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THE OPERATOR EQUATION $AXB = X$ AND THE FUGLEDE–PUTNAM TYPE PROPERTY

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Abstract. In this paper, we study some connections between solutions A and B satisfying the operator equation $AXB = X$. We also investigate several properties between such solutions A and B. In particular, we show that if A has the single valued extension property, then so does B when X is injective. Moreover, we consider the (weak) Fuglede-Putnam type property (defined below) and investigate the local spectral properties between the solutions A and B under the Fuglede-Putnam type property.

1. Introduction

Let $\mathscr{L}(\mathscr{H})$ be the algebra of all bounded linear operators on a separable complex Hilbert space $\mathscr{H}$. If $A \in \mathscr{L}(\mathscr{H})$, we write $\sigma(A)$, $\sigma_{su}(A)$, $\sigma_p(A)$, and $\sigma_{ap}(A)$ for the spectrum, the surjective spectrum, the point spectrum, and the approximate point spectrum of A, respectively, while $r(A)$ denotes the spectral radius of A.

A subspace $\mathscr{M}$ of $\mathscr{H}$ is an *invariant subspace* under the operator A if $A\mathscr{M} \subseteq \mathscr{M}$. In addition, if both $\mathscr{M}$ and $\mathscr{M}^\perp$ are invariant subspaces for A, then we say $\mathscr{M}$ is a *reducing subspace* for A. The collection of all subspaces of $\mathscr{H}$ invariant under A is denoted by $LatA$. A *hyperinvariant subspace* for A is a subspace $\mathscr{M}$ of $\mathscr{H}$ such that $S\mathscr{M} \subseteq \mathscr{M}$ for every operator S which commutes with A. The collection of all subspaces of $\mathscr{H}$ hyperinvariant under A is denoted by $HLatA$.

An operator T in $\mathscr{L}(\mathscr{H})$ has the unique polar decomposition $T = U|T|$, where $|T| = (T^*T)^{\frac{1}{2}}$ and U is the appropriate partial isometry satisfying $ker(U) = ker(|T|) = ker(T)$ and $ker(U^*) = ker(T^*)$. Associated with T is a related operator $|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ called the *Aluthge transform* of T, denoted throughout this paper by $\tilde{T}$. In many cases, the Aluthge transforms of T have the better properties than T (see [14] and [15] for more details).

An operator $T \in \mathscr{L}(\mathscr{H})$ is said to be a *quasinormal* operator if T and T^*T commute. An operator $T \in \mathscr{L}(\mathscr{H})$ is said to be a p*-hyponormal* operator if $(T^*T)^p \geqslant$

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$(TT^*)^p$, where $0 < p < \infty$. If $p = 1$, T is called *hyponormal*. An operator $T \in \mathscr{L}(\mathscr{H})$ is said to be a *subnormal* operator if T has a normal extension which means that there exists a Hilbert space K such that H can be embedded in K and there exists a normal operator N such that $N|_{\mathscr{H}} = T$.

We next consider the following operator equation. This type of the operator equation has been studied by many authors (see [6], [8], [12], etc.)

Let $X \in \mathscr{L}(\mathscr{H})$ be given. If A and B in $\mathscr{L}(\mathscr{H})$ satisfy the operator equation $AXB = X$, then (A,B) is said to be a solution of the operator equation $AXB = X$.

For example, if X is a Toeplitz operator, then (U^*,U) is a solution of $U^*XU = X$ where U is the unilateral shift. Moreover, if X is a generalized Toeplitz operator with respect to given contractions A and B, then $AXB^* = X$ holds. Hence (A,B^*) is a solution of $AXB^* = X$. For another example, let T be a contraction, i.e., $\|T\| \leqslant 1$, on a complex Hilbert space $\mathscr{H}$. Since the sequence $\{T^{*n}T^n\}$ is monotonically decreasing, it converges strongly to a positive contraction X. Hence $T^*XT = X$ holds, and then (T^*,T) is a solution of $T^*XT = X$ (see [6] for more details). We next consider other example. Let $X = U$ be the unilateral shift and W_α be the weighted shift defined by $W_\alpha e_n = \alpha_n e_{n+1}$ for $\alpha_n > 0$, $n = 1,2,\ldots$. Then $W_\alpha^* U W_\beta = U$ if and only if for all $n = 1,2,\ldots$,

$$W_\alpha^* U W_\beta e_n = \beta_n \overline{\alpha_{n+1}} e_{n+1} = e_{n+1} = U e_n.$$

Hence (W_α^*, W_β) is a solution of $W_\alpha^* U W_\beta = U$ if and only if $\beta_n \overline{\alpha_{n+1}} = 1$ for all $n = 1,2,\ldots$.

We next consider the generalized derivation type. Define $\Delta_{A,B} : \mathscr{L}(\mathscr{H}) \to \mathscr{L}(\mathscr{H})$ by $\Delta_{A,B}(X) = AXB - X$. Then $\Delta_{A,B}^2(X) = A\Delta_{A,B}(X)B - \Delta_{A,B}(X)$. By the induction, we get that

$$\Delta_{A,B}^n(X) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} A^{n-k} X B^{n-k}.$$

In particular, if $A = B^*$, $X = I$, and $\Delta_{A,B}^n(X) = 0$, then B is an n-isometry.

We next define the (weak) Fuglede-Putnam type property ((W)FPT). We say that (A,B) satisfies the weak Fuglede-Putnam type property (WFPT) if $\Delta_{A^*,B^*}(Y) = 0$ for some nonzero Y in $\mathscr{L}(\mathscr{H})$ whenever $\Delta_{A,B}(X) = 0$ for some nonzero X in $\mathscr{L}(\mathscr{H})$. In particular, if $Y = X$, we say that (A,B) satisfies the Fuglede-Putnam type property (FPT) with X.

For example, let U be the unilateral shift defined by $Ue_n = e_{n+1}$ where $\{e_n\}$ is an orthonormal basis for $\mathscr{H}$. Set $X = \begin{pmatrix} 0 & 0 \\ I - UU^* & 0 \end{pmatrix}$. If $A = U \oplus I$ and $B = I \oplus U^*$, then (A,B) satisfies the Fuglede-Putnam type property (FPT) since $\Delta_{A,B}(X) = 0 = \Delta_{A^*,B^*}(X)$.

In this paper, we study some connections between solutions A and B satisfying the operator equation $AXB = X$. We also investigate several properties between such solutions A and B. In particular, we show that if A has the single valued extension property, then so does B. Moreover, we consider the (weak) Fuglede-Putnam type property and investigate the local spectral properties between the solutions A and B under the Fuglede-Putnam type property.

2. Preliminaries

An operator $T \in \mathscr{L}(\mathscr{H})$ has the *single valued extension property (i.e., SVEP)* at $\lambda_0 \in \mathbb{C}$ if for every open neighborhood U of λ_0 the only analytic function $f : U \longrightarrow \mathscr{H}$ which satisfies the equation $(T-\lambda)f(\lambda) \equiv 0$ is the constant function $f \equiv 0$ on U. The operator T is said to have the single valued extension property if T has the single valued extension property at every $\lambda \in \mathbb{C}$. For an operator $T \in \mathscr{L}(\mathscr{H})$ and for a vector $x \in \mathscr{H}$, the *local resolvent set* $\rho_T(x)$ of T at x is defined as the union of every open subset G of $\mathbb{C}$ on which there is an analytic function $f : G \to \mathscr{H}$ such that $(T-\lambda)f(\lambda) \equiv x$ on G. The *local spectrum* of T at x is given by $\sigma_T(x) = \mathbb{C} \setminus \rho_T(x)$. We define the *local spectral subspace* of an operator $T \in \mathscr{L}(\mathscr{H})$ by $\mathscr{H}_T(F) = \{x \in \mathscr{H} : \sigma_T(x) \subset F\}$ for a subset F of $\mathbb{C}$. An operator $T \in \mathscr{L}(\mathscr{H})$ is said to have *Dunford's property* (C) if $\mathscr{H}_T(F)$ is closed for each closed subset F of $\mathbb{C}$. An operator $T \in \mathscr{L}(\mathscr{H})$ is said to have *Bishop's property* (β) if for every open subset G of $\mathbb{C}$ and every sequence $\{f_n\}$ of $\mathscr{H}$-valued analytic functions on G such that $(T-\lambda)f_n(\lambda)$ converges uniformly to 0 in norm on compact subsets of G, we get that $f_n(\lambda)$ converges uniformly to 0 in norm on compact subsets of G. An operator $T \in \mathscr{L}(H)$ is said to be *decomposable* if for every open cover $\{U,V\}$ of $\mathbb{C}$ there are T-invariant subspaces $\mathscr{X}$ and $\mathscr{Y}$ such that

$$\mathscr{H} = \mathscr{X} + \mathscr{Y}, \ \ \sigma(T|_{\mathscr{X}}) \subset \overline{U}, \ \text{ and } \ \sigma(T|_{\mathscr{Y}}) \subset \overline{V}.$$

It is well known that

$$\text{Bishop's property } (\beta) \Rightarrow \text{Dunford's property } (C) \Rightarrow \text{SVEP}.$$

Any of the converse implications does not hold, in general (see [19] for more details).

3. Connections between solutions

Let $X \in \mathscr{L}(\mathscr{H})$ be given. Recall that if A and B in $\mathscr{L}(\mathscr{H})$ satisfy the operator equation $AXB = X$, then (A,B) is said to be a solution of $AXB = X$. In this section we study some connections between solutions A and B satisfying the operator equation $AXB = X$. We first consider the local spectral properties for this program.

THEOREM 3.1. *Let $X \in \mathscr{L}(\mathscr{H})$ be given with $0 \notin \sigma_p(X)$ and let (A,B) be a solution in $\mathscr{L}(\mathscr{H})$ satisfying the operator equation $AXB = X$. If A has the single valued extension property, then B has the single valued extension property.*

Proof. Let $f : G \to \mathbb{C}$ be an analytic function on G such that $(B-\lambda I)(f(\lambda)) \equiv 0$ on G, where G is a domain of f. Multiplying both sides by AX, we have

$$AX(B-\lambda I)f(\lambda) = (AXB - \lambda AX)f(\lambda) \equiv 0 \ \text{ on } \ G.$$

Since $AXB = X$, $(I - \lambda A)Xf(\lambda) \equiv 0$ on G.

(i) If $0 \notin G$, then $(\frac{1}{\lambda}I - A)Xf(\lambda) \equiv 0$ on G. Consider an analytic function g given by $g(z) = \frac{1}{z}$ for all $z \in G$. Set $\mu = \frac{1}{\lambda}$. Then $(\mu - A)X(f(g)(\mu)) \equiv 0$ on $G' = \{\frac{1}{\lambda} : \lambda \in G\}$. Since A has the single valued extension property, $X(f \circ g)(\mu) \equiv 0$ on G'. Hence $Xf(\lambda) \equiv 0$ on G. Since X is injective, $f(\lambda) \equiv 0$ on G.

(ii) Assume $0 \in G$. When $\lambda = 0$, since $(I - \lambda A)Xf(\lambda) \equiv 0$ on G and $kerX = \{0\}$, $f(0) = 0$. Since f is analytic at 0 and $f \neq 0$, by Taylor expansion at 0, we may assume that f has zeros with finite multiplicities, say k at 0. Then $f(z) = z^k h(z)$ on some neighborhood N of 0 in G, where $h(0) \neq 0$ on N. Set $N' = N \setminus \{0\}$. Then

$$\Big(\frac{1}{\lambda}I - A\Big)Xf(\lambda) = \Big(\frac{1}{\lambda}I - A\Big)X\lambda^k h(\lambda) \equiv 0 \text{ on } N'.$$

Since $N' = N \setminus \{0\}$, we get

$$\Big(\frac{1}{\lambda}I - A\Big)Xh(\lambda) \equiv 0 \text{ on } N'.$$

By (i), $h(\lambda) \equiv 0$ on $N' \subset G$. By the Identity theorem, $h(\lambda) \equiv 0$ on G. Since $f(\lambda) = \lambda^k h(\lambda)$, $f(\lambda) \equiv 0$ on G. By (i) and (ii), B has the single valued extension property. □

REMARK 3.2. The condition $0 \notin \sigma_p(X)$ in Theorem 3.1 is necessary.

EXAMPLE 3.3. Let U be the unilateral shift defined by $Ue_n = e_{n+1}$ where $\{e_n\}$ is an orthonormal basis for $\mathscr{H}$. Set $X = \begin{pmatrix} 0 & 0 \\ I - UU^* & 0 \end{pmatrix}$. Then $0 \in \sigma_p(X)$. If $A = U \oplus I$ and $B = I \oplus U^*$, then $AXB = X$ holds. Moreover, since $A = U \oplus I$ is subnormal, it has the single valued extension property. However, $B = I \oplus U^*$ does not have the single valued extension property.

REMARK 3.4. The converse of Theorem 3.1 does not hold.

EXAMPLE 3.5. Let $X = U$ (in Theorem 3.1) be the unilateral shift defined by $Ue_n = e_{n+1}$ where $\{e_n\}$ is an orthonormal basis for $\mathscr{H}$. Then (U^*, U) is a solution of $U^*XU = X$ and U has the single valued extension property. However, U^* does not have the single valued extension property .

As applications of Theorem 3.1, we get the following corollaries.

COROLLARY 3.6. *Let $X \in \mathscr{L}(\mathscr{H})$ be given with $0 \notin \sigma_p(X)$ and let (A,B) be a solution in $\mathscr{L}(\mathscr{H})$ satisfying the operator equation $AXB = X$. If A is hyponormal (i.e. $A^*A \geqslant AA^*$), then B has the single valued extension property.*

Proof. If A satifies $A^*A \geqslant AA^*$, then it is known that A has the single valued extension property. Hence the proof follows from Theorem 3.1. □

COROLLARY 3.7. *Let $X = U$ be the unilateral shift and W_α, W_β be the weighted shift defined by $W_\alpha e_n = \alpha_n e_{n+1}$ and $W_\beta e_n = \beta_n e_{n+1}$ for all $n = 1, 2, \ldots$ where $\{\alpha_n\}$ and $\{\beta_n\}$ are positive sequences. If (W_α, W_β^*) satisfies $W_\alpha X W_\beta^* = X$, then*

$$\lim_{n\to\infty} \sup(\alpha_2 \cdots \alpha_{n+1})^{\frac{1}{n}} = \infty.$$

Proof. Since $W_\alpha X W_\beta^* = X$, $\beta_n = \frac{1}{\alpha_{n+1}}$ for all $n = 1, 2, \cdots$. Since $\sigma_p(W_\alpha) = \emptyset$, W_α has the single valued extension property. Then by Theorem 3.1, W_β^* has the single valued extension property. It follows from Theorem 2.89 in [1] that

$$\lim_{n\to\infty} \inf \left(\frac{1}{\alpha_2} \cdots \frac{1}{\alpha_{n+1}} \right)^{\frac{1}{n}} = 0.$$

Hence $\lim_{n\to\infty} \sup(\alpha_2 \cdots \alpha_{n+1})^{\frac{1}{n}} = \infty$. $\square$

COROLLARY 3.8. *Let $X \in \mathcal{L}(\mathcal{H})$ be given with $0 \notin \sigma_p(X^*)$ and let (A, B) be a solution in $\mathcal{L}(\mathcal{H})$ satisfying the operator equation $AXB = X$. If B^* has the single valued extension property, then A^* has the single valued extension property.*

Proof. If we take the adjoint of the operator equation $AXB = X$, then the proof follows from Theorem 3.1. $\square$

COROLLARY 3.9. *Let $X \in \mathcal{L}(\mathcal{H})$ be given with $0 \notin \sigma_p(X)$ and let (A, B) be a solution in $\mathcal{L}(\mathcal{H})$ satisfying the operator equation $AXB = X$. If A has the single valued extension property, then for a vector $x \in \mathcal{H}$, $\rho_B(x)^{-1} \subset \rho_A(AXx)$ where $\rho_B(x)^{-1} := \{\frac{1}{\lambda} : \lambda \in \rho_B(x)\}$.*

Proof. If A has the single valued extension property, then B has the single valued extension property from Theorem 3.1. If $\lambda \in \rho_B(x)$, then there exist a neighborhood D of λ and a $\mathcal{H}$-valued analytic function f on D such that $(B - \lambda I) f(\lambda) = x$ defined on D. If $0 \notin D$, then

$$(\frac{1}{\lambda} I - A) \lambda X f(\lambda) = (AXB - \lambda AX) f(\lambda) = AXx$$

for any $\lambda \in D$. Since $\lambda X f(\lambda)$ is analytic on D, $\frac{1}{\lambda} \in \rho_A(AXx)$.

If $0 \in D$, choose a proper open subset D_0 of D. Then for any $\lambda \in D_0$,

$$\left(\frac{1}{\lambda} I - A \right) \lambda X f(\lambda) = (AXB - \lambda AX) f(\lambda) = AXx.$$

Hence $\frac{1}{\lambda} \in \rho_A(AXx)$. $\square$

COROLLARY 3.10. *Let $X \in \mathcal{L}(\mathcal{H})$ be given with $0 \notin \sigma_p(X)$ and let (A,B) be a solution in $\mathcal{L}(\mathcal{H})$ satisfying the operator equation $AXB = X$. If A is an isometry, then the following statements hold.*

(i) For any closed set F in $\mathbb{C}$,

$$XH_B(F) \subset H_{A^*}(F) \text{ and } \sigma_{A^*}(Xx) \subset \sigma_B(x)$$

where $H_S(F) = \{x \in \mathcal{H} : \sigma_S(x) \subset F\}$.

(ii) If there exists $\lambda_0 \in \sigma(A^*)\backslash\sigma(B)$, then $H_{A^*}(F)$ is dense in $\mathcal{H}$.

(iii) $\cup_{x\in\mathcal{H}}\sigma_{A^*}(Xx) \subset \sigma(B)$.

Proof. (i) Since A is an isometry, it has the single valued extension property. In fact, let $f : G \to \mathbb{C}$ be an analytic function on G such that $(A-\lambda I)(f(\lambda)) \equiv 0$ on G, where G is a domain of f. Then

$$0 = \|(A-\lambda I)f(\lambda)\| \geqslant \Big|\|Af(\lambda)\| - \|\lambda f(\lambda)\|\Big| = |1-|\lambda|| \|f(\lambda)\|$$

for any $\lambda \in G$. Hence $f(\lambda) = 0$ on G. Thus A has the single valued extension property. By Theorem 3.1, B has also the single valued extension property. Since $A^*A = I$ and $AXB = X$, $XB = A^*X$. If $x \in H_B(F)$, then $\sigma_B(x) \subset F$, i.e., $F^c \subset \rho_B(x)$. Hence there exists a $\mathcal{H}$-valued analytic function f defined on F^c such that

$$(B-\lambda I)f(\lambda) = x, \quad \lambda \in F^c.$$

Since $XB = A^*X$, we get

$$(A^*-\lambda I)Xf(\lambda) = X(B-\lambda I)f(\lambda) = Xx.$$

Hence $\lambda \in \rho_{A^*}(Xx)$, i.e., $\sigma_{A^*}(Xx) \subset F$. That implies $Xx \in H_{A^*}(F)$, i.e., $XH_B(F) \subset H_{A^*}(F)$.

For any $\lambda_0 \in \rho_B(x)$, there exist a neighborhood D of λ_0 and a $\mathcal{H}$-valued analytic function f on D such that $(B-\lambda I)f(\lambda) = x$ for any $\lambda \in D$. Hence

$$(A^*-\lambda I)Xf(\lambda) = (XB-\lambda X)f(\lambda) = Xx.$$

Hence $\rho \in \rho_{A^*}(Xx)$. Thus $\rho_B(x) \subset \rho_{A^*}(Xx)$, i.e., $\sigma_{A^*}(Xx) \subset \sigma_B(x)$.

(ii) If there exists $\lambda_0 \in \sigma(A^*)\backslash\sigma(B)$, then $d_0 = dist(\lambda_0, \sigma(B)) > 0$. Set $F = \{z \in \mathbb{C} : |\lambda-\lambda_0| \geqslant \frac{d_0}{3}\}$. Then $\sigma(B) \subset F$. Since A has the single valued extension property, by Theorem 3.1 B has the single valued extension property. Since $\sigma_B(x) \subset \sigma(B) \subset F$ for any $x \in \mathcal{H}$, $\mathcal{H} \subset H_B(F)$. By (i),

$$\mathcal{H} = \overline{X\mathcal{H}} \subset \overline{XH_B(F)} \subset \overline{H_{A^*}(F)}.$$

Since $H_{A^*}(F) \subset \mathcal{H}$ clearly, $\overline{H_{A^*}(F)} = \overline{\mathcal{H}} = \mathcal{H}$.

(iii) By Theorem 3.1, B has the single valued extension property. Since $\sigma_{A^*}(Xx) \subset \sigma_B(x)$ by (i),

$$\cup_{x\in\mathcal{H}}\sigma_{A^*}(Xx) \subset \cup_{x\in\mathcal{H}}\sigma_B(x) = \sigma(B).$$

So we complete the proof. □

Recall that a conjugation on $\mathcal{H}$ is an antilinear operator $C : \mathcal{H} \to \mathcal{H}$ which satisfies $\langle Cx, Cy \rangle = \langle y, x \rangle$ for all $x, y \in \mathcal{H}$ and $C^2 = I$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be complex symmetric if there exists a conjugation C on $\mathcal{H}$ such that $T = CT^*C$. In this case, we say that T is a complex symmetric operator with conjugation C.

THEOREM 3.11. *Let $X \in \mathcal{L}(\mathcal{H})$ with $0 \notin \sigma_p(X)$ and let A and B be complex symmetric operators with a conjugation C satisfying $AXB = X$. If A has the single valued extension property, then B and B^* have the single valued extension property.*

Proof. By Theorem 3.1, B has the single valued extension property. Since $CA^*C = A$ and $CB^*C = B$,

$$CXC = CAXBC = (CAC)(CXC)(CBC) = A^*(CXC)B^*.$$

Note that $\sigma_p(X) = \sigma_p(CXC)^*$. In fact, if $\gamma \in \sigma_p(X)$, there exists a nonzero x such that $Xx = \gamma x$. Hence

$$0 = C(X-\gamma)x = CXx - \overline{\gamma}Cx = CXC^2x - \overline{\gamma}Cx = (CXC - \overline{\gamma})Cx.$$

Since $Cx \neq 0$, $\overline{\gamma} \in \sigma_p(CXC)$. Hence $\gamma \in \sigma_p(CXC)^*$. Therefore, $\sigma_p(X) \subset \sigma_p((CXC)^*)$. Similarly, $\sigma_p(CXC)^* \subset \sigma_p(X)$. Thus $\sigma_p(X) = \sigma_p(CXC)$

Now it suffices to show that B^* has the single valued extension property. If $(B^* - \gamma)f(\gamma) = 0$ for an analytic function f on a domain D, then $(CBC - \gamma)f(\gamma) = 0$ on D. Then

$$0 = (BC - \overline{\gamma}C)f(\gamma) = (B - \overline{\gamma})Cf(\gamma) \text{ on } D$$

Take $z = \overline{\gamma}$. Then $0 = (B - z)Cf(\overline{z})$ on D^* where $D^* = \{\overline{z} : z \in D\}$. Since $f(\gamma)$ is analytic on D, $f(\gamma) = \sum_{n=0}^{\infty} a_n(\gamma - \gamma_0)^n$ for $\gamma_0 \in D$. Hence

$$\begin{aligned} h(z) = Cf(\overline{z}) &= C(\sum_{n=0}^{\infty} a_n(\overline{z} - \gamma_0)^n) \\ &= \sum_{n=0}^{\infty} Ca_n(z - \overline{\gamma_0})^n, \end{aligned}$$

which means that $h(z)$ is analytic at $\overline{\gamma_0}$. From this, we know that $Cf(\overline{z})$ is analytic on D^*. Since B has the single valued extension property, $Cf(\overline{z}) = 0$ on D^*. Hence $f(\overline{z}) = 0$ on D^*, i.e., $f(\gamma) = 0$ on D. Hence B^* has the single valued extension property. □

EXAMPLE 3.12. Let $X = U$ (in Theorem 3.1) be the unilateral shift defined by $Ue_n = e_{n+1}$ where $\{e_n\}$ is an orthonormal basis for $\mathcal{H}$. If A and B are diagonal operators defined by $Ae_n = d_ne_n$ and $Be_n = \frac{1}{d_n}e_n$ for each n, respectively, then A and B are complex symmetric operators, A has the single valued extension property, and (A, B) is a solution of $AXB = X$. Moreover, B and B^* have the single valued extension property.

COROLLARY 3.13. *Let $X \in \mathcal{L}(\mathcal{H})$ be with $0 \notin \sigma_p(X)$. If A is normal and B is a complex symmetric operator with a conjugation C satisfying $AXB = X$, then B and B^* have the single valued extension property.*

Proof. Since A is normal, it has known that A has the single valued extension property. Hence B has the single valued extension property from Theorem 3.1. As an application of the proof of Theorem 3.11, B^* has the single valued extension property. □

COROLLARY 3.14. *Let $X \in \mathcal{L}(\mathcal{H})$ with $0 \notin \sigma_p(X)$ and let A and B be complex symmetric operators with a conjugation C satisfying $AXB = X$. If A has the single valued extension property, then*

$$\sigma(B) = \sigma_{su}(B) = \sigma_{ap}(B).$$

Proof. Since B and B^* have the single valued extension property, from Theorem 3.11, the proof follows from [1]. □

In the following proposition, we consider the spectra of a solution (A,B) satisfying $AXB = X$.

PROPOSITION 3.15. *Let $X \in \mathcal{L}(\mathcal{H})$ be given, and let (A,B) be a solution of $AXB = X$. Set $G^{-1} = \{\frac{1}{\lambda} : \lambda \in G\}$. Then the following statements hold.*

(i) If $0 \notin \sigma_p(X)$, then $0 \notin \sigma_p(B)$ and $\sigma_p(B)^{-1} \subset \sigma_p(A)$.

(ii) If $0 \notin \sigma_{ap}(X)$, then $0 \notin \sigma_{ap}(B)$ and $\sigma_{ap}(B)^{-1} \subset \sigma_{ap}(A)$.

(iii) If $0 \notin \sigma(X)$, then $0 \notin \sigma_{ap}(B)$ and A is surjective.

Proof. In order to prove (i) and (ii), it suffices to show that (ii) holds. If $0 \notin \sigma_{ap}(X)$, then there exists $c > 0$ such that $||Xx|| \geqslant c||x||$ for all $x \in \mathcal{H}$. If $\lambda \in \sigma_{ap}(B)$, then there exists a sequence $\{x_n\}$ with $||x_n|| = 1$ such that $\lim_{n\to\infty} ||(B-\lambda)x_n|| = 0$. Since $AXB = X$,

$$\begin{aligned} 0 = \lim_{n\to\infty} ||AX(B-\lambda I)x_n|| &= \lim_{n\to\infty} ||(AXB-\lambda AX)x_n|| \\ &= \lim_{n\to\infty} ||(I-\lambda A)Xx_n||. \end{aligned} \tag{1}$$

If $\lambda = 0$, then $0 = \lim_{n\to\infty} ||Xx_n|| \geqslant \lim_{n\to\infty} c||x_n|| = c > 0$. Therefore, $0 \notin \sigma_{ap}(B)$. Then from (1), we get that $\lim_{n\to\infty} ||(\frac{1}{\lambda} - A)Xx_n|| = 0$. Since $||Xx_n|| \geqslant c||x_n|| = c > 0$ for all n, $\lim_{n\to\infty} ||(\frac{1}{\lambda} - A)\frac{Xx_n}{||Xx_n||}|| = 0$. Hence, $\frac{1}{\lambda} \in \sigma_{ap}(A)$. Since $\lambda \in \sigma_{ap}(B)$, $\sigma_{ap}(B)^{-1} \subset \sigma_{ap}(A)$.

(iii) If $0 \notin \sigma(X)$, then B is left invertible and A is right invertible. Hence $0 \notin \sigma_{ap}(B)$ and A is surjective. □

PROPOSITION 3.16. *Let $X \in \mathcal{L}(\mathcal{H})$ be given, and let (A,B) be a solution of $AXB = X$. Then the following statements hold.*

(i) (A^n, B^n) *are also solutions of* $AXB = X$ *for* $n \geqslant 1$.

(ii) $X \ker B \subset \ker A$ *and* $X \ker(B - \lambda) \subset \ker(A - \frac{1}{\lambda})$ *if* $\lambda \neq 0$.

(iii) $(\widetilde{A}, \widetilde{B})$ *is a solution of* $\widetilde{A} Y \widetilde{B} = Y$ *where* $Y = |A|^{\frac{1}{2}} X U_B |B|^{\frac{1}{2}}$ *and* $\widetilde{A}$ *and* $\widetilde{B}$ *are the Aluthge transforms of* A *and* B*, respectively.*

Proof. (i) The proof is trivial.

(ii) If $x \in \ker B$, then $0 = AXBx = Xx$. Hence $AXx = 0$, i.e., $Xx \in \ker A$. Thus $X \ker B \subset \ker A$. If $x \in \ker(B - \lambda)$, then

$$0 = (AXB - \lambda AX)x = (X - \lambda AX)x = (I - \lambda A)Xx.$$

Since $\lambda \neq 0$, $(A - \frac{1}{\lambda})Xx = 0$. Thus $Xx \in \ker(A - \frac{1}{\lambda})$, and hence $X \ker(B - \lambda) \subset \ker(A - \frac{1}{\lambda})$.

(iii) Let $A = U_A|A|$ and $B = U_B|B|$ be the polar decomposition of A and B, respectively. Since $AXB = X$, $\widetilde{A} Y \widetilde{B} = Y$ where $Y = |A|^{\frac{1}{2}} X U_B |B|^{\frac{1}{2}}$. $\square$

We next study the (weak) Fuglede-Putnam type property ((W)FPT). Define $\Delta_{A,B} : \mathscr{L}(\mathscr{H}) \to \mathscr{L}(\mathscr{H})$ by $\Delta_{A,B}(X) = AXB - X$. We first recall the (weak) Fuglede-Putnam type property ((W)FPT).

DEFINITION 3.17. We say that (A, B) satisfies the weak Fuglede-Putnam type property (WFPT) if $\Delta_{A^*,B^*}(Y) = 0$ for some nonzero Y in $\mathscr{L}(\mathscr{H})$ whenever $\Delta_{A,B}(X) = 0$ for some $X \neq 0$ in $\mathscr{L}(\mathscr{H})$. In particular, if $Y = X$, we say that (A, B) satisfies the Fuglede-Putnam type property (FPT) with X.

We next give some basic properties for the Fuglede-Putnam type property (FPT). Recall that if x and y are vectors in $\mathscr{H}$, then the rank one operator $x \otimes y$ on $\mathscr{H}$ is defined by $(x \otimes y)z = \langle z, y \rangle x$ for $z \in \mathscr{H}$.

PROPOSITION 3.18. *(i) If* A *and* B^* *are isometries, then* (A, B) *satisfies the Fuglede-Putnam type property (FPT) with* X.

(ii) If $A^* x = \gamma A x$ *and* $B^* y = \overline{\gamma} B y$ *for some nonzero* $\gamma \in \mathbb{C}$*, then* (A, B) *satisfies the Fuglede-Putnam type property (FPT) with* $x \otimes y$.

(iii) If (A, B) *satisfies the Fuglede-Putnam type property (FPT) with* X*, then* $(A \oplus A, B \oplus B)$ *satisfies the Fuglede-Putnam type property (FPT) with* $X \oplus X$.

Proof. (i) Assume that $\Delta_{A,B}(X) = 0$ for some $X \neq 0$ in $\mathscr{L}(\mathscr{H})$. Then $AXB = X$, and hence $A^* X B^* = A^*(AXB)B^* = (A^*A)X(BB^*) = X$. Hence $\Delta_{A^*,B^*}(X) = 0$.

(ii) Assume $\Delta_{A,B}(x \otimes y) = 0$. Then

$$\begin{aligned} \Delta_{A^*,B^*}(x \otimes y) &= A^*(x \otimes y)B^* - x \otimes y = A^* x \otimes B y - x \otimes y \\ &= \gamma A x \otimes \frac{1}{\overline{\gamma}} B^* y - x \otimes y = A x \otimes B^* y - x \otimes y \\ &= A(x \otimes y)B - x \otimes y = x \otimes y - x \otimes y = 0. \end{aligned}$$

(iii) Since (A,B) satisfies the Fuglede-Putnam type property (FPT), $\Delta_{A^*,B^*}(X)=0$ whenever $\Delta_{A,B}(X)=0$ for some $X\neq 0$ in $\mathcal{L}(\mathcal{H})$. If $\Delta_{A,B}(X)=0$ for some $X\neq 0$, then $\Delta_{A\oplus A,B\oplus B}(X\oplus X)=0$. Since $\Delta_{A^*,B^*}(X)=0$, $\Delta_{A^*\oplus A^*,B^*\oplus B^*}(X\oplus X)=0$. □

COROLLARY 3.19. *Let* $\Delta_{A,B}(X)=0$ *for all* X *in* $\mathcal{L}(\mathcal{H})$. *If* A *and* B^* *are isometries, then*

$$\|AYB-Y+X\|\geqslant\|X\|$$

for all $Y\in\mathcal{L}(\mathcal{H})$.

Proof. Since A and B are contractions and (A,B) satisfies the Fuglede-Putnam type property (FPT) from Proposition 3.18, the proof follows from [17] or [22]. □

REMARK 3.20. If (A,B) satisfies the Fuglede-Putnam type property (FPT) with X, then (A,B) satisfies the weak Fuglede-Putnam type property (WFPT). But the converse is not true.

EXAMPLE 3.21. Let $X=\begin{pmatrix} a & -a \\ a & -a\end{pmatrix}\in\mathcal{L}(\mathbb{C}^2)$ where $a\neq 0$. If $A=\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}$ and $B=\begin{pmatrix} 2 & -1 \\ 1 & 0\end{pmatrix}$, then $AXB=X$ holds. If $Y=\begin{pmatrix} a & a \\ a & a\end{pmatrix}\in\mathcal{L}(\mathbb{C}^2)$ where $a\neq 0$, then $A^*YB^*=Y$ holds. Hence (A,B) satisfies the weak Fuglede-Putnam type property (WFPT). However, since $A^*XB^*\neq X$, (A,B) does not satisfy the Fuglede-Putnam type property (FPT) with X

We observe from Example 3.21 that $(WFPT)$ does not preserve the normality, indeed, A is normal, but B is not. We next study the basic properties of the (weak) Fuglede-Putnam type property ((W)FPT).

PROPOSITION 3.22. *(i) If* A *is similar to* B *via* $A=SBS^{-1}$ *where* S *is invertible, then* (A,B) *satisfies the weak Fuglede-Putnam type property (WFPT).*

(ii) If A *and* B *are complex symmetric operators, then* (A,B) *satisfies the weak Fuglede-Putnam type property (WFPT).*

Proof. (i) If $\Delta_{A,B}(X)=0$ for some $X\neq 0$ in $\mathcal{L}(\mathcal{H})$, then

$$X=AXB=SBS^{-1}XB=SB(S^{-1}XS)S^{-1}B.$$

Therefore we get that

$$S^{-1}X=B(S^{-1}X)B=B(S^{-1}X)S^{-1}AS.$$

Then $S^{-1}XS^{-1}=B(S^{-1}XS^{-1})A$. Hence $A^*(S^{-1}XS^{-1})^*B^*=(S^{-1}XS^{-1})^*$, and $\Delta_{A^*,B^*}(S^{-1}XS^{-1})=0$.

(ii) Assume that $\Delta_{A,B}(X)=0$ for some $X\neq 0$ in $\mathcal{L}(\mathcal{H})$. Since $CA^*C=A$ and $DB^*D=B$ where C and D are conjugations, $X=AXB=(CA^*C)X(DB^*D)$. Hence $A^*(CXD)B^*=CXD$. Thus $\Delta_{A^*,B^*}(CXD)=0$. □

PROPOSITION 3.23. *If (A,B) satisfies the Fuglede-Putnam type property (FPT) with X, then the following statements hold.*

(i) If R and S are similar to A and B, respectively, then (R,S) satisfies the weak Fuglede-Putnam type property (WFPT). In particular, if R and S are unitarily equivalent to A and B, respectively, then (R,S) satisfies the Fuglede-Putnam type property (FPT) with X.

(ii) $(\widetilde{A},\widetilde{B})$ satisfies the weak Fuglede-Putnam type property (WFPT). In particular, if A and B are quasinormal, then $(\widetilde{A},\widetilde{B})$ satisfies the Fuglede-Putnam type property (FPT) with X.

Proof. (i) If R and S are similar to A and B, respectively, then there exist invertible operators U and V such that $R = UAU^{-1}$ and $S = VBV^{-1}$. If $\Delta_{R,S}(X) = 0$ for some $X \neq 0$ in $\mathscr{L}(\mathscr{H})$, then $A(U^{-1}XV)B = U^{-1}XV$. Since (A,B) satisfies the Fuglede-Putnam type property (FPT), $A^*(U^{-1}XV)B^* = U^{-1}XV$. Since $R = UAU^{-1}$ and $S = VBV^{-1}$, $R^*((UU^*)^{-1}X(VV^*))S^* = (UU^*)^{-1}X(VV^*)$. Hence (R,S) satisfies the weak Fuglede-Putnam type property (WFPT). In particular, if R and S are unitarily equivalent to A and B, respectively, then $UU^* = I = VV^*$. Hence we get the result.

(ii) We know that $(\widetilde{A},\widetilde{B})$ is a solution of $\widetilde{A}Y\widetilde{B} = Y$ where $Y = |A|^{\frac{1}{2}}XU_B|B|^{\frac{1}{2}}$ by Proposition 3.16. Since $(\widetilde{A})^*|A|^{\frac{1}{2}}U_A^* = |A|^{\frac{1}{2}}U_A^*A^*$ and $B^*|B|^{\frac{1}{2}} = |B|^{\frac{1}{2}}(\widetilde{B})^*$, $(\widetilde{A})^*Z(\widetilde{B})^* = Z$ where $Z = |A|^{\frac{1}{2}}U_A^*X|B|^{\frac{1}{2}}$. In particular, if A and B are quasinormal, then $\widetilde{A} = A$ and $\widetilde{B} = B$ from [14]. So we complete the proof. □

In the following example, we show that (R,S) in Proposition 3.23 may not satisfy the Fuglede-Putnam type property (FPT) with the same X, even if (A,B) satisfies the Fuglede-Putnam type property (FPT) with X.

EXAMPLE 3.24. Let $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $U = V = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$ be in $\mathscr{L}(\mathbb{C}^2)$. If $R = UAU^{-1} = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$ and $S = VBV^{-1} = \begin{pmatrix} -1 & -2 \\ 0 & 1 \end{pmatrix}$, then R and S are similar to A and B, respectively. If $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, then $AXB = X$ and $A^*XB^* = X$ hold. Thus (A,B) satisfies the Fuglede-Putnam type property (FPT). But $RXS = X$. On the other hand, $R^*XS^* \neq X$. Hence (R,S) does not satisfiy the Fuglede-Putnam type property (FPT) with X. On the other hand, if $Y = \begin{pmatrix} a & -a \\ b & -a \end{pmatrix}$ where a or b is nonzero, then $R^*YS^* = Y$. Hence (R,S) satisfies the weak Fuglede-Putnam type property (WFPT).

THEOREM 3.25. *Assume that B is normal and A is similar to B via $A = TBT^{-1}$ where T is invertible. If $\Delta_{A,B}(X) = 0$ for some $X \neq 0$ in $\mathscr{L}(\mathscr{H})$, then the following statements hold.*

(i) (B,B) satisfies the Fuglede-Putnam type property (FPT) with $T^{-1}X$.

(ii) (A,B) satisfies the weak Fuglede-Putnam type property (WFPT).

Proof. (i) Since $\Delta_{A,B}(X)=0$ for some $X\neq 0$ in $\mathscr{L}(\mathscr{H})$, $B(T^{-1}X)B=T^{-1}X$. Hence $T^{-1}X\in\ker\Delta_{B,B}$. Note that $\ker\Delta_{B,B}=\ker\Delta_{B^*,B^*}$. In fact,

$$\begin{aligned}\Delta_{B^*,B^*}(\Delta_{B,B}(T^{-1}X)) &= B^*\Delta_{B,B}(T^{-1}X)B^*-\Delta_{B,B}(T^{-1}X)\\ &= B^*[B(T^{-1}X)B-T^{-1}X]B^*-B(T^{-1}X)B+T^{-1}X\end{aligned}$$

and

$$\begin{aligned}\Delta_{B,B}(\Delta_{B^*,B^*}(T^{-1}X)) &= B\Delta_{B^*,B^*}(T^{-1}X)B-\Delta_{B^*,B^*}(T^{-1}X)\\ &= B[B^*(T^{-1}X)B^*-T^{-1}X]B-B^*(T^{-1}X)B^*+T^{-1}X.\end{aligned}$$

Since B is normal, $\Delta_{B,B}^*=\Delta_{B^*,B^*}$, and

$$\Delta_{B^*,B^*}(\Delta_{B,B}(T^{-1}X))=\Delta_{B,B}(\Delta_{B^*,B^*}(T^{-1}X)),$$

$\Delta_{B,B}$ is normal. Hence $\ker\Delta_{B,B}=\ker\Delta_{B^*,B^*}$. Thus $T^{-1}X\in\ker\Delta_{B^*,B^*}$, and then $B^*(T^{-1}X)B^*=T^{-1}X$. Hence (B,B) satisfies the Fuglede-Putnam type property (FPT) with $T^{-1}X$.

(ii) Since $A=TBT^{-1}$, $B^*=T^*A^*(T^{-1})^*$. Since $B^*(T^{-1}X)B^*=T^{-1}X$ by (i),

$$A^*[(T^{-1})^*T^{-1}X]B^*=(T^{-1})^*T^{-1}X.$$

Thus $\Delta_{A^*,B^*}(|T^{-1}|^2X)=0$. Hence (A,B) satisfies the weak Fuglede-Putnam type property (WFPT). □

COROLLARY 3.26. *Assume that A and B^* are subnormal satisfying $\Delta_{A,B}(X)=0$ for some $X\neq 0$ in $\mathscr{L}(\mathscr{H})$. Then their normal extensions (S,T) satisfies the weak Fuglede-Putnam type property (WFPT).*

Proof. Since A and B^* are subnormal, their normal extensions S and T are followings;

$$S=\begin{pmatrix}A & A_1\\ 0 & A_2\end{pmatrix} \quad\text{and}\quad T=\begin{pmatrix}B & 0\\ B_1 & B_2\end{pmatrix}.$$

Take $Y=\begin{pmatrix}X & 0\\ 0 & 0\end{pmatrix}$. Then $SYT=Y$. Since S and T are normal, (S,T) satisfies the weak Fuglede-Putnam type property (WFPT) from Theorem 3.25. □

Recall that an operator $T\in\mathscr{L}(\mathscr{H})$ has the Bishop's property (β) modulo a closed set $S\subset\mathbb{C}$ if for all open subsets $V\subseteq\mathbb{C}\backslash S$ the mapping on the space

$$\mathscr{O}(V,\mathscr{H})\to\mathscr{O}(V,\mathscr{H}),\quad f\mapsto(T-z)f$$

is injective with closed range on the space $\mathscr{O}(V,\mathscr{H})$ of all analytic functions on V with values in $\mathscr{H}$. If this condition is satisfied with $S=\emptyset$, the T will be said to possess the Bishop's property (β). We also recall that T has the property (δ) modulo S if for every open cover $\{U,V\}$ of $\mathbb{C}$, the decomposition $\mathscr{H}=H_T(\overline{V})+H_T(\mathbb{C}\backslash U)$ holds for $S\subset U\subset\overline{U}\subset V$.

In the following theorem, we show that the Fuglede-Putnam type property preserves the Bishop's property (β) modulo a closed set $S\subset\mathbb{C}$ of an operator.

THEOREM 3.27. *Assume that* (A,B) *satisfies the Fuglede-Putnam type property (FPT) with* X *bounded below. If* A *has the Bishop's property* (β) *modulo* $\{0\}$*, then* B *has also the Bishop's property* (β) *modulo* $\{0\}$.

Proof. Assume that A has the Bishop's property (β) modulo $\{0\}$. Let $V \subseteq \mathbb{C}\backslash\{0\}$ be open and let $\{f_n\}$ be a sequence in $\mathscr{O}(V,\mathscr{H})$ with

$$\lim_{n\to\infty}(B-z)f_n(z) = 0.$$

Since (A,B) satisfies the Fuglede-Putnam type property (FPT), $AXB = X$. Hence

$$\begin{aligned}\lim_{n\to\infty} AX(B-zI)f_n(z) &= \lim_{n\to\infty}(AXB - zAX)f_n(z)\\ &= \lim_{n\to\infty}(I-zA)Xf_n(z) = 0\end{aligned}$$

in $\mathscr{O}(V,\mathscr{H})$. Since $0 \notin V$,

$$\lim_{n\to\infty}(\frac{1}{z}I - A)Xf_n(z) = 0$$

in $\mathscr{O}(V,\mathscr{H})$. Consider an analytic function g given by $g(z) = \frac{1}{z}$ for all $z \in V$. Set $\mu = \frac{1}{z}$. Then $\lim_{n\to\infty}(\mu I - A)X(f_n \circ g)(\mu)) = 0$ in $\mathscr{O}(V',\mathscr{H})$ where $V' = \{\frac{1}{z} : z \in V\}$. Since A has the Bishop's property (β) modulo $\{0\}$, $\lim_{n\to\infty} X(f_n \circ g)(\mu) = 0$ in $\mathscr{O}(V',\mathscr{H})$. Hence $\lim_{n\to\infty} Xf_n(z) = 0$ in $\mathscr{O}(V,\mathscr{H})$. Since X is bounded below, $\lim_{n\to\infty} f_n(z) = 0$ in $\mathscr{O}(V,\mathscr{H})$. Hence B has the Bishop's property (β) modulo $\{0\}$. □

COROLLARY 3.28. *Assume that* (A,B) *satisfies the Fuglede-Putnam type property (FPT) with* X *bounded below. If* A *is decomposable modulo* $\{0\}$*, then* B *is also decomposable modulo* $\{0\}$.

Proof. Since A and A^* have the Bishop's property (β) modulo $\{0\}$, B and B^* have the Bishop's property (β) modulo $\{0\}$ from Theorem 3.27. Hence B is decomposable modulo $\{0\}$. □

COROLLARY 3.29. *Assume that* (A,B) *satisfies the Fuglede-Putnam type property (FPT) with* X *bounded below. If* A *is normal or compact, then* B *is decomposable modulo* $\{0\}$.

Proof. Since A is decomposable, B is decomposable modulo $\{0\}$ from Corollary 3.28. □

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