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COMPLEX WEYL SYMBOLS OF THE EXTENDED METAPLECTIC REPRESENTATION OPERATORS

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Abstract. We consider the extended metaplectic representation of the semi-direct product of the Heisenberg group and the symplectic group (the Jacobi group). We give explicit formulas for the Berezin symbols and for the complex Weyl symbols of the corresponding representation operators. Then we deduce formulas for the symbols of the representation operators in the classical Weyl calculus. As an application, we find the classical Weyl symbol of the exponential of an operator whose Weyl symbol is a polynomial on $\mathbb{R}^{2n}$ of degree $\leqslant 2$, recovering a result of L. Hörmander.

1. Introduction

The central object of this note is the extended metaplectic representation of the Jacobi group. Our main reference is [18, Chap. IV].

Let $S := SU(n,n) \cap Sp(n,\mathbb{C})$. Then S is a subgroup of $SU(n,n)$ which is isomorphic to $Sp(n,\mathbb{R})$ [18, p. 175]. Let H_n be the $(2n+1)$-dimensional (real) Heisenberg group. Then S acts naturally on H_n and we can form the semi-direct product $G := H_n \rtimes S$ called the (multi-dimensional) Jacobi group [7]. Note that the name *Jacobi group* comes from [17].

By combining the metaplectic representation σ of S on the Fock space $\mathscr{F}$ with a non-degenerate unitary irreducible representation ρ of H_n on $\mathscr{F}$, we obtain the extended metaplectic representation π of G,

$$\pi(h,k) = \rho(h)\sigma(k), \qquad h \in H_n, \; k \in S.$$

There are different ways to construct σ, see in particular [18] and [25]. In [14], we recovered the formulas for the kernel of $\sigma(k)$, $k \in S$ given in [18] by using some functional equation satisfied by this kernel, see also [13]. We gave also some explicit formulas for the complex Weyl symbol of $\sigma(k)$ for $k \in S$. Let us recall that the complex Weyl calculus W_0 is the correspondence between operators on $\mathscr{F}$ and functions on $\mathbb{C}^n$ obtained by translating the usual Weyl correspondence (see [18], [23]) by means of the Bargmann transform.

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In the present note, we extend the results of [14] to π. More precisely, we first obtain a formula for the (covariant) Berezin symbol of $\pi(h,k)$. Then we deduce formulas for $W_0(\pi(h,k))$. Moreover, by translating π to the extended metaplectic representation π' of $G'=H_n\rtimes Sp(n,\mathbb{R})$ on $L^2(\mathbb{R}^n)$ by using the Bargmann transform, we obtain explicit formulas for the classical Weyl symbols $W_1(\pi'(g'))$ of $\pi'(g')$ for $g'\in G'$. Similar results are also obtain for the differential representations.

This note is organized as follows. We begin with some generalities on the Berezin correspondence on $\mathscr{F}$ (Section 2) and on the complex Weyl calculus (Section 3). In Section 4, we review some results on σ and we compute the Berezin symbol of $\pi(g)$ for $g\in G$. In Section 5, we give explicit formulas for $W_0(\pi(g))$, $g\in G$ and for $W_0(d\pi(X))$, X in the Lie algebra of G. From this, we deduce formulas for $W_1(\pi(g))$, $g\in G'$ and for $W_1(d\pi(X))$, X in the Lie algebra of G' (Section 6). The main results of this note are then Theorem 5.4 and Theorem 6.1. Finally, we apply the preceding results to the problem of computing the classical Weyl symbol of the exponential of an operator whose Weyl symbol is a polynomial on $\mathbb{R}^{2n}$ of degree $\leqslant 2$ (Section 7) and to the problem of computing the Moyal star exponential of such a polynomial (Section 8).

2. Berezin quantization on the Fock space

This section and the next section are mostly of expository nature. We follow closely [8], [10] and [12].

We first introduce the Fock space. Let $\lambda>0$ and let $\mathscr{F}_\lambda$ be the Hilbert space of all holomorphic functions f on $\mathbb{C}^n$ such that

$$\|f\|^2:=\int_{\mathbb{C}^n}|f(z)|^2\,e^{-\lambda|z|^2/2}\,d\mu_\lambda(z)<+\infty$$

where $d\mu_\lambda(z):=(2\pi)^{-n}\lambda^n\,dm(z)$. Here $z=x+iy$ with x and y in $\mathbb{R}^n$ and $dm(z):=dx\,dy$ is the standard Lebesgue measure on $\mathbb{C}^n$.

For each $z\in\mathbb{C}^n$, let $e_z(w)=\exp(\lambda\,\bar{z}w/2)$. Then we have the reproducing property $f(z)=\langle f,e_z\rangle$ for each $f\in\mathscr{F}_\lambda$.

Let us introduce the Berezin calculus on $\mathscr{F}_\lambda$ [5], [6], [8]. The Berezin (covariant) symbol of an operator A on $\mathscr{F}_\lambda$ is the function $S_\lambda(A)$ defined on $\mathbb{C}^n$ by

$$S_\lambda(A)(z):=\frac{\langle A\,e_z\,,\,e_z\rangle}{\langle e_z\,,\,e_z\rangle}$$

and the double Berezin symbol s_λ is defined by

$$s_\lambda(A)(z,w):=\frac{\langle A\,e_w\,,\,e_z\rangle}{\langle e_w\,,\,e_z\rangle}$$

for each $(z,w)\in\mathbb{C}^n\times\mathbb{C}^n$ such that $\langle e_w\,,\,e_z\rangle\neq 0$.

Since $s_\lambda(A)(z,w)$ is holomorphic in the variable z and anti-holomorphic in the variable w, this function is determined by its restriction to the diagonal of $\mathbb{C}^n\times\mathbb{C}^n$, that is, by $S_\lambda(A)$. On the other hand, A can be recovered from $s_\lambda(A)$ as follows. We have

$$\begin{aligned} Af(z) &= \langle Af, e_z\rangle = \langle f, A^* e_z\rangle \\ &= \int_{\mathbb{C}^n} f(w)\overline{A^* e_z(w)}\, e^{-\lambda|w|^2/2}\, d\mu_\lambda(w) \\ &= \int_{\mathbb{C}^n} f(w)\overline{\langle A^* e_z, e_w\rangle}\, e^{-\lambda|w|^2/2}\, d\mu_\lambda(w) \\ &= \int_{\mathbb{C}^n} f(w)\, s_\lambda(A)(z,w)\langle e_w, e_z\rangle\, e^{-\lambda|w|^2/2}\, d\mu_\lambda(w). \end{aligned}$$

This implies in particular that the map $A \to S_\lambda(A)$ is injective and that the kernel of A is the function

$$k_A(z,w) = \langle Ae_w, e_z\rangle = s_\lambda(A)(z,w)\langle e_w, e_z\rangle. \tag{2.1}$$

It is also known that S_λ is a bounded operator from the space $\mathscr{L}_2(\mathscr{F}_\lambda)$ of all Hilbert-Schmidt operators on $\mathscr{F}_\lambda$ (endowed with the Hilbert-Schmidt norm) to $L^2(\mathbb{C}^n, \mu_\lambda)$ which is one-to-one and has dense range [29]. Let S_λ^* be the adjoint operator of S_λ. Recall that the Berezin transform is the operator B_λ on $L^2(\mathbb{C}^n, \mu_\lambda)$ defined by $B_\lambda := S_\lambda S_\lambda^*$. We have the integral formula

$$(B_\lambda f)(z) = \int_{\mathbb{C}^n} f(w)\, e^{-\lambda|z-w|^2/2}\, d\mu_\lambda(w),$$

see [5], [6], [29]. Note also that we have $B_\lambda = \exp(\Delta/2\lambda)$ where $\Delta = 4\sum_{k=1}^n \partial^2/\partial z_k \partial \bar{z}_k$, see [29], [26].

Now, we introduce the non-degenerate unitary irreducible representations of the Heisenberg group.

For each $z, w \in \mathbb{C}^n$, we denote $zw := \sum_{k=1}^n z_k w_k$. For each $z, z', w, w' \in \mathbb{C}^n$, let

$$\omega((z,w),(z',w')) = \tfrac{i}{2}(zw' - z'w).$$

The $(2n+1)$-dimensional real Heisenberg group is

$$H_n := \{((z,\bar{z}),c) : z \in \mathbb{C}^n, c \in \mathbb{R}\}$$

equipped with the multiplication

$$((z,\bar{z}),c)\cdot((z',\bar{z}'),c') = ((z+z',\bar{z}+\bar{z}'),c+c'+\tfrac{1}{2}\omega((z,\bar{z}),(z',\bar{z}'))).$$

By the Stone-von Neumann theorem, there exists a unique (up to unitary equivalence) unitary irreducible representation ρ_λ of H_n whose restriction to the center of H_n is the character $((0,0),c) \to e^{i\lambda c}$ [28]. The Bargmann-Fock realization of ρ_λ on $\mathscr{F}_\lambda$ is defined as follows [2]. We have

$$(\rho_\lambda(h)f)(z) = \exp\left(i\lambda c_0 + \tfrac{\lambda}{2}\bar{z}_0 z - \tfrac{\lambda}{4}|z_0|^2\right) f(z - z_0)$$

for each $h = ((z_0, \bar{z_0}), c_0) \in H_n$ and $z \in \mathbb{C}^n$.

We also need later another realization of the unitary irreducible representation of H_n with central character $((0,0),c)\to e^{i\lambda c}$, that is, the Schrödinger representation ρ'_λ defined on $L^2(\mathbb{R}^n)$ by

$$(\rho'_\lambda((a+ib,a-ib),c)\phi)(x)=\exp\left(i\lambda(c-bx+\tfrac{1}{2}ab)\right)\phi(x-a)$$

for each $a,b\in\mathbb{R}^n$, $c\in\mathbb{R}$ and $x\in\mathbb{R}^n$.

A unitary intertwining operator between ρ_λ and ρ'_λ is then the Bargmann transform $\mathscr{B}:L^2(\mathbb{R}^n)\to\mathscr{F}_\lambda$ defined by

$$(\mathscr{B}f)(z)=\left(\frac{\lambda}{\pi}\right)^{n/4}\int_{\mathbb{R}^n}\exp\left(-\frac{\lambda}{4}z^2+\lambda zx-\frac{\lambda}{2}x^2\right)\phi(x)\,dx,$$

see in particular [8], [18].

Let us mention that the Heisenberg group and its unitary irreducible representations play a prominent role in a modern branch of harmonic analysis called *time-frequency analysis*, see [21, Chap. 9]. A recent application of this theory can be found in [22]. Moreover, the metaplectic group, which will be introduced here in Section 4 can be seen as the fundamental symmetry group in time-frequency analysis [19], [21]. These facts and references were brought to my attention by the referee.

3. Complex Weyl quantization on the Fock space

Here we first recall the definition of the complex Weyl correspondence W_0 on $\mathscr{F}_\lambda$, see [1, Example 2.2 and Example 4.2] and [10].

DEFINITION 3.1. The complex Weyl symbol of an operator A on $\mathscr{F}_\lambda$ with kernel k_A is the function $W_0(A)$ on $\mathbb{C}^n$ defined by

$$W_0(A)(z)=2^n\int_{\mathbb{C}^n}k_A(z+w,z-w)\exp\left(\frac{\lambda}{2}(-z\bar{z}-w\bar{w}+z\bar{w}-\bar{z}w)\right)d\mu_\lambda(w). \quad (3.1)$$

Recall that that $W_0:\mathscr{L}_2(\mathscr{F}_\lambda)\to L^2(\mathbb{C}^n,\mu_\lambda)$ is the unitary part in the polar decomposition of S_λ, that is we have $S_\lambda=B_\lambda^{1/2}W_0$, see [26, Theorem 6], [8], [12].

DEFINITION 3.2. The classical Weyl symbol of an operator A on $L^2(\mathbb{R}^n)$ is the function $W_1(A)$ on $\mathbb{R}^{2n}$ defined by

$$W_1(A)(x,y)=W_0(\mathscr{B}A\mathscr{B}^{-1})(x+iy).$$

Now we connect W_1 to the classical Weyl correspondence on $\mathbb{R}^{2n}$ which can be defined as follows [18], [23]. For each function f in the Schwartz space $\mathscr{S}(\mathbb{R}^{2n})$, we define the operator $\mathscr{W}(f)$ acting on the Hilbert space $L^2(\mathbb{R}^n)$ by

$$(\mathscr{W}(f)\phi)(x)=(2\pi)^{-n}\int_{\mathbb{R}^{2n}}e^{iyt}f(x+\tfrac{1}{2}y,t)\,\phi(x+y)\,dy\,dt. \quad (3.2)$$

Then we have the following proposition.

PROPOSITION 3.3. ([11], [14]) Let $f \in \mathscr{S}(\mathbb{R}^{2n})$. Then, for each $x, y \in \mathbb{R}^n$, we have

$$W_1(\mathscr{W}(f))(x,y) = f(x, \lambda y)$$

and, if $\lambda = 1$, then W_1 and $\mathscr{W}$ are inverse to each other.

4. The extended metaplectic representation

In this section, we first review some results from [18] about the metaplectic representation.

Let $S := Sp(n,\mathbb{C}) \cap SU(n,n)$. Then the map $M \to UMU^{-1}$ where $U := \begin{pmatrix} I_n & iI_n \\ I_n & -iI_n \end{pmatrix}$ is an isomorphism from S to $Sp(n,\mathbb{R})$ [18, p. 175]. Note that S consists of all matrices

$$k = \begin{pmatrix} P & Q \\ \bar{Q} & \bar{P} \end{pmatrix}, \quad P, Q \in M_n(\mathbb{C}), \quad PP^* - QQ^* = I_n, \quad PQ^t = QP^t.$$

Here the superscript 't' denotes transposition. We also have

$$P^*P - Q^t\bar{Q} = I_n, \qquad P^*Q = Q^t\bar{P}.$$

and we see that P is invertible and that $P^{-1}Q$ and $\bar{Q}P^{-1}$ are symmetric.

Consider the natural action of $k = \begin{pmatrix} P & Q \\ \bar{Q} & \bar{P} \end{pmatrix} \in S$ on $(z, \bar{z})$, $z \in \mathbb{C}^n$:

$$k(z, \bar{z}) = (Pz + Q\bar{z}, \bar{Q}z + \bar{P}\bar{z}).$$

Then S also acts on H_n by

$$k \cdot ((z, \bar{z}), c) = (k(z, \bar{z}), c).$$

For each $k = \begin{pmatrix} P & Q \\ \bar{Q} & \bar{P} \end{pmatrix} \in S$, let $\sigma(k)$ be the operator on $\mathscr{F}_\lambda$ with kernel

$$b_k(z,w) = (\operatorname{Det} P)^{-1/2} \exp\left(\frac{\lambda}{4}\left(z(\bar{Q}P^{-1}z) + 2(P^{-1}z)\bar{w} - \bar{w}(P^{-1}Q\bar{w})\right)\right). \tag{4.1}$$

where $z^{1/2}$ as the principal determination of the square-root. Then we have the following result.

PROPOSITION 4.1. ([14], [18])

1. For each $k, k' \in S$, we have $\sigma(kk') = \pm\sigma(k)\sigma(k')$;
2. For each $k \in S$, $\sigma(k)$ is unitary;
3. For each $k \in S$ and $h \in H_n$, we have $\rho_\lambda(k \cdot h)\sigma(k) = \sigma(k)\rho_\lambda(h)$;
4. For each $k = \begin{pmatrix} P & Q \\ \bar{Q} & \bar{P} \end{pmatrix} \in S$, we have

$$S_\lambda(\sigma(k))(z) = (\operatorname{Det} P)^{-1/2} \exp\left(\frac{\lambda}{4}\left(z(\bar{Q}P^{-1}z) + 2((P^{-1} - I_n)z)\bar{z} - \bar{z}(P^{-1}Q\bar{z})\right)\right).$$

We have also a similar result for $d\sigma$. Let $\mathfrak{s}$ be the Lie algebra of S.

PROPOSITION 4.2. ([14]) Let $X = \begin{pmatrix} A & B \\ \bar{B} & \bar{A} \end{pmatrix} \in \mathfrak{s}$.

1. We have
$$(d\sigma(X)f)(z) = (-\tfrac{1}{2}\operatorname{Tr}(A) + \tfrac{\lambda}{4}z(\bar{B}z))f(z) - \sum_{j=1}^{n}(Az)_j\frac{\partial f}{\partial z_j} - \tfrac{1}{\lambda}\sum_{j,k} b_{jk}\frac{\partial^2 f}{\partial z_j \partial z_k}$$
where $B = (b_{jk})$;
2. The kernel of $d\sigma(X)$ is
$$b_X(z,w) = \left(-\tfrac{1}{2}\operatorname{Tr}(A) + \tfrac{\lambda}{4}z(\bar{B}z) - \tfrac{\lambda}{2}(Az)\bar{w} - \tfrac{\lambda}{4}\bar{w}(B\bar{w})\right)\exp\left(\tfrac{\lambda}{2}z\bar{w}\right);$$
3. We have
$$S_\lambda(d\sigma(X))(z) = -\tfrac{1}{2}\operatorname{Tr}(A) + \tfrac{\lambda}{4}z(\bar{B}z) - \tfrac{\lambda}{2}(Az)\bar{z} - \tfrac{\lambda}{4}\bar{z}(B\bar{z}).$$

The (multi-dimensional) Jacobi group is the semi-direct product $G := H_n \rtimes S$ with respect to the action of S on H_n [7], [9], [27]. The elements of G are written as $((z,\bar{z}),c,k)$ where $z \in \mathbb{C}^n$, $c \in \mathbb{R}$ and $k \in S$. The multiplication of G is then given by
$$((z,\bar{z}),c,k)\cdot((z',\bar{z}'),c',k') = ((z,\bar{z}) + k(z',\bar{z}'), c + c' + \tfrac{1}{2}\omega((z,\bar{z}),k(z',\bar{z}')), kk').$$

We denote by $\mathfrak{g}$ the Lie algebra of G. The elements of $\mathfrak{g}$ can be written as $\left((z,\bar{z}),c,\begin{pmatrix} A & B \\ \bar{B} & \bar{A} \end{pmatrix}\right)$.

The extended metaplectic representation π of G is defined by
$$\pi(h,k) := \rho_\lambda(h)\sigma(k), \qquad (h,k) \in G.$$

The fact that π is a (projective) unitary representation of G follows from Proposition 4.1. Moreover, π is irreducible since ρ_λ is.

Now, we compute the Berezin symbols of $\pi(g)$, $g \in G$ and $d\pi(X)$, $X \in \mathfrak{g}$.

PROPOSITION 4.3. *Let $g = (h,k) \in G$ with $h = ((z_0,\bar{z_0}),c_0) \in H_n$ and $k = \begin{pmatrix} P & Q \\ \bar{Q} & \bar{P} \end{pmatrix} \in S$. Then the kernel of $\pi(g)$ is*
$$\begin{aligned} B_g(z,w) =& (\operatorname{Det} P)^{-1/2}\exp(i\lambda c_0)\exp\left(\tfrac{\lambda}{4}\left(-|z_0|^2 + z_0(\bar{Q}P^{-1}z_0)\right)\right)\\ &\times\exp\left(\tfrac{\lambda}{2}\left(\bar{z_0}z - z(\bar{Q}P^{-1}z_0) - (P^{-1}z_0)\bar{w}\right)\right)\\ &\times\exp\left(\tfrac{\lambda}{4}\left(z(\bar{Q}P^{-1}z) + 2(P^{-1}z)\bar{w} - \bar{w}(P^{-1}Q\bar{w})\right)\right). \end{aligned}$$

Moreover, we have
$$\begin{aligned} S_\lambda(\pi(g))(z) =& (\operatorname{Det} P)^{-1/2}\exp(i\lambda c_0)\exp\left(\tfrac{\lambda}{4}\left(-|z_0|^2 + z_0(\bar{Q}P^{-1}z_0)\right)\right)\\ &\times\exp\left(\tfrac{\lambda}{2}\left(\bar{z_0}z - z(\bar{Q}P^{-1}z_0) - (P^{-1}z_0)\bar{z}\right)\right)\\ &\times\exp\left(\tfrac{\lambda}{4}\left(z(\bar{Q}P^{-1}z) + 2((P^{-1} - I_n)z)\bar{z} - \bar{z}(P^{-1}Q\bar{z})\right)\right). \end{aligned}$$

Proof. Let $g=(h,k)\in G$ with $h=((z_0,\bar{z_0}),c_0)\in H_n$ and $k=\begin{pmatrix} P & Q\\ \bar{Q} & \bar{P}\end{pmatrix}\in S$. By Equation 2.1, the kernel $B_g(z,w)$ of $\pi(g)$ is given by

$$B_g(z,w)=\langle \pi(g)e_w,e_z\rangle=\langle \rho_\lambda(h)\sigma(k)e_w,e_z\rangle=\langle \sigma(k)e_w,\rho_\lambda(h)^{-1}e_z\rangle.$$

But we have

$$\begin{aligned}(\rho_\lambda(h)^{-1}e_z)(w)&=\exp\left(-i\lambda c_0-\tfrac{\lambda}{2}\bar{z_0}w-\tfrac{\lambda}{4}|z_0|^2\right)e_z(w+z_0)\\ &=\exp\left(-i\lambda c_0+\tfrac{\lambda}{2}\bar{z}z_0-\tfrac{\lambda}{4}|z_0|^2\right)e_{z-z_0}(w).\end{aligned}$$

This gives

$$\rho_\lambda(h)^{-1}e_z=\exp\left(-i\lambda c_0+\tfrac{\lambda}{2}\bar{z}z_0-\tfrac{\lambda}{4}|z_0|^2\right)e_{z-z_0}$$

hence

$$\begin{aligned}B_g(z,w)&=\exp\left(i\lambda c_0+\tfrac{\lambda}{2}\bar{z_0}z-\tfrac{\lambda}{4}|z_0|^2\right)\langle\sigma(k)e_w,e_{z-z_0}\rangle\\ &=\exp\left(i\lambda c_0+\tfrac{\lambda}{2}\bar{z_0}z-\tfrac{\lambda}{4}|z_0|^2\right)b_k(z-z_0,w).\end{aligned}$$

The desired formula for $B_g(z,w)$ follows from Proposition 4.1. To prove the second assertion of the proposition, we have just to write that

$$S_\lambda(\pi(g))=B_g(z,z)\langle e_z,e_z\rangle^{-1}.\quad\square$$

Passing to the differential of π and using Proposition 4.2, we easily obtain the following proposition.

PROPOSITION 4.4. *Let $X=\left((z_0,\bar{z_0}),c_0,\begin{pmatrix} A & B\\ \bar{B} & \bar{A}\end{pmatrix}\right)\in\mathfrak{g}$. Then we have*

$$S_\lambda(d\pi(X))(z)=i\lambda c_0+\tfrac{\lambda}{2}(\bar{z_0}z-z_0\bar{z})-\tfrac{1}{2}\operatorname{Tr}(A)+\tfrac{\lambda}{4}z(\bar{B}z)-\tfrac{\lambda}{2}(Az)\bar{z}-\tfrac{\lambda}{4}\bar{z}(B\bar{z}).$$

5. Complex Weyl symbols of representation operators

In this section, we compute $W_0(\pi(g))$ for $g\in G$. We start with some technical lemmas.

LEMMA 5.1. *Let A,B,D be $n\times n$ complex matrices such that $A^t=A,D^t=D$. Let $M=\begin{pmatrix} A & B^t\\ B & D\end{pmatrix}$, $U=\begin{pmatrix} I_n & iI_n\\ I_n & -iI_n\end{pmatrix}$ and $N=U^tMU$. Assume that* $\operatorname{Re}(N)$ *is positive definite. Let $u,v\in\mathbb{C}^n$. Then we have*

$$\begin{aligned}&\int_{\mathbb{C}^n}\exp\left(-(w(Aw)+\bar{w}(D\bar{w})+2\bar{w}(Bw))\right)\exp(uw+v\bar{w})\,dm(w)\\ &=\pi^n(\operatorname{Det}N)^{-1/2}\exp\left(\tfrac{1}{4}\begin{pmatrix} u & v\end{pmatrix}M^{-1}\begin{pmatrix} u\\ v\end{pmatrix}\right).\end{aligned}$$

Proof. See [18, App A, Theorem 3] or [14]. □

LEMMA 5.2. *Let a,b,p be $n\times n$ complex matrices such that $\begin{pmatrix} -a & I_n+p^t \\ I_n+p & d\end{pmatrix}$ has inverse matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta\end{pmatrix}$. Then we have the following equations*

$$\begin{pmatrix} a & I_n-p^t \\ p-I_n & d\end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta\end{pmatrix}\begin{pmatrix} a & p^t-I_n \\ I_n-p & d\end{pmatrix}=\begin{pmatrix} 4\delta-a & 3I_n-4\gamma-p^t \\ 3I_n-4\beta-p & 4\alpha+d\end{pmatrix};$$
$$\begin{pmatrix} a & I_n-p^t \\ p-I_n & d\end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta\end{pmatrix}\begin{pmatrix} -a & I_n \\ p & 0\end{pmatrix}=\begin{pmatrix} a-2\delta & 2\gamma-I_n \\ 2\beta+p-2I_n & -2\alpha\end{pmatrix};$$
$$\begin{pmatrix} -a & p^t \\ I_n & 0\end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta\end{pmatrix}\begin{pmatrix} -a & I_n \\ p & 0\end{pmatrix}=\begin{pmatrix} \delta-a & I_n-\gamma \\ I_n-\beta & \alpha\end{pmatrix}.$$

Proof. From the equalities

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta\end{pmatrix}\begin{pmatrix} -a & I_n+p^t \\ I_n+p & d\end{pmatrix}=\begin{pmatrix} -a & I_n+p^t \\ I_n+p & d\end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta\end{pmatrix}=I_{2n}$$

we deduce the series of equations

$$\begin{aligned}\alpha a &=\beta(I_n+p)-I_n; & \gamma a &= \delta(I_n+p);\\ \beta d &=-\alpha(I_n+p^t); & \delta d &= I_n-\gamma(I_n+p^t);\\ a\alpha &=(I_n+p^t)\gamma-I_n; & a\beta &= (I_n+p^t)\delta;\\ d\gamma &=-(I_n+p)\alpha; & d\delta &= I_n-(I_n+p)\beta.\end{aligned}$$

By using these equations, we verify that

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta\end{pmatrix}\begin{pmatrix} a & -I_n+p^t \\ I_n-p & d\end{pmatrix}=\begin{pmatrix} 2\beta-I_n & -2\alpha \\ 2\delta & I_n-2\gamma\end{pmatrix}$$

which implies that

$$\begin{pmatrix} a & I_n-p^t \\ -I_n+p & d\end{pmatrix}\begin{pmatrix} 2\beta-I_n & -2\alpha \\ 2\delta & I_n-2\gamma\end{pmatrix}=\begin{pmatrix} 4\delta-a & 3I_n-4\gamma-p^t \\ 3I_n-4\beta-p & 4\alpha+d\end{pmatrix}.$$

Hence we have proved the first equation of the lemma. Similarly, one can use the preceding series of equations in order to verify that

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta\end{pmatrix}\begin{pmatrix} -a & I_n \\ p & 0\end{pmatrix}=\begin{pmatrix} I_n-\beta & \alpha \\ -\delta & \gamma\end{pmatrix}$$

and

$$\begin{pmatrix} a & I_n-p^t \\ p-I_n & d\end{pmatrix}\begin{pmatrix} I_n-\beta & \alpha \\ -\delta & \gamma\end{pmatrix}=\begin{pmatrix} a-2\delta & 2\gamma-I_n \\ 2\beta+p-2I_n & -2\alpha\end{pmatrix}.$$

The second equation of the lemma follows. We also have

$$\begin{pmatrix} -a & p^t \\ I_n & 0\end{pmatrix}\begin{pmatrix} I_n-\beta & \alpha \\ -\delta & \gamma\end{pmatrix}=\begin{pmatrix} \delta-a & I_n-\gamma \\ I_n-\beta & \alpha\end{pmatrix}.$$

This gives the third equation of the lemma. □

LEMMA 5.3. *Let* $k = \left(\begin{smallmatrix} P & Q \\ \bar{Q} & \bar{P} \end{smallmatrix}\right) \in S$.

1. *Let*

$$M_0 = \begin{pmatrix} -\bar{Q}P^{-1} & I_n + (P^t)^{-1} \\ I_n + P^{-1} & P^{-1}Q \end{pmatrix}.$$

Then we have

$$\mathrm{Det}(M_0) = (-1)^n (\mathrm{Det}\,P)^{-1}\,\mathrm{Det}(k + I_{2n}).$$

2. *Let* $J = \left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right)$. *With the notation of Lemma 5.2, take* $a = \bar{Q}P^{-1}$, $d = P^{-1}Q$ *and* $p = P^{-1}$. *Then we have*

$$\tfrac{1}{2}J(k - I_{2n})(k + I_{2n})^{-1} = \begin{pmatrix} \delta & \frac{1}{2}I_n - \gamma \\ \frac{1}{2}I_n - \beta & \alpha \end{pmatrix}$$

and

$$J(k + I_{2n})^{-1} = \begin{pmatrix} -\delta & \gamma \\ \beta - I_n & -\alpha \end{pmatrix}.$$

3. *Let* $J_0 = \left(\begin{smallmatrix} 0 & I_n \\ I_n & 0 \end{smallmatrix}\right)$, $M_0' = M_0 - J_0$ *and* $N_0 = U^t M_0 U$. *Assume that* $\mathrm{Det}(k + I_{2n}) \neq 0$. *Then* $\mathrm{Re}(N_0)$ *is a positive definite matrix.*

Proof. Part of this lemma has already been proved in [14]. Netherless, we detail here the whole proof for completeness. We remark that we have

$$\begin{pmatrix} -\bar{Q}P^{-1} & I_n + (P^t)^{-1} \\ I_n + P^{-1} & P^{-1}Q \end{pmatrix} \begin{pmatrix} -P & -Q \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} \bar{Q} & \bar{Q}P^{-1}Q + I_n + (P^t)^{-1} \\ -I_n - P & -Q \end{pmatrix}$$
$$= \begin{pmatrix} \bar{Q} & I_n + \bar{P} \\ -I_n - P & -Q \end{pmatrix} = J(k + I_{2n})$$

since

$$\bar{Q}P^{-1}Q + (P^t)^{-1} = (\bar{Q}P^{-1}QP^t + I_n)(P^t)^{-1} = \bar{P}.$$

Then, by taking the determinant, we get (1). Moreover, we also obtain

$$\begin{pmatrix} -P & -Q \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} J(k + I_{2n})$$

hence

$$\begin{pmatrix} \delta & \frac{1}{2}I_n - \gamma \\ \frac{1}{2}I_n - \beta & \alpha \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} - J\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} J$$
$$= \frac{1}{2}\begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} - J\begin{pmatrix} -P & -Q \\ 0 & I_n \end{pmatrix}(k + I_{2n})^{-1}$$

and the first equation of (2) follows. Similarly, we can write

$$\begin{aligned}\begin{pmatrix}-\delta & \gamma\\ \beta-I_n & -\alpha\end{pmatrix} &= \begin{pmatrix}0 & 0\\ -I_n & 0\end{pmatrix} + J\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}J\\ &= \begin{pmatrix}0 & 0\\ -I_n & 0\end{pmatrix} + J\begin{pmatrix}-P & -Q\\ 0 & I_n\end{pmatrix}(k+I_{2n})^{-1}\\ &= \left(\begin{pmatrix}0 & 0\\ -I_n & 0\end{pmatrix}(k+I_{2n}) + J\begin{pmatrix}-P & -Q\\ 0 & I_n\end{pmatrix}\right)(k+I_{2n})^{-1}\\ &= J(k+I_{2n})^{-1}.\end{aligned}$$

This ends the proof of (2). In order to prove (3), we first note that

$$UN_0U^* = (UU^t)M_0(UU^*) = 4J_0M_0 = 4J_0(J_0+M_0') = 4(I_{2n}+J_0M_0').$$

By using the relations given at the beginning of Section 4, we can verify that M_0' is unitary, hence J_0M_0' is unitary. Then there exist a unitary matrix T_0 and $t_1,t_2,\ldots,t_{2n} \in \mathbb{R}$ such that

$$J_0M_0' = T_0\,\mathrm{Diag}(e^{it_1},e^{it_2},\ldots,e^{it_{2n}})T_0^*.$$

Thus we have

$$UN_0U^* = 4T_0\,\mathrm{Diag}(1+e^{it_1},1+e^{it_2},\ldots,1+e^{it_{2n}})T_0^*.$$

Consequently, we can write

$$N_0 = T\,\mathrm{Diag}(1+e^{it_1},1+e^{it_2},\ldots,1+e^{it_{2n}})T^*$$

where T is invertible. Noting that N_0 is symmetric since M_0 is symmetric, we deduce that

$$\mathrm{Re}(N_0) = \tfrac{1}{2}(N_0+\overline{N_0}) = \tfrac{1}{2}(N_0+N_0^*) = T\,\mathrm{Diag}(1+\cos(t_1),1+\cos(t_2),\ldots,1+\cos(t_{2n}))T^*.$$

Now, on the one hand, we have

$$\mathrm{Det}\,N_0 = (\mathrm{Det}\,U)^2\,\mathrm{Det}\,M_0 = (\mathrm{Det}\,U)^2(-1)^n(\mathrm{Det}\,P)^{-1}\,\mathrm{Det}(k+I_{2n}) \neq 0$$

and, on the other hand,

$$\mathrm{Det}\,N_0 = |\,\mathrm{Det}\,T|^2\prod_{j=1}^{2n}(1+e^{it_j}).$$

This shows that, for each $j=1,2,\ldots,2n$, we have $1+e^{it_j}\neq 0$ hence $1+\cos(t_j)>0$. This implies that $\mathrm{Re}(N_0)$ is positive definite. $\square$

Let us denote by $\mathrm{Arg}(z)$ the principal argument of $z\in\mathbb{C}$. For each $k=\begin{pmatrix}P & Q\\ \bar{Q} & \bar{P}\end{pmatrix}\in S$, define $c_n(k)$ as follows.

$c_n(k) := 2^n(\mathrm{Det}(I_{2n}+k))^{-1/2}$ if $\mathrm{Det}(I_{2n}+k)>0$;

$c_n(k) := -i2^n|\,\mathrm{Det}(I_{2n}+k)|^{-1/2}$ if $\mathrm{Det}(I_{2n}+k)<0$ and $\mathrm{Arg}(\mathrm{Det}(P))\in]0,\pi[$;

$c_n(k) := i2^n|\,\mathrm{Det}(I_{2n}+k)|^{-1/2}$ if $\mathrm{Det}(I_{2n}+k)<0$ and $\mathrm{Arg}(\mathrm{Det}(P))\in]-\pi,0[$.

THEOREM 5.4. *Let* $k=\begin{pmatrix} P & Q\\ \bar{Q} & \bar{P}\end{pmatrix}\in S$ *such that* $\mathrm{Det}(I_{2n}+k)\neq 0$. *Let* $g=((z_0,\bar{z_0}), c_0,k)\in G$. *Let* $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ *be the inverse matrix of* $\begin{pmatrix} -\bar{Q}P^{-1} & I_n+(P^t)^{-1}\\ I_n+P^{-1} & P^{-1}Q\end{pmatrix}$.

1. *For each* $z\in\mathbb{C}^n$, *we have*

$$\begin{aligned}W_0(\pi(g))(z)=&c_n(k)\exp(i\lambda c_0)\exp\left(\lambda\,(z\;\bar{z})\begin{pmatrix}\delta & \frac{1}{2}I_n-\gamma\\ \frac{1}{2}I_n-\beta & \alpha\end{pmatrix}\begin{pmatrix}z\\ \bar{z}\end{pmatrix}\right)\\ &\times\exp\left(\lambda\,(z\;\bar{z})\begin{pmatrix}-\delta & \gamma\\ \beta-I_n & -\alpha\end{pmatrix}\begin{pmatrix}z_0\\ \bar{z_0}\end{pmatrix}\right)\\ &\times\exp\left(\frac{\lambda}{4}\,(z_0\;\bar{z_0})\begin{pmatrix}\delta & \frac{1}{2}I_n-\gamma\\ \frac{1}{2}I_n-\beta & \alpha\end{pmatrix}\begin{pmatrix}z_0\\ \bar{z_0}\end{pmatrix}\right).\end{aligned}$$

2. *Alternatively, for each* $z\in\mathbb{C}^n$, *we have*

$$\begin{aligned}W_0(\pi(g))(z)=&c_n(k)\exp(i\lambda c_0)\exp\left(\frac{\lambda}{2}\,(z\;\bar{z})\,J(k-I_{2n})(k+I_{2n})^{-1}\begin{pmatrix}z\\ \bar{z}\end{pmatrix}\right)\\ &\times\exp\left(\lambda\,(z\;\bar{z})\,J(k+I_{2n})^{-1}\begin{pmatrix}z_0\\ \bar{z_0}\end{pmatrix}\right)\\ &\times\exp\left(\frac{\lambda}{8}\,(z_0\;\bar{z_0})\,J(k-I_{2n})(k+I_{2n})^{-1}\begin{pmatrix}z_0\\ \bar{z_0}\end{pmatrix}\right).\end{aligned}$$

Proof. Let $g=((z_0,\bar{z_0}),c_0,k)\in G$ with $k=\begin{pmatrix} P & Q\\ \bar{Q} & \bar{P}\end{pmatrix}\in S$. We use the following formula, see Definition 3.1:

$$W_0(\pi(g))(z)=\left(\frac{\lambda}{\pi}\right)^n\int_{\mathbb{C}^n}B_g(z+w,z-w)\exp\left(\frac{\lambda}{2}\,(-z\bar{z}-w\bar{w}+z\bar{w}-\bar{z}w)\right)dm(w)$$

where the kernel B_g of $\pi(g)$ is given by Proposition 4.3. This gives

$$\begin{aligned}W_0(\pi(g))(z)=&\left(\frac{\lambda}{\pi}\right)^n(\mathrm{Det}\,P)^{-1/2}\exp\left(i\lambda c_0-\frac{\lambda}{4}|z_0|^2+\frac{\lambda}{4}z_0(\bar{Q}P^{-1}z_0)\right)\\ &\times\exp\left(\frac{\lambda}{2}z(\bar{z_0}-\bar{Q}P^{-1}z_0)-\frac{\lambda}{2}(P^{-1}z_0)\,\bar{z}\right)\\ &\times\exp\left(\frac{\lambda}{4}\left(z(\bar{Q}P^{-1}z)+2\,\bar{z}(P^{-1}-I_n)z-\bar{z}(P^{-1}Q\,\bar{z})\right)\right)\\ &\times\int_{\mathbb{C}^n}\exp\left(\frac{\lambda}{4}\left(w\bar{Q}P^{-1}w-\bar{w}(P^{-1}Q\bar{w})-2\bar{w}(I_n+P^{-1})w\right)\right)\\ &\times\exp\left(\frac{\lambda}{2}(\bar{Q}P^{-1}z+((P^t)^{-1}-I_n)\,\bar{z}+\bar{z_0}-\bar{Q}P^{-1}z_0)w\right)\\ &\times\exp\left(\frac{\lambda}{2}((I_n-P^{-1})z+P^{-1}Q\,\bar{z}+P^{-1}z_0)\bar{w})\right)dm(w).\end{aligned}$$

The integral in this formula, which we denote by $I(g)$, can be computed by applying Lemma 5.1 to

$$M=\frac{\lambda}{4}\begin{pmatrix}-\bar{Q}P^{-1} & I_n+(P^t)^{-1}\\ I_n+P^{-1} & P^{-1}Q\end{pmatrix}$$

and

$$u=\tfrac{\lambda}{2}(\bar{Q}P^{-1}z+((P^t)^{-1}-I_n)\,\bar{z}+\bar{z_0}-\bar{Q}P^{-1}z_0);$$

$$v=\tfrac{\lambda}{2}((I_n-P^{-1})z+P^{-1}Q\,\bar{z}+P^{-1}z_0).$$

Note that we have

$$\begin{pmatrix}u\\v\end{pmatrix}=\tfrac{\lambda}{2}\begin{pmatrix}\bar{Q}P^{-1} & (P^t)^{-1}-I_n\\ I_n-P^{-1} & P^{-1}Q\end{pmatrix}\begin{pmatrix}z\\ \bar{z}\end{pmatrix}+\tfrac{\lambda}{2}\begin{pmatrix}-\bar{Q}P^{-1} & I_n\\ P^{-1} & 0\end{pmatrix}\begin{pmatrix}z_0\\ \bar{z_0}\end{pmatrix}$$

and, consequently,

$$\begin{aligned}&\begin{pmatrix}u & v\end{pmatrix}M^{-1}\begin{pmatrix}u\\v\end{pmatrix}\\
=&\lambda\,\begin{pmatrix}z & \bar{z}\end{pmatrix}\begin{pmatrix}\bar{Q}P^{-1} & I_n-(P^t)^{-1}\\ -I_n+P^{-1} & P^{-1}Q\end{pmatrix}\begin{pmatrix}-\bar{Q}P^{-1} & I_n+(P^t)^{-1}\\ I_n+P^{-1} & P^{-1}Q\end{pmatrix}^{-1}\\
&\times\begin{pmatrix}\bar{Q}P^{-1} & -I_n+(P^t)^{-1}\\ I_n-P^{-1} & P^{-1}Q\end{pmatrix}\begin{pmatrix}z\\ \bar{z}\end{pmatrix}\\
&+2\lambda\,\begin{pmatrix}z & \bar{z}\end{pmatrix}\begin{pmatrix}\bar{Q}P^{-1} & I_n-(P^t)^{-1}\\ -I_n+P^{-1} & P^{-1}Q\end{pmatrix}\begin{pmatrix}-\bar{Q}P^{-1} & I_n+(P^t)^{-1}\\ I_n+P^{-1} & P^{-1}Q\end{pmatrix}^{-1}\\
&\times\begin{pmatrix}-\bar{Q}P^{-1} & I_n\\ P^{-1} & 0\end{pmatrix}\begin{pmatrix}z_0\\ \bar{z_0}\end{pmatrix}\\
&+\lambda\,\begin{pmatrix}z_0 & \bar{z_0}\end{pmatrix}\begin{pmatrix}-\bar{Q}P^{-1} & (P^t)^{-1}\\ I_n & 0\end{pmatrix}\begin{pmatrix}-\bar{Q}P^{-1} & I_n+(P^t)^{-1}\\ I_n+P^{-1} & P^{-1}Q\end{pmatrix}^{-1}\\
&\times\begin{pmatrix}-\bar{Q}P^{-1} & I_n\\ P^{-1} & 0\end{pmatrix}\begin{pmatrix}z_0\\ \bar{z_0}\end{pmatrix}.\end{aligned}$$

Recalling the notation $a=\bar{Q}P^{-1}$, $d=P^{-1}Q$, $p=P^{-1}$ and

$$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}=\begin{pmatrix}-a & I_n+p^t\\ I_n+p & d\end{pmatrix}^{-1}=\begin{pmatrix}-\bar{Q}P^{-1} & I_n+(P^t)^{-1}\\ I_n+P^{-1} & P^{-1}Q\end{pmatrix}^{-1}$$

and taking Lemma 5.2 into account, we get

$$\begin{aligned}\begin{pmatrix}u & v\end{pmatrix}M^{-1}\begin{pmatrix}u\\v\end{pmatrix}=&\lambda\,\begin{pmatrix}z & \bar{z}\end{pmatrix}\begin{pmatrix}4\delta-a & 3I_n-4\gamma-p^t\\ 3I_n-4\beta-p & 4\alpha+d\end{pmatrix}\begin{pmatrix}z\\ \bar{z}\end{pmatrix}\\
&+2\lambda\,\begin{pmatrix}z & \bar{z}\end{pmatrix}\begin{pmatrix}a-2\delta & 2\gamma-I_n\\ 2\beta+p-2I_n & -2\alpha\end{pmatrix}\begin{pmatrix}z_0\\ \bar{z_0}\end{pmatrix}\\
&+\lambda\,\begin{pmatrix}z_0 & \bar{z_0}\end{pmatrix}\begin{pmatrix}\delta-a & I_n-\gamma\\ I_n-\beta & \alpha\end{pmatrix}\begin{pmatrix}z_0\\ \bar{z_0}\end{pmatrix}.\end{aligned}$$

Thus we obtain

$$I(g) = \pi^n (\operatorname{Det} U^t M U)^{-1/2} \exp\left(\tfrac{1}{4}\,(u\ v)\,M^{-1}\begin{pmatrix} u \\ v\end{pmatrix}\right)$$
$$= \pi^n (\operatorname{Det} U^t M U)^{-1/2} \exp\left(\tfrac{\lambda}{4}\,(z\ \bar{z})\begin{pmatrix} 4\delta - a & 3I_n - 4\gamma - p^t \\ 3I_n - 4\beta - p & 4\alpha + d\end{pmatrix}\begin{pmatrix} z \\ \bar{z}\end{pmatrix}\right)$$
$$\times \exp\left(\tfrac{\lambda}{2}\,(z\ \bar{z})\begin{pmatrix} a - 2\delta & 2\gamma - I_n \\ 2\beta + p - 2I_n & -2\alpha\end{pmatrix}\begin{pmatrix} z_0 \\ \bar{z_0}\end{pmatrix}\right)$$
$$\times \exp\left(\tfrac{\lambda}{4}\,(z_0\ \bar{z_0})\begin{pmatrix} \delta - a & I_n - \gamma \\ I_n - \beta & \alpha\end{pmatrix}\begin{pmatrix} z_0 \\ \bar{z_0}\end{pmatrix}\right).$$

On the other hand, the factor in front of $I(g)$ in the formula for $W_0(\pi(g))$ can be written as

$$\left(\tfrac{\lambda}{\pi}\right)^n (\operatorname{Det} P)^{-1/2} \exp(i\lambda c_0) \exp\left(\tfrac{\lambda}{4}\,(z\ \bar{z})\begin{pmatrix} a & p^t - I_n \\ p - I_n & -d\end{pmatrix}\begin{pmatrix} z \\ \bar{z}\end{pmatrix}\right)$$
$$\times \exp\left(\tfrac{\lambda}{2}\,(z\ \bar{z})\begin{pmatrix} -a & I_n \\ -p & 0\end{pmatrix}\begin{pmatrix} z_0 \\ \bar{z_0}\end{pmatrix}\right) \exp\left(\tfrac{\lambda}{4}\,(z_0\ \bar{z_0})\begin{pmatrix} a & -\frac{1}{2}I_n \\ -\frac{1}{2}I_n & 0\end{pmatrix}\begin{pmatrix} z_0 \\ \bar{z_0}\end{pmatrix}\right).$$

Putting this expression with the formula found for $I(g)$, we obtain

$$W_0(\pi(g))(z) = \lambda^n (\operatorname{Det} P)^{-1/2} (\operatorname{Det} U^t M U)^{-1/2}$$
$$\times \exp(i\lambda c_0)\, \exp\left(\lambda\,(z\ \bar{z})\begin{pmatrix} \delta & \frac{1}{2}I_n - \gamma \\ \frac{1}{2}I_n - \beta & \alpha\end{pmatrix}\begin{pmatrix} z \\ \bar{z}\end{pmatrix}\right)$$
$$\times \exp\left(\lambda\,(z\ \bar{z})\begin{pmatrix} -\delta & \gamma \\ \beta - I_n & -\alpha\end{pmatrix}\begin{pmatrix} z_0 \\ \bar{z_0}\end{pmatrix}\right)$$
$$\times \exp\left(\tfrac{\lambda}{4}\,(z_0\ \bar{z_0})\begin{pmatrix} \delta & \frac{1}{2}I_n - \gamma \\ \frac{1}{2}I_n - \beta & \alpha\end{pmatrix}\begin{pmatrix} z_0 \\ \bar{z_0}\end{pmatrix}\right).$$

But by using Lemma 5.3 we notice that

$$\operatorname{Det}(U^t M U) = (-1)^n 2^{2n} \operatorname{Det}(M) = 2^{-2n}\lambda^{2n} \operatorname{Det}(k + I_{2n})(\operatorname{Det} P)^{-1}$$

where $\operatorname{Det}(k+I_{2n})$ is real, since there exists $k' \in Sp(n,\mathbb{R})$ such that $k = U^{-1}k'U$ hence $\operatorname{Det}(k+I_{2n}) = \operatorname{Det}(k'+I_{2n})$. This proves that $2^n(\operatorname{Det} P)^{-1/2}(\operatorname{Det}(k+I_{2n})(\operatorname{Det} P)^{-1})^{-1/2}$ takes the announced value $c_n(k)$. This ends the proof of (1).

(2) is just a reformulation of (1) taking Lemma 5.3 into account. □

We compute now $W_0(d\pi(X))$ for $X \in \mathfrak{g}$.

PROPOSITION 5.5. *Let* $X = ((z_0, \bar{z_0}), c_0, Y) \in \mathfrak{g}$ *with* $Y = \begin{pmatrix} A & B \\ \bar{B} & \bar{A}\end{pmatrix} \in \mathfrak{s}$. *Then we have*

$$W_0(d\pi(X))(z) = i\lambda c_0 + \tfrac{\lambda}{2}(\bar{z}_0 z - z_0 \bar{z}) + \tfrac{\lambda}{4}(z(\bar{B}z) - \bar{z}(B\bar{z}) - 2(Az)\bar{z})$$

for each $z \in \mathbb{C}^n$.

Proof. We can use the formula $W_0 = B_\lambda^{-1/2} S_\lambda$, see Section 3, where $B_\lambda = \exp(\Delta/2\lambda)$ with $\Delta = 4\sum_{k=1}^n \partial^2/\partial z_k \partial \bar{z}_k$, see Section 2. Then we get $B_\lambda^{-1/2} = \exp(-\frac{1}{\lambda}\sum_{k=1}^n \partial^2/\partial z_k \partial \bar{z}_k)$.

Thus, by using the formula for $S_\lambda(d\pi(X))$ given in Proposition 4.3, we find

$$B_\lambda^{-1/2}(S_\lambda(d\pi(X)))(z) = -\frac{1}{\lambda}\sum_{k=1}^n \partial^2/\partial z_k \partial \bar{z}_k (S_\lambda(d\pi(X))) = \frac{1}{2}\sum_{k=1}^n (Ae_k)e_k = \frac{1}{2}\operatorname{Tr}(A),$$

hence the result follows. $\square$

6. Classical Weyl symbols of representation operators

For each $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$ and each $c \in \mathbb{R}$, let us denote by (x,y,c) the element $((x+iy, x-iy), c)$ of H_n. Then the multiplication of H_n can be written as

$$(x,y,c).(x',y',c') = (x+x', y+y', c+c'+\tfrac{1}{2}(xy'-x'y)).$$

We can thus consider the semidirect product $G' := H_n \rtimes Sp(n,\mathbb{R})$ with respect to the action of $Sp(n,\mathbb{R})$ on H_n given by

$$k' \cdot (x,y,c) = (x',y',c), \quad \text{where} \quad \begin{pmatrix} x' \\ y' \end{pmatrix} = k' \begin{pmatrix} x \\ y \end{pmatrix}.$$

Let $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$ and $(x',y') \in \mathbb{R}^n \times \mathbb{R}^n$. Let $z = x+iy$ and $z' = x'+iy'$. For $k \in S$, let $k' = U^{-1}kU$. Note that we have

$$\begin{pmatrix} z' \\ \bar{z}' \end{pmatrix} = k \begin{pmatrix} z \\ \bar{z} \end{pmatrix} \Leftrightarrow U \begin{pmatrix} x' \\ y' \end{pmatrix} = kU \begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = k' \begin{pmatrix} x \\ y \end{pmatrix}.$$

From this, we see that the action of S on H_n corresponds to the action of $Sp(n,\mathbb{R})$ on $\mathbb{R}^{2n}$.

Recall that the metaplectic representation σ' of $Sp(n,\mathbb{R})$ can be defined as follows, see [18, Chapter IV]. For each $k' \in Sp(n,\mathbb{R})$, we define $\sigma'(k') := \mathscr{B}^{-1}\sigma(Uk'U^{-1})\mathscr{B}$. Similarly, we can introduce the extended metaplectic representation of G' via

$$\pi'((x,y,c),k') := \mathscr{B}^{-1}\pi((x+iy,x-iy),c,Uk'U^{-1})\mathscr{B}.$$

Then we have

$$\pi'((x,y,c),k') = (\mathscr{B}^{-1}\rho_\lambda((x+iy,x-iy),c)\mathscr{B})(\mathscr{B}^{-1}\sigma(Uk'U^{-1})\mathscr{B}) = \rho'_\lambda(x,y,c)\sigma'(k').$$

From now on, we take $\lambda = 1$. We deduce from Section 5 formulas for the classical Weyl symbol of $\pi'(g')$ for $g' \in G'$.

For $k' = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in Sp(n,\mathbb{R})$, we define $c'_n(k')$ by $c'_n(k') = c_n(Uk'U^{-1})$. Then, from the definition of c_n, we easily deduce that

$$\begin{aligned} c'_n(k') &= 2^n(\operatorname{Det}(I_{2n}+k'))^{-1/2} \text{ if } \operatorname{Det}(I_{2n}+k') > 0; \\ c'_n(k') &= -i2^n|\operatorname{Det}(I_{2n}+k')|^{-1/2} \text{ if } \operatorname{Det}(I_{2n}+k') < 0 \\ &\qquad \text{and} \quad \operatorname{Arg}(\operatorname{Det}(A+D+i(C-B))) \in]0,\pi[; \\ c'_n(k') &= i2^n|\operatorname{Det}(I_{2n}+k')|^{-1/2} \text{ if } \operatorname{Det}(I_{2n}+k') < 0 \\ &\qquad \text{and} \quad \operatorname{Arg}(\operatorname{Det}(A+D+i(C-B))) \in]-\pi,0[. \end{aligned}$$

THEOREM 6.1. *Let* $g' = ((x_0, y_0, c_0), k') \in G'$ *such that* $\operatorname{Det}(I_{2n} + k') \neq 0$. *Then we have*

$$W_1(\pi'(g'))(x,y) = c_n'(k') \exp(ic_0) \exp\left(-i \begin{pmatrix} x & y \end{pmatrix} J(k' - I_{2n})(k' + I_{2n})^{-1} \begin{pmatrix} x \\ y \end{pmatrix}\right)$$
$$\times \exp\left(-2i \begin{pmatrix} x & y \end{pmatrix} J(k + I_{2n})^{-1} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}\right)$$
$$\times \exp\left(-\tfrac{i}{4} \begin{pmatrix} x_0 & y_0 \end{pmatrix} J(k' - I_{2n})(k' + I_{2n})^{-1} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}\right).$$

Proof. This is a consequence of (2) of Theorem 5.4 since for each $g' = ((x_0, y_0, c_0), k') \in G'$ we have

$$\begin{aligned} W_1(\pi'(x_0,y_0,c_0),k')(x,y) =& W_1(\mathscr{B}^{-1}\pi((x_0+iy_0, x_0-iy_0), c_0, Uk'U^{-1})\mathscr{B})(x,y) \\ =& W_0(\pi((x_0+iy_0, x_0-iy_0), c_0, Uk'U^{-1}))(x+iy). \quad \square \end{aligned}$$

In particular, for $g' = ((0,0,0),k') \in G'$, we recover some known results, see for instance [14], [15], [16]. By the same way, from Proposition 5.5 we can deduce a formula for $W_1(d\pi'(X'))$, $X' \in \mathfrak{g}'$.

PROPOSITION 6.2. *For each* $X' = ((x_0, y_0, c_0), Y') \in \mathfrak{g}'$ *with* $Y' = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in sp(n, \mathbb{R})$, *we have*

$$\begin{aligned} W_1(d\pi'(X'))(x,y) =& ic_0 + i(x_0 y - x y_0) + \tfrac{i}{2}(2y(Ax) + y(By) - x(Cx)) \\ =& ic_0 + i(x_0 y - x y_0) - \tfrac{i}{2} \begin{pmatrix} x & y \end{pmatrix} JY' \begin{pmatrix} x \\ y \end{pmatrix}. \end{aligned}$$

7. Weyl symbols of the exponential of some operators

In the preceding section, we have seen that for each $X' \in \mathfrak{g}'$, $W_1(d\pi'(X'))$ is a polynomial on $\mathbb{R}^{2n}$ of degree $\leqslant 2$ (Proposition 6.2). Conversely, each polynomial on $\mathbb{R}^{2n}$ of degree $\leqslant 2$ can be written as $W_1(d\pi'(X'))$ for some $X' \in \mathfrak{g}'$. Since we have $\pi'(\exp X') = \exp(d\pi'(X'))$, we can deduce from Theorem 5.4 and Theorem 6.1 a formula for the Weyl symbol of the exponential of an operator whose Weyl symbol is a polynomial on $\mathbb{R}^{2n}$ of degree $\leqslant 2$, see Corollary 7.4. Then we recover Theorem 4.7 in [24]. We begin with a lemma on the exponential map in G.

LEMMA 7.1. *Let* $X = ((z_0, \bar{z_0}), c_0, Y) \in \mathfrak{g}$. *Let* $Z_0 = \begin{pmatrix} z_0 \\ \bar{z_0} \end{pmatrix}$. *Then, for each* $t \in \mathbb{R}$, *we have*

$$\exp(tX) = ((z(t), \overline{z(t)}), c(t), \exp(tY))$$

where

$$\begin{pmatrix} z(t) \\ \overline{z(t)} \end{pmatrix} = \frac{\exp(tY) - I_n}{Y} Z_0$$

and

$$c(t) = c_0 t + \tfrac{1}{2}\omega\left(Z_0, \frac{\sinh(tY) - tY}{Y^2} Z_0\right).$$

Proof. Let $X = ((z_0, \bar{z_0}), c_0, Y) \in \mathfrak{g}$. We can write

$$\exp(tX) = ((z(t), \overline{z(t)}), c(t), \exp(tY))$$

for $t \in \mathbb{R}$. Let $Z(t) := \begin{pmatrix} z(t) \\ \overline{z(t)} \end{pmatrix}$. Then the relation

$$\exp(t+s)X = \exp(tX)\exp(sX), \quad s,t \in \mathbb{R} \tag{7.1}$$

gives

$$Z(t+s) = Z(t) + \exp(tY)Z(s), \quad s,t \in \mathbb{R}.$$

Taking the derivative, we find $Z'(t) = \exp(tY)Z_0$ hence

$$Z(t) = \frac{\exp(tY) - I_n}{Y} Z_0.$$

Equation 7.1 also gives

$$c(t+s) = c(t) + c(s) + \tfrac{1}{2}\omega(Z(t), \exp(tY)Z(s)).$$

By differentiating at $t = 0$, we get

$$c'(s) = c_0 + \tfrac{1}{2}\omega(Z_0, Z(s)).$$

Thus

$$c(t) = c_0 t + \tfrac{1}{2}\omega\left(Z_0, \frac{\exp(tY) - I_n - tY}{Y^2} Z_0\right).$$

Noting that $\omega(Z_0, Y^{2k}Z_0) = 0$ for each non-negative integer k, we finally find

$$c(t) = c_0 t + \tfrac{1}{2}\omega\left(Z_0, \frac{\sinh(tY) - tY}{Y^2} Z_0\right). \quad \square$$

Now we give formulas for $W_0(\pi(\exp X))$, $X \in \mathfrak{g}$ and $W_1(\pi'(\exp X'))$, $X' \in \mathfrak{g}'$.

PROPOSITION 7.2. *Let* $X = ((z_0, \bar{z_0}), c_0, Y) \in \mathfrak{g}$. *Let* $t \in \mathbb{R}$ *such that* $\operatorname{Det}(I_{2n} + \exp(tY)) \neq 0$. *Then for each* $z \in \mathbb{C}^n$ *we have*

$$\begin{aligned} W_0(\pi(\exp(tX)))(z) = {} & \exp(itc_0)\operatorname{Det}(\cosh(\tfrac{1}{2}tY))^{-1/2} \\ & \times \exp\left(\tfrac{1}{2}\begin{pmatrix} z & \bar{z}\end{pmatrix} J \tanh(\tfrac{1}{2}tY)\begin{pmatrix} z \\ \bar{z}\end{pmatrix}\right) \\ & \times \exp\left(\begin{pmatrix} z & \bar{z}\end{pmatrix} J \frac{\tanh(\frac{1}{2}tY)}{Y}\begin{pmatrix} z_0 \\ \bar{z_0}\end{pmatrix}\right) \\ & \times \exp\left(\tfrac{1}{2}\begin{pmatrix} z_0 & \bar{z_0}\end{pmatrix} J \frac{\frac{1}{2}tY - \tanh(\frac{1}{2}tY)}{Y^2}\begin{pmatrix} z_0 \\ \bar{z_0}\end{pmatrix}\right). \end{aligned}$$

Proof. Let $X=((z_0,\overline{z_0}),c_0,Y)\in\mathfrak{g}$ and $t\in\mathbb{R}$. Let $Z_0=\binom{z_0}{\bar{z}_0}$. We write as before

$$\exp(tX)=((z(t),\overline{z(t)}),c(t),\exp(tY)),$$

and

$$Z(t):=\begin{pmatrix} z(t) \\ \overline{z(t)} \end{pmatrix}=\frac{\exp(tY)-I_n}{Y}Z_0.$$

We apply (2) of Theorem 5.4 to $g=\exp(tX)$. We get

$$\begin{aligned} W_0(\pi(\exp(tX)))(z) &= c_n(\exp(tY))\exp(ic(t)) \\ &\times\exp\left(\tfrac{1}{2}\left(z\ \bar{z}\right)J(\exp(tY)-I_{2n})(\exp(tY)+I_{2n})^{-1}\begin{pmatrix} z \\ \bar{z}\end{pmatrix}\right) \\ &\times\exp\left(\left(z\ \bar{z}\right)J(\exp(tY)+I_{2n})^{-1}Z(t)\right) \\ &\times\exp\left(\tfrac{1}{8}Z(t)^tJ(\exp(tY)-I_{2n})(\exp(tY)+I_{2n})^{-1}Z(t)\right). \end{aligned}$$

But we have the following relations

$$c_n(\exp(tY))=2^n\operatorname{Det}(\exp(tY)+I_{2n})^{-1/2}=\operatorname{Det}(\cosh(\tfrac{1}{2}tY))^{-1/2};$$

$$ic(t)=ic_0t+\tfrac{i}{2}\omega\left(Z_0,\frac{\sinh(tY)-tY}{Y^2}Z_0\right);$$

$$(\exp(tY)-I_{2n})(\exp(tY)+I_{2n})^{-1}=\tanh(\tfrac{1}{2}tY)$$

and also

$$\left(z\ \bar{z}\right)J(\exp(tY)+I_{2n})^{-1}Z(t)=\left(z\ \bar{z}\right)J\frac{\tanh(\frac{1}{2}tY)}{Y}Z_0.$$

Note that for each $Z=(z,w),Z'=(z',w')\in\mathbb{C}^{2n}$, we have

$$Z(JZ')=zw'-z'w=\tfrac{2}{i}\omega(Z,Z')$$

and, for each $Y\in\mathfrak{s}$, we have $\omega(YZ,Z')=-\omega(Z,YZ')$. Then we can write

$$\begin{aligned} &\tfrac{1}{8}Z(t)^tJ(\exp(tY)-I_{2n})(\exp(tY)+I_{2n})^{-1}Z(t) \\ &=\tfrac{1}{4i}\omega(Z(t),(\exp(tY)-I_{2n})(\exp(tY)+I_{2n})^{-1}Z(t)) \\ &=-\tfrac{i}{4}\omega\left(\frac{\exp(tY)-I_{2n}}{Y}Z_0,(\exp(tY)-I_{2n})(\exp(tY)+I_{2n})^{-1}\frac{\exp(tY)-I_{2n}}{Y}Z_0\right) \\ &=-\tfrac{i}{4}\omega\left(Z_0,\frac{I_{2n}-\exp(-tY)}{Y}(\exp(tY)-I_{2n})(\exp(tY)+I_{2n})^{-1}\frac{\exp(tY)-I_{2n}}{Y}Z_0\right) \\ &=-\tfrac{i}{2}\omega\left(Z_0,\frac{\sinh(tY)-2\tanh(\frac{1}{2}tY)}{Y^2}Z_0\right). \end{aligned}$$

The result follows. □

PROPOSITION 7.3. *Let $X' = ((x_0, y_0, c_0), Y') \in \mathfrak{g}'$. Let $t \in \mathbb{R}$ such that* $\mathrm{Det}(I_{2n} + \exp(tY')) \neq 0$. *Then, for each $(x,y) \in \mathbb{R}^{2n}$, we have*

$$\begin{aligned} W_1(\pi'(\exp(tX')))(x,y) = \; & \exp(itc_0)\,\mathrm{Det}(\cosh(\tfrac{1}{2}tY'))^{-1/2} \\ & \times \exp\left(-i\,(x\ y)\,J\tanh(\tfrac{1}{2}tY')\begin{pmatrix} x \\ y \end{pmatrix}\right) \\ & \times \exp\left(-2i\,(x\ y)\,J\frac{\tanh(\frac{1}{2}tY')}{Y'}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}\right) \\ & \times \exp\left(-i\,(x_0\ y_0)\,J\frac{\frac{1}{2}tY' - \tanh(\frac{1}{2}tY')}{Y'^2}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}\right). \end{aligned}$$

Proof. Let $z_0 = x_0 + iy_0$, $Y = UY'U^{-1}$ and $X = ((z_0, \bar{z_0}), c_0, Y) \in \mathfrak{g}$. Then we have

$$W_1(\pi'(\exp(tX')))(x,y) = W_1(\mathscr{B}^{-1}\pi(\exp(tX))\mathscr{B})(x,y) = W_0(\pi(\exp(tX)))(z)$$

and the result follows from Proposition 7.2. □

COROLLARY 7.4. *Let M be a real symmetric $(2n)\times(2n)$-matrix, $a,b \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Let A be the differential operator on $\mathbb{R}^n$ of classical Weyl symbol $ip(x,y)$ where*

$$p(x,y) := ax + by + c - (x\ y)\,M\begin{pmatrix} x \\ y \end{pmatrix}.$$

Assume that $\mathrm{Det}(I_{2n} + \exp(2JM)) \neq 0$. *Then the classical Weyl symbol of* $\exp(A)$ *is given by*

$$\begin{aligned} W_1(\exp(A))(x,y) = \; & \exp(ic)\,\mathrm{Det}(\cosh(JM))^{-1/2} \\ & \times \exp\left(i\,(x\ y)\,J\tanh(JM)\begin{pmatrix} x \\ y \end{pmatrix}\right) \\ & \times \exp\left(-i\,(x\ y)\,J\frac{\tanh(JM)}{JM}\begin{pmatrix} b \\ -a \end{pmatrix}\right) \\ & \times \exp\left(\tfrac{i}{4}\,(b\ -a)\,J\frac{JM - \tanh(JM)}{(JM)^2}\begin{pmatrix} b \\ -a \end{pmatrix}\right). \end{aligned}$$

Proof. By Proposition 6.2, we see that $A = d\pi'(X')$ where $X' = ((b,-a),c,-2JM)$. Then $\exp(A) = \pi'(\exp(X'))$ and the result follows from Proposition 7.3. □

EXAMPLE. We take $p(x,y) = ax + by + c - s(x^2 + y^2)$ where $a,b \in \mathbb{R}^n$ and $c,s \in \mathbb{R}$. Then $JM = sJ$, $\cosh(JM) = \cos(s)I_{2n}$, $\tanh(JM) = \tan(s)J$ and

$$\frac{\tanh(JM)}{JM} = \frac{\tan(s)}{s}I_{2n}, \quad \frac{JM - \tanh(JM)}{(JM)^2} = \frac{\tan(s) - s}{s^2}J.$$

The exponential of the operator A with Weyl symbol $ip(x,y)$ is then given by

$$W_1(\exp(A))(x,y) = \exp(ic)(\cos s)^{-n}\exp\left(-i\tan(s)(x^2+y^2)\right) \times \exp\left(i\tfrac{\tan(s)}{s}(ax+by)\right)\exp\left(-\tfrac{i}{4}\tfrac{\tan(s)-s}{s^2}(a^2+b^2)\right).$$

8. Star exponentials

Here we give a reformulation of Corollary 7.4 in terms of star exponential for the Moyal star product. The Moyal star product on $\mathbb{R}^{2n}$ is defined as follows.

Take coordinates $x=(p,q)$ on $\mathbb{R}^{2n}\cong\mathbb{R}^n\times\mathbb{R}^n$. We have $x_i=p_i$ for $1\leqslant i\leqslant n$ and $x_i=q_{i-n}$ for $n+1\leqslant i\leqslant 2n$. For $u,v\in C^\infty(\mathbb{R}^{2n})$, let $P^0(u,v):=uv$,

$$P^1(u,v):=\sum_{k=1}^{n}\left(\frac{\partial u}{\partial p_k}\frac{\partial v}{\partial q_k}-\frac{\partial u}{\partial q_k}\frac{\partial v}{\partial p_k}\right)=\sum_{1\leqslant i,j\leqslant n}\Lambda^{ij}\partial_{x_i}u\partial_{x_j}v$$

(the Poisson brackets) and, more generally, for $l\geqslant 2$,

$$P^l(u,v):=\sum_{1\leqslant i_1,\dots,i_l,j_1,\dots,j_l\leqslant n}\Lambda^{i_1j_1}\Lambda^{i_2j_2}\dots\Lambda^{i_lj_l}\partial^l_{x_{i_1}\dots x_{i_l}}u\,\partial^l_{x_{j_1}\dots x_{j_l}}v.$$

Then the Moyal product $*$ is the following formal deformation of the pointwise multiplication of $C^\infty(\mathbb{R}^{2n})$

$$u*v:=\sum_{l\geqslant 0}\frac{t^l}{l!}P^l(u,v),$$

t being a formal parameter.

An important problem is the computation of the star exponential $\exp_*(f):=\sum_{l\geqslant 0}\frac{1}{l!}f^{*,l}$ for some functions f. Such computations are usually done by solving some differential systems, see [3], [4].

We can restrict $*$ to polynomials on $\mathbb{R}^{2n}$ and take $t=-i/2$. Then $*$ induces an associative product on the polynomials which we also denote by $*$.

The classical Weyl correspondence $\mathscr{W}$ (Equation 3.2) can be extended to polynomials as follows. For each function $f(p,q)=u(p)q^\alpha$ where u is a polynomial on $\mathbb{R}^n$, we have

$$(\mathscr{W}(f)\varphi)(p)=\left(i\frac{\partial}{\partial s}\right)^\alpha\left(u(p+\tfrac{1}{2}s)\,\varphi(p+s)\right)\Big|_{s=0},$$

see [23], [30] for instance. Then, for each polynomial f, $\mathscr{W}(f)$ is a differential operator with polynomial coefficients and $*$ corresponds to the composition of operators in the Weyl quantization, that is, for each polynomials f_1,f_2 on $\mathbb{R}^{2n}$, we have $\mathscr{W}(f_1*f_2)=\mathscr{W}(f_1)\mathscr{W}(f_2)$ or, equivalently, for each differential operators A_1,A_2 with polynomial coefficients, we have $W_1(A_1)*W_1(A_2)=W_1(A_1A_2)$.

PROPOSITION 8.1. *Let M be a real symmetric $(2n)\times(2n)$-matrix, $a,b\in\mathbb{R}^n$ and $c\in\mathbb{R}$. Let*

$$p(x,y):=ax+by+c-\begin{pmatrix}x & y\end{pmatrix}M\begin{pmatrix}x\\ y\end{pmatrix}.$$

Assume that $\mathrm{Det}(I_{2n}+\exp(2JM))\neq 0$. *Then we have*

$$\begin{aligned}\exp_*(ip)(x,y)=\ &\exp(itc)\,\mathrm{Det}(\cosh(JM))^{-1/2}\\ &\times\exp\left(i\,(x\ y)\,J\tanh(JM)\begin{pmatrix}x\\ y\end{pmatrix}\right)\\ &\times\exp\left(-i\,(x\ y)\,J\frac{\tanh(JM)}{JM}\begin{pmatrix}b\\ -a\end{pmatrix}\right)\\ &\times\exp\left(\tfrac{i}{4}\,(b\ -a)\,J\frac{JM-\tanh(JM)}{(JM)^2}\begin{pmatrix}b\\ -a\end{pmatrix}\right).\end{aligned}$$

Proof. Let $X'=((b,-a),c),-2JM)\in\mathfrak{g}'$. Then we have $ip(x,y)=W_1(d\pi'(X'))(x,y)$ hence

$$\exp_*(ip)=\exp_*(W_1(d\pi'(X')))=W_1(\exp(d\pi'(X')))=W_1(\pi'(\exp(X')))$$

and the result follows from Corollary 7.4. □

Note that a similar formula can be found in [4].

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