}(t)\mathrm{E}^{\mathrm{T}}\mathrm{PE}\dot{x}(t) \\
&= x^{\mathrm{T}}(t)\left(\mathrm{A}^{\mathrm{T}}\mathrm{PE} + \mathrm{E}^{\mathrm{T}}\mathrm{PA}\right)x(t) + 2x^{\mathrm{T}}(t)\mathrm{E}^{\mathrm{T}}\mathrm{P}\Pi f(t,x(t)) \\
&= -x^{\mathrm{T}}(t)\mathrm{Q}x(t) + 2x^{\mathrm{T}}(t)\mathrm{E}^{\mathrm{T}}\mathrm{P}\Pi f(t,x(t)) \\
&= -x^{\mathrm{T}}(t)\mathrm{Q}x(t) + (\mathrm{E}x(t))^{\mathrm{T}}\mathrm{P}(\Pi f(t,x(t)) + (\Pi f(t,x(t)))^{\mathrm{T}}\mathrm{P}(\mathrm{E}x(t)).
\end{aligned}$$

Besides, it implies from Lemma 3.5 that

$$\dot{\mathcal{V}}(x(t)) \leqslant -x^{\mathrm{T}}(t)\mathrm{Q}x(t) + m(\mathrm{E}x(t))^{\mathrm{T}}\mathrm{PE}x(t) + \frac{1}{m}(\Pi f(t,x(t)))^{\mathrm{T}}\mathrm{P}(\Pi f(t,x(t))).$$

By Eq. (3.5) and Eq. (3.6), we arrive at

$$\dot{\mathcal{V}}(x(t)) \leqslant -q_1 x^{\mathrm{T}}(t)x(t) + m\rho_2 x^{\mathrm{T}}(t)x(t) + \frac{\lambda_{\max}(\mathrm{P})}{m}||\Pi||^2||f(t,x(t))||^2.$$

Using assumption ($\mathcal{H}_2$), it derives that

$$\dot{\mathcal{V}}(x(t)) \leqslant -q_1 x^{\mathrm{T}}(t)x(t) + m\rho_2 x^{\mathrm{T}}(t)x(t) + \frac{\lambda_{\max}(\mathrm{P})}{m}\lambda_{\max}(\Pi^{\mathrm{T}}\Pi)\left(\theta(t)||x(t)|| + \delta(t)\right)^2.$$

Based on the inequality, $(d+c)^n \leqslant 2^{n-1}(d^n + c^n)$, for all $d,c \geqslant 0,\ n \geqslant 1,$ we get

$$\dot{\mathcal{V}}(x(t)) \leqslant -q_1 x^{\mathrm{T}}(t)x(t) + m\rho_2 x^{\mathrm{T}}(t)x(t) + \frac{2}{m}\lambda_{\max}(\mathrm{P})\lambda_{\max}(\Pi^{\mathrm{T}}\Pi)\theta^2(t)x^{\mathrm{T}}(t)x(t)$$
$$+ \frac{2}{m}\lambda_{\max}(\mathrm{P})\lambda_{\max}(\Pi^{\mathrm{T}}\Pi)\delta^2(t).$$

Since $t \mapsto \delta(t)$ is a continuous non–negative bounded function, there exists $\bar{\delta} > 0,$ such that

$$\delta(t) \leqslant \bar{\delta}, \quad \forall t \geqslant 0. \tag{3.9}$$

By virtue of Eq. (3.7) and Eq. (3.9), it follows

$$\begin{aligned}\dot{\mathcal{V}}(x(t)) &\leqslant -q_1 x^{\mathrm{T}}(t)x(t) + m\rho_2 x^{\mathrm{T}}(t)x(t) + \frac{2}{m}\lambda_{\max}(\mathrm{P})\lambda_{\max}(\Pi^{\mathrm{T}}\Pi)\widetilde{\theta}^2 x^{\mathrm{T}}(t)x(t)\\ &\quad + \frac{2}{m}\lambda_{\max}(\mathrm{P})\lambda_{\max}(\Pi^{\mathrm{T}}\Pi)\bar{\delta}^2\\ &= -\left(q_1 - m\rho_2 - \frac{2}{m}\lambda_{\max}(\mathrm{P})\lambda_{\max}(\Pi^{\mathrm{T}}\Pi)\widetilde{\theta}^2\right)x^{\mathrm{T}}(t)x(t)\\ &\quad + \frac{2}{m}\lambda_{\max}(\mathrm{P})\lambda_{\max}(\Pi^{\mathrm{T}}\Pi)\bar{\delta}^2.\end{aligned}$$

We may take m, such as $q_1 > m\rho_2 + \frac{2}{m}\lambda_{\max}(\mathrm{P})\lambda_{\max}(\Pi^{\mathrm{T}}\Pi)\widetilde{\theta}^2.$

Set $\beta = q_1 - m\rho_2 - \frac{2}{m}\lambda_{\max}(\mathrm{P})\lambda_{\max}(\Pi^{\mathrm{T}}\Pi)\widetilde{\theta}^2$. We see that,

$$\dot{\mathcal{V}}(x(t)) \leqslant -\frac{\beta}{\rho_2}(\mathrm{E}x(t))^{\mathrm{T}}\mathrm{P}(\mathrm{E}x(t)) + \frac{2}{m}\lambda_{\max}(\mathrm{P})\lambda_{\max}(\Pi^{\mathrm{T}}\Pi)\bar{\delta}^2. \tag{3.10}$$

Integrating (3.10) from $u \in [0,t]$ to $t \geqslant 0,$ on both sides of the inequality, it yields

$$\mathcal{V}(x(t)) - \mathcal{V}(x(u)) \leqslant \int_u^t -\beta\mathcal{V}(x(s)) + \frac{2}{m}\lambda_{\max}(\mathrm{P})\lambda_{\max}(\Pi^{\mathrm{T}}\Pi)\bar{\delta}^2\, ds.$$

By Lemma 3.8, we obtain

$$\mathcal{V}(x(t)) \leqslant \mathcal{V}(x_0)\exp(-\beta t) + \frac{2\lambda_{\max}(\mathrm{P})\lambda_{\max}(\Pi^{\mathrm{T}}\Pi)\bar{\delta}^2}{\beta m}.$$

It then follows,

$$\begin{aligned}\rho_1||x(t)||^2 \leqslant x^{\mathrm{T}}(t)\mathrm{E}^{\mathrm{T}}\mathrm{P}\mathrm{E}x(t) &\leqslant \mathcal{V}(x_0)\exp(-\beta t) + \frac{2\lambda_{\max}(\mathrm{P})\lambda_{\max}(\Pi^{\mathrm{T}}\Pi)\bar{\delta}^2}{\beta m}\\ &\leqslant \rho_2||x_0||^2\exp(-\beta t) + \frac{2\lambda_{\max}(\mathrm{P})\lambda_{\max}(\Pi^{\mathrm{T}}\Pi)\bar{\delta}^2}{\beta m}.\end{aligned}$$

That is,

$$||x(t)||^2 \leqslant \frac{\rho_2}{\rho_1}||x_0||^2 \exp(-\beta t) + \frac{2\lambda_{\max}(\mathrm{P})\lambda_{\max}(\Pi^{\mathrm{T}}\Pi)\bar{\delta}^2}{\beta m \rho_1}.$$

Then for all $t \geqslant 0$, and all consistent initial conditions x_0,

$$||x(t)|| \leqslant \left(\frac{\rho_2}{\rho_1}\right)^{\frac{1}{2}} ||x_0|| \exp\left(-\frac{\beta}{2}t\right) + \frac{\sqrt{2\lambda_{\max}(\mathrm{P})}\bar{\delta}||\Pi||}{\sqrt{\beta m \rho_1}},$$

which means that the singular perturbed system (3.1) is practically uniformly exponentially stable. □

An extension can be done via Gamidov's type inequality, if we replace the assumption $(\mathcal{H}_2)$ by the following:

$(\mathcal{H}_2')$ Assume that, there exists a continuous non–negative known function $\theta'(t)$, such that

$$||f(t,x)|| \leqslant \theta'(t)||x||^q, \quad \forall (t,x) \in \mathrm{R}_+ \times \mathcal{W}_{k^*},$$

where $q \in]0,1[$ and $\theta'(t)$ is a known non–negative continuous function, with

$$\left(\int_0^{+\infty} \theta'(t)^2 dt\right)^{\frac{1}{2}} \leqslant \bar{\theta'} < +\infty,$$

for a certain non–negative constant $\bar{\theta'}$.

THEOREM 3.9. *The perturbed singular system* (3.1) *is practically uniformly exponentially stable, if there exists a positive definite symmetric matrix* P*, being the solution of Lyapunov matrix equation* (3.2)*, where the matrix* $\mathrm{Q} = \mathrm{Q}^{\mathrm{T}} > 0$ *satisfies* (3.3). *Moreover, the perturbation term satisfies assumption* $(\mathcal{H}_2')$.

Proof. Consider the following Lyapunov–like function:

$$\mathcal{V}(x) = x^{\mathrm{T}}\mathrm{E}^{\mathrm{T}}\mathrm{PE}x,$$

which is a positive quadratic form on $\mathcal{W}_{k^*}$.

The total derivative $\mathcal{V}(x(\cdot))$ with respect to time along the perturbed singular system (3.1) is given by the following:

$$\begin{aligned}\dot{\mathcal{V}}(x(t)) &= \dot{x}^{\mathrm{T}}(t)\mathrm{E}^{\mathrm{T}}\mathrm{PE}x(t) + x^{\mathrm{T}}(t)\mathrm{E}^{\mathrm{T}}\mathrm{PE}\dot{x}(t)\\ &= x^{\mathrm{T}}(t)\left(\mathrm{A}^{\mathrm{T}}\mathrm{PE} + \mathrm{E}^{\mathrm{T}}\mathrm{PA}\right)x(t) + 2x^{\mathrm{T}}(t)\mathrm{E}^{\mathrm{T}}\mathrm{P}\Pi f(t,x(t))\\ &= -x^{\mathrm{T}}(t)\mathrm{Q}x(t) + 2x^{\mathrm{T}}(t)\mathrm{E}^{\mathrm{T}}\mathrm{P}\Pi f(t,x(t)).\end{aligned}$$

Using Eq. (3.3) and assumption $(\mathcal{H'}_2)$, we arrive at

$$\begin{aligned}\dot{\mathcal{V}}(x(t)) &\leqslant -q_1||x(t)||^2 + 2||x(t)||\ ||\mathrm{E}||\ ||\mathrm{P}||\ ||\Pi||\ ||f(t,x(t))||\\ &\leqslant -q_1||x(t)||^2 + 2||\mathrm{E}||\ ||\mathrm{P}||\ ||\Pi||\ \theta'(t)\ ||x(t)||^{q+1}.\end{aligned}$$

By (3.5), it derives that

$$\dot{\mathcal{V}}(x(t)) \leqslant -\frac{q_1}{\rho_2}\mathcal{V}(x(t)) + \frac{2\lambda_{\max}(\mathrm{P})||\mathrm{E}||\,||\Pi||}{\rho_1^{\frac{q+1}{2}}}\theta'(t)\mathcal{V}^{\frac{q+1}{2}}(x(t)). \tag{3.11}$$

Setting, $\mu = \frac{q_1}{\rho_2} > 0,\ \rho(t) = \frac{2\lambda_{\max}(\mathrm{P})||\mathrm{E}||\,||\Pi||}{\rho_1^{\frac{q+1}{2}}}\theta'(t),$ we have

$$\dot{\mathcal{V}}(x(t)) \leqslant -\mu\mathcal{V}(x(t)) + \rho(t)\mathcal{V}^{\frac{q+1}{2}}(x(t)).$$

Let $u(t) = \mathcal{V}^{\frac{1-q}{2}}(x(t))$, it comes that

$$\dot{u}(t) = \frac{1-q}{2}\dot{\mathcal{V}}(x(t))\mathcal{V}^{-\frac{1+q}{2}}(x(t)). \tag{3.12}$$

Hence, when Eq. (3.11) and Eq. (3.12) are combined, it gives

$$\frac{1-q}{2}\dot{\mathcal{V}}(x(t))\mathcal{V}^{-\frac{1+q}{2}}(x(t)) \leqslant -\mu\frac{1-q}{2}\mathcal{V}^{\frac{1-q}{2}}(x(t)) + \frac{1-q}{2}\rho(t).$$

That is,

$$\dot{u}(t) \leqslant -\mu\frac{1-q}{2}u(t) + \frac{1-q}{2}\rho(t).$$

By using, the comparison lemma [15], one has

$$u(t) \leqslant u(0)e^{-\mu\frac{1-q}{2}t} + \frac{1-q}{2}e^{-\mu\frac{1-q}{2}t}\int_0^t \rho(s)e^{\mu\frac{1-q}{2}s}ds.$$

By Cauchy-Schwarz inequality, we then derive that

$$\begin{aligned}\rho_1^{\frac{1-q}{2}}||x(t)||^{1-q} &\leqslant \rho_2^{\frac{1-q}{2}}||x_0||^{1-q}e^{-\mu\frac{1-q}{2}t} \\ &\quad + \frac{1-q}{2}e^{-\mu\frac{1-q}{2}t}\left(\int_0^\infty \rho^2(s)ds\right)^{\frac{1}{2}}\left(\int_0^\infty e^{2\mu\frac{1-q}{2}s}ds\right)^{\frac{1}{2}}.\end{aligned}$$

That is, we obtain

$$\begin{aligned}\rho_1^{\frac{1-q}{2}}||x(t)||^{1-q} &\leqslant \rho_2^{\frac{1-q}{2}}||x_0||^{1-q}e^{-\mu\frac{1-q}{2}t} + (1-q)e^{-\mu\frac{1-q}{2}t}\frac{\lambda_{\max}(\mathrm{P})||\mathrm{E}||\,||\Pi||}{\rho_1^{\frac{q+1}{2}}}\bar{\theta'} \\ &\quad \times \frac{1}{(1-q)\mu}\left(e^{\mu(1-q)t}-1\right)^{\frac{1}{2}}.\end{aligned}$$

It is easily observed that,

$$\begin{aligned}||x(t)||^{1-q} &\leqslant \frac{\rho_2^{\frac{1-q}{2}}}{\rho_1^{\frac{1-q}{2}}}||x_0||^{1-q}e^{-\mu\frac{1-q}{2}t} + \frac{\bar{\theta'}}{\mu}\frac{\lambda_{\max}(\mathrm{P})||\mathrm{E}||\,||\Pi||}{\rho_1} \\ &= \frac{\rho_2^{\frac{1-q}{2}}}{\rho_1^{\frac{1-q}{2}}}||x_0||^{1-q}e^{-\mu\frac{1-q}{2}t} + \frac{\bar{\theta'}}{\mu}\frac{\lambda_{\max}(\mathrm{P})\sqrt{\lambda_{\max}(\mathrm{E}^{\mathrm{T}}\mathrm{E})}\sqrt{\lambda_{\max}(\Pi^{\mathrm{T}}\Pi)}}{\rho_1}.\end{aligned}$$

Thus, we obtain for all $t \geqslant 0$, and all consistent initial conditions x_0,

$$||x(t)|| \leqslant \frac{\sqrt{\rho_2}}{\sqrt{\rho_1}}||x_0||e^{-\mu\frac{1}{2}t} + \left(\frac{\bar{\theta'}}{\mu}\frac{\lambda_{\max}(\mathrm{P})\sqrt{\lambda_{\max}(\mathrm{E^TE})}\sqrt{\lambda_{\max}(\Pi^\mathrm{T}\Pi)}}{\rho_1^{\frac{p+3}{2}}}\right)^{\frac{1}{1-q}},$$

that is the singular perturbed system (3.1) is practically uniformly exponentially stable. □

$(\mathcal{H}_3)$ Assume that, the function $f(t,x)$ satisfies the following condition:

$$||f(t,x)|| \leqslant \Theta'(t)||x||^q + \varphi(t)||x||, \quad \forall (t,x) \in \mathrm{R}_+ \times \mathcal{W}_{k^*},$$

where $q \in]0,1[$, $\varphi(t)$ is a continuous non–negative bounded function, and $\Theta'(t)$ is a known non–negative continuous function, with

$$\left(\int_0^{+\infty} \Theta'(t)^2 dt\right)^{\frac{1}{2}} \leqslant \bar{\Theta'} < \infty,$$

for a certain non–negative constant $\bar{\Theta'}$.

THEOREM 3.10. *The perturbed singular system* (3.1) *is practically uniformly exponentially stable, if there exists a positive definite symmetric matrix* P, *being the solution of Lyapunov matrix equation* (3.2)*, where the matrix* $\mathrm{Q} = \mathrm{Q^T} > 0$ *satisfies* (3.3)*. Moreover, the perturbation term satisfies assumption* $(\mathcal{H}_3)$.

Proof. We consider the following Lyapunov function:

$$\mathcal{V}(x) = x^\mathrm{T}\mathrm{E^TPE}x,$$

which is a positive quadratic form on $\mathcal{W}_{k^*}$. The total derivative $\mathcal{V}(x(\cdot))$ with respect to time along the perturbed singular system (3.1) equal to the following:

$$\dot{\mathcal{V}}(x(t)) = -x^\mathrm{T}(t)\mathrm{Q}x(t) + 2x^\mathrm{T}\mathrm{E^TP}\Pi f(t,x(t)).$$

Using Eq. (3.3) and assumption $(\mathcal{H}_3)$, we see that

$$\begin{aligned}\dot{\mathcal{V}}(x(t)) &\leqslant -q_1||x(t)||^2 + 2||x(t)||\,||\mathrm{E}||\,||\mathrm{P}||\,||\Pi||\,||f(t,x(t))|| \\ &\leqslant -q_1||x(t)||^2 + 2||\mathrm{E}||\,||\mathrm{P}||\,||\Pi||\,\Theta'(t)\,||x(t)||^{q+1} + 2||\mathrm{E}||\,||\mathrm{P}||\,||\Pi||\,\varphi(t)\,||x(t)||^2.\end{aligned}$$

By (3.5), it follows that

$$\begin{aligned}\dot{\mathcal{V}}(x(t)) \leqslant -\frac{q_1}{\rho_2}\mathcal{V}(x(t)) &+ \frac{2\lambda_{\max}(\mathrm{P})||\mathrm{E}||\,||\Pi||}{\rho_1^{\frac{q+1}{2}}}\Theta'(t)\mathcal{V}^{\frac{q+1}{2}}(x(t)) \\ &+ \frac{2\lambda_{\max}(\mathrm{P})||\mathrm{E}||\,||\Pi||}{\rho_1}\varphi(t)\mathcal{V}(x(t)).\end{aligned}$$

Since, $\varphi(t)$ is a non–negative continuous bounded function, there exists $\bar{m}$ such that

$$\varphi(t) \leqslant \bar{m}, \quad \forall t \geqslant 0.$$

Thus, one obtains

$$\begin{aligned}
\dot{\mathcal{V}}(x(t)) &\leqslant -\frac{q_1}{\rho_2}\mathcal{V}(x(t)) + \frac{2\lambda_{\max}(\mathrm{P})||\mathrm{E}||\,||\Pi||}{\rho_1^{\frac{q+1}{2}}}\Theta'(t)\mathcal{V}^{\frac{q+1}{2}}(x(t)) \qquad (3.13)\\
&\quad + \frac{2\lambda_{\max}(\mathrm{P})||\mathrm{E}||\,||\Pi||}{\rho_1}\bar{m}\mathcal{V}(x(t))\\
&\leqslant -\left(\frac{q_1}{\rho_2} - \frac{2\lambda_{\max}(\mathrm{P})||\mathrm{E}||\,||\Pi||\bar{m}}{\rho_1}\right)\mathcal{V}(x(t)) \qquad (3.14)\\
&\quad + \frac{2\lambda_{\max}(\mathrm{P})||\mathrm{E}||\,||\Pi||}{\rho_1^{\frac{q+1}{2}}}\Theta'(t)\mathcal{V}^{\frac{q+1}{2}}(x(t)). \qquad (3.15)
\end{aligned}$$

Without loss of generality, we may assume that $\dfrac{q_1}{\rho_2} > \dfrac{2\lambda_{\max}(\mathrm{P})||\mathrm{E}||\,||\Pi||\bar{m}}{\rho_1}$.

Now, we set

$$\mu' = \frac{q_1}{\rho_2} - \frac{2\lambda_{\max}(\mathrm{P})||\mathrm{E}||\,||\Pi||\bar{m}}{\rho_1} > 0, \quad \rho_1(t) = \frac{2\lambda_{\max}(\mathrm{P})||\mathrm{E}||\,||\Pi||}{\rho_1^{\frac{q+1}{2}}}\Theta'(t).$$

Hence, we see

$$\dot{\mathcal{V}}(x(t)) \leqslant -\mu'\mathcal{V}(x(t)) + \rho_1(t)\mathcal{V}^{\frac{q+1}{2}}(x(t)).$$

Let $\mathrm{U}(t) = \mathcal{V}^{\frac{1-q}{2}}(x(t))$, it comes that

$$\dot{\mathrm{U}}(t) = \frac{1-q}{2}\dot{\mathcal{V}}(x(t))\mathcal{V}^{-\frac{1+q}{2}}(x(t)). \qquad (3.16)$$

Combining Eq. (3.13) and Eq. (3.16), one obtains

$$\frac{1-q}{2}\dot{\mathcal{V}}(x(t))\mathcal{V}^{-\frac{1+q}{2}}(x(t)) \leqslant -\mu\frac{1-q}{2}\mathcal{V}^{\frac{1-q}{2}}(x(t)) + \frac{1-q}{2}\rho_1(t).$$

That is,

$$\dot{\mathrm{U}}(t) \leqslant -\mu'\frac{1-q}{2}\mathrm{U}(t) + \frac{1-q}{2}\rho_1(t).$$

Thus, one has

$$\mathrm{U}(t) \leqslant \mathrm{U}(0)e^{-\mu'\frac{1-q}{2}t} + \frac{1-q}{2}e^{-\mu'\frac{1-q}{2}t}\int_0^t \rho_1(s)e^{\mu'\frac{1-q}{2}s}ds.$$

From the Cauchy-Schwarz inequality, it follows that

$$\begin{aligned}
\rho_1^{\frac{1-q}{2}}||x(t)||^{1-q} &\leqslant \rho_2^{\frac{1-q}{2}}||x_0||^{1-q}e^{-\mu'\frac{1-q}{2}t}\\
&\quad + \frac{1-q}{2}e^{-\mu'\frac{1-q}{2}t}\left(\int_0^\infty \rho_1^2(s)ds\right)^{\frac{1}{2}}\left(\int_0^\infty e^{2\mu'\frac{1-q}{2}s}ds\right)^{\frac{1}{2}}.
\end{aligned}$$

Hence, we obtain

$$\rho_1^{\frac{1-q}{2}}||x(t)||^{1-q} \leqslant \rho_2^{\frac{1-q}{2}}||x_0||^{1-q}e^{-\mu'\frac{1-q}{2}t} + (1-q)e^{-\mu'\frac{1-q}{2}t}\frac{\lambda_{\max}(\mathrm{P})||\mathrm{E}||\,||\Pi||}{\rho_1^{\frac{q+1}{2}}}\bar{\Theta}' \times \frac{1}{(1-q)\mu'}\left(e^{\mu'(1-q)t}-1\right)^{\frac{1}{2}}.$$

It is easily observed that,

$$\begin{aligned}||x(t)||^{1-q} &\leqslant \frac{\rho_2^{\frac{1-q}{2}}}{\rho_1^{\frac{1-q}{2}}}||x_0||^{1-q}e^{-\mu'\frac{1-q}{2}t} + \frac{\bar{\Theta}'}{\mu'}\frac{\lambda_{\max}(\mathrm{P})||\mathrm{E}||\,||\Pi||}{\rho_1}\\ &= \frac{\rho_2^{\frac{1-q}{2}}}{\rho_1^{\frac{1-q}{2}}}||x_0||^{1-q}e^{-\mu'\frac{1-q}{2}t} + \frac{\bar{\Theta}'}{\mu'}\frac{\lambda_{\max}(\mathrm{P})\sqrt{\lambda_{\max}(\mathrm{E}^{\mathrm{T}}\mathrm{E})}\sqrt{\lambda_{\max}(\Pi^{\mathrm{T}}\Pi)}}{\rho_1}.\end{aligned}$$

Thus, we obtain for all $t \geqslant 0$, and all consistent initial conditions x_0,

$$||x(t)|| \leqslant \frac{\sqrt{\rho_2}}{\sqrt{\rho_1}}||x_0||e^{-\mu'\frac{1}{2}t} + \left(\frac{\bar{\Theta}'}{\mu'}\frac{\lambda_{\max}(\mathrm{P})\sqrt{\lambda_{\max}(\mathrm{E}^{\mathrm{T}}\mathrm{E})}\sqrt{\lambda_{\max}(\Pi^{\mathrm{T}}\Pi)}}{\rho_1}\right)^{\frac{1}{1-q}},$$

which in turn gives that the singular perturbed system (3.1) is practically uniformly exponentially stable. □

4. Example

Consider the following perturbed singular system:

$$\mathrm{E}\dot{x}(t) = \mathrm{A}x(t) + \Pi f(t, x(t)), \tag{4.1}$$

where $x = (x_1, x_2, x_3) \in \mathrm{R}^3$.

$$\mathrm{E} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathrm{A} = \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & 0 \\ -1 & 0 & -1 \end{pmatrix}, \quad f(t,x) = \begin{pmatrix} f_1(t,x) \\ f_2(t,x) \\ f_3(t,x) \end{pmatrix},$$

with

$$\begin{cases} f_1(t,x) = 0 \\ f_2(t,x) = \dfrac{1}{\mathrm{ch}(t)} \\ f_3(t,x) = e^{-\Lambda t}\sqrt{|x_3|}, \quad \Lambda > 0. \end{cases}$$

The system (4.1) might be viewed as a perturbed singular system of the following linear time–invariant singular system:

$$\mathrm{E}\dot{x}(t) = \mathrm{A}x(t). \tag{4.2}$$

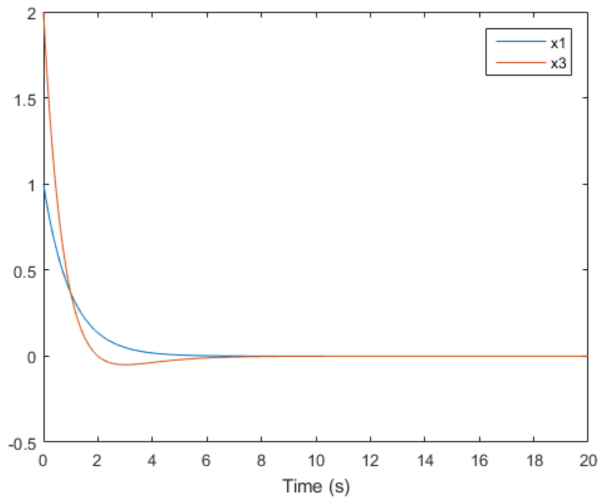

Figure 1: *The initial response of the nominal system* (4.2)*, with the initial condition* $x_0 = [1,\ 0,\ 2]^{\mathrm{T}}$

Note that, with this term of perturbation f, the fact that the function $\sqrt{|x_3|}$ is not Lipschitzian around zero does not pose a problem for the uniqueness of the solutions because our study is done outside a small ball centered at the origin.

In fact, we have $\det(z\mathrm{E}-\mathrm{A}) = (z+1)^2 \neq 0$ for some $z \in \mathcal{C}$, and $\deg\left(\det(z\mathrm{E}-\mathrm{A})\right) = \mathrm{rank}(\mathrm{E}) = 2$. Then the linear time–invariant singular system (4.1) is regular and impulse–free.

Our objective now is to find the subspace of consistent initial conditions within the method of Campbell. In fact, we have

$$\check{\mathrm{E}} = (\lambda\mathrm{E}+\mathrm{A})^{-1}_{|\lambda=0}.\,\mathrm{E},$$

$$\check{\mathrm{E}} = \mathrm{A}^{-1}\mathrm{E} = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}.$$

Additionally,

$$\lambda(\check{\mathrm{E}}) = \{0,\ -1,\ -1\}.$$

Applying the method of Campbell [3], one obtains

$$\check{\mathrm{E}}^{\mathrm{D}} = \check{\mathrm{E}}^2(3\mathrm{I}+2\check{\mathrm{E}}) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & -1 \end{pmatrix}.$$

Thus, it yields that

$$\breve{E}\breve{E}^{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and

$$\aleph\left(\mathrm{I} - \breve{E}\breve{E}^{D}\right) = (\mathrm{I} - \breve{E}\breve{E}^{D})x_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} x_0 = 0.$$

Then the subspace of consistent initial conditions is given by the following:

$$\aleph(\mathrm{I} - \breve{E}\breve{E}^{D}) = \mathcal{W}_{k^*} = \{x \in \mathrm{R}^3 : x_1 \in \mathrm{R},\ x_2 = 0,\ x_3 \in \mathrm{R}\}. \tag{4.3}$$

Let's consider the matrices P and Q in the general form:

$$\mathrm{P} = \begin{pmatrix} \rho_{1_1} & \rho_{1_2} & \rho_{1_3} \\ \rho_{1_2} & \rho_{2_2} & \rho_{2_3} \\ \rho_{1_3} & \rho_{2_3} & \rho_{3_3} \end{pmatrix} = \mathrm{P}^{\mathrm{T}}, \quad \mathrm{Q} = \begin{pmatrix} \mathrm{q}_{1_1} & \mathrm{q}_{1_2} & \mathrm{q}_{1_3} \\ \mathrm{q}_{1_2} & \mathrm{q}_{2_2} & \mathrm{q}_{2_3} \\ \mathrm{q}_{1_3} & \mathrm{q}_{2_3} & \mathrm{q}_{3_3} \end{pmatrix} = \mathrm{Q}^{\mathrm{T}}.$$

By a superficial calculation, one receives

$$\begin{aligned} \mathrm{A}^{\mathrm{T}}\mathrm{PE} + \mathrm{E}^{\mathrm{T}}\mathrm{PA} &= \begin{pmatrix} -1 & 0 & -1 \\ -1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \rho_{1_1} & \rho_{1_2} & \rho_{1_3} \\ \rho_{1_2} & \rho_{2_2} & \rho_{2_3} \\ \rho_{1_3} & \rho_{2_3} & \rho_{3_3} \end{pmatrix} \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & 0 \\ -1 & 0 & -1 \end{pmatrix} \\ &+ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \rho_{1_1} & \rho_{1_2} & \rho_{1_3} \\ \rho_{1_2} & \rho_{2_2} & \rho_{2_3} \\ \rho_{1_3} & \rho_{2_3} & \rho_{3_3} \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} -2\rho_{1_1} - 2\rho_{1_3} & -\rho_{1_1} - \rho_{1_2} & -2\rho_{1_3} - \rho_{3_3} \\ -\rho_{1_1} - \rho_{1_2} & 0 & -\rho_{1_3} - \rho_{2_3} \\ -2\rho_{1_3} - \rho_{3_3} & -\rho_{1_3} - \rho_{2_3} & -2\rho_{3_3} \end{pmatrix} = -\mathrm{Q}. \end{aligned}$$

That is, $\mathrm{q}_{2_2} = 0$, for $\rho_{1_2} = \rho_{2_3} = \rho_{1_3} = 0$, then the matrix Q is the following:

$$\mathrm{Q} = \begin{pmatrix} \mathrm{q}_{1_1} & \mathrm{q}_{1_2} & \mathrm{q}_{1_3} \\ \mathrm{q}_{1_2} & 0 & 0 \\ \mathrm{q}_{1_3} & 0 & \mathrm{q}_{3_3} \end{pmatrix},$$

where $\mathrm{q}_{1_1} = 2\rho_{1_1} \neq 0$, $\mathrm{q}_{1_2} = \rho_{1_1} \neq 0$, $\mathrm{q}_{1_3} = \rho_{3_3} \neq 0$, $\mathrm{q}_{3_3} = 2\rho_{3_3} \neq 0$.

We select P and Q, as follows:

$$\mathrm{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathrm{P}^{\mathrm{T}}, \quad \mathrm{Q} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 2 \end{pmatrix} = \mathrm{Q}^{\mathrm{T}}.$$

Then it yields

$$
\begin{aligned}
x^{\mathrm{T}}\mathrm{Q}x &= (x_1\ x_2\ x_3)\begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \\
&= \left(2x_1^2 + 2x_1x_2 + 2x_1x_3 + 2x_3^2\right)_{x_2=0} \\
&= 2x_1^2 + 2x_1x_3 + 2x_3^2 > 0, \quad \forall x \in \mathcal{W}_{k^*}\backslash\{0\}.
\end{aligned}
$$

Utilizing MATLAB, one obtains $q_1 = 1,\ q_2 = 3$.

On the other side, we have

$$
\begin{aligned}
\mathcal{V}(x) = x^{\mathrm{T}}\mathrm{E}^{\mathrm{T}}\mathrm{PE}x &= (x_1\ x_2\ x_3)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \\
&= x_1^2 + x_3^2 > 0, \quad \forall x \in \mathcal{W}_{k^*}\backslash\{0\}.
\end{aligned}
$$

Consequently, $\mathcal{V}(x)$ can be used as a Lyapunov function candidate for the system (4.1).

Based on the set of consistent initial conditions (4.3), we might choose Π as the following:

$$
\Pi = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}.
$$

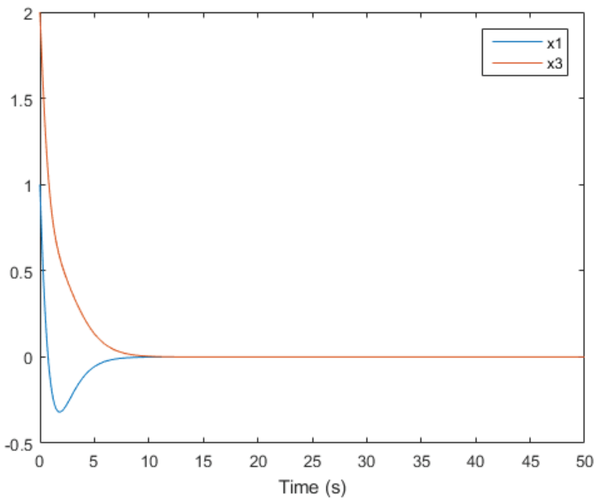

Figure 2: *The initial response of the perturbed singular system* (4.1), *with the initial condition* $x_0 = [1,\ 0,\ 3]^{\mathrm{T}}$

By easy calculation, one gets

$$\Pi f(t,x) = \begin{pmatrix} f_2(t,x) + f_3(t,x) \\ 0 \\ f_2(t,x) + f_3(t,x) \end{pmatrix}.$$

Thus, for all $t \geqslant 0$, and all $x \in \mathcal{W}_{k^*}$, we obtain

$$||f(t,x)|| \leqslant e^{-\Lambda t}\sqrt{|x_3|}.$$

It is clear that assumption $(\mathcal{H}_2')$ is satisfied with $\theta'(t) = e^{-\Lambda t}$. Then all assumptions of Theorem 3.9 are satisfied, thus the linear time–invariant perturbed singular system (4.1) is practically uniformly exponentially stable, as shown in Figure 2, for $\Lambda = 2$.

5. Conclusion

We managed to use Gamidov's type inequality to establish stability results. The proposed approach for stability analysis is relied on the bounds of perturbations that characterize the asymptotic convergence of the solutions to a closed ball centered at the origin. We developed an example to show the validity of our main findings.

Conflict of interest. The authors certify that they have no affiliations with or involvement in any organization or entity with any financial interest, or non financial interest in the subject matter or materials discussed in this manuscript.

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REFERENCES

[1] A. BENABDALLAH, W. HDIDI, *Improving the performance of semiglobal output controllers for nonlinear systems*, Kybernetika, **53** (2017), 296–330.

[2] A. BENABDALLAH, W. HDIDI, *Uniting two local output controllers for linear system subject to input saturation: LMI approach*, Journal of the Franklin Institute, **355** (2018), 6969–6991.

[3] S. L. CAMPBELL, C. D. MEYER, N. J. ROSE, *Applications of the Drazin inverse to linear systems of differential equations with singular constant coefficients*, SIAM Journal on Applied Mathematics, **31** (1976), 411–425.

[4] S. L. CAMPBELL, *Singular systems of differential equations*, Pitman, Marshfield, Mass, 1980.

[5] S. L. CAMPBELL, *Singular systems of differential equations II*, Pitman, Marshfield, Mass, 1982.

[6] D. CONG HUONG, N. HUU SAU, *Finite-time stability and stabilisation of singular discrete-time linear positive systems with time-varying delay*, International Journal of Systems Science, **50** (2019), 1–15.

[7] T. CARABALLO, F. EZZINE, M. A. HAMMAMI, *On the exponential stability of stochastic perturbed singular systems in mean square*, Applied Mathematics and Optimization, **3** (2021), 1–23.

[8] T. CARABALLO, F. EZZINE, M. A. HAMMAMI, *New stability criteria for stochastic perturbed singular systems in mean square*, Nonlinear Dynamics, **1** (2021), 241–256.

[9] T. CARABALLO, F. EZZINE, M. A. HAMMAMI, *A new result on stabilization analysis for stochastic nonlinear affine systems via Gamidov's inequality*, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A, Matemáticas, **4** (2022), 116–177.

[10] R. J. CAVERLYL, J. R. FORBES, *LMI Properties and Applications in Systems, Stability, and Control Theory*, arXiv:1903.08599, 2021.
[11] L. DAI, *Singular Control systems*, Springer, New York, 1989.
[12] D. LJ. DEBELJKOVIC, B. JOVANOVIĆ, V. DRAKULIĆ, *Singular system theory in chemical engineering theory: Stability in the sense of Lyapunov: A survey*, Hemijska Industrija, **6** (2001), 260–272.
[13] D. LJ. DEBELJKOVIC, I. BUZUROVIC, *Lyapunov stability of linear continuous singular systems: An overview*, International Journal of Information and System Sciences, **1** (2011), 247–268.
[14] F. EZZINE, M. A. HAMMAMI, *Growth conditions for the stability of a class of time–varying perturbed singular systems*, Kybernetika, **58** (2022), 1–24.
[15] H. K. KHALIL, *Nonlinear Systems*, Mac–Millan, 2nd edition, 1996.
[16] D. G. LUENBERGER, *Dynamic equation in descriptor form*, IEEE Transactions on Automatic Control, **22** (1977), 310–319.
[17] D. G. LUENBERGER, *Time–invariant descriptor systems*, Automatica, **14** (1978), 473–480.
[18] A. M. LYAPUNOV, *The general problem of the stability of motion*, International Journal of Control, **55** (1992), 521–790.
[19] J. MIN JIAO, *Robust stability and stabilization of discrete singular systems with interval time-varying delay and linear fractional uncertainty*, International Journal of Automation and Computing, **9** (2012), 1–8.
[20] D. H. OWNES, D. LJ. DEBELJKOVIĆ, *Stability of linear descriptor systems: A geometric analysis*, IMA Journal of Mathematical Control and Information, **2** (1985), 139–151.
[21] Q. C. PHAM, N. TABAREAU, J. E. SLOTINE, *A contraction theory approach to stochastic Incremental stability*, IEEE transcactions on Automatic Control, **54** (2009), 1285–1290.
[22] H. H. ROSENBROCK, *Structure properties of linear dynamical systems*, International Journal of Control, **20** (1974), 191–202.
[23] H. WANG, A. XUE, R. LU, *New stability criteria for singular systems with time–varying delay and nonlinear perturbations*, International Journal of Systems Science, **45** (2014), 1–15.

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