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AN ELEMENTARY PROOF OF A HLAWKA TYPE INEQUALITY

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Abstract. In this paper, we provided an elementary proof of a multiple Hlawka type inequality proved by W. Berndt and S. Sra in [1]. Our proof does not use the tensor product method.

1. Introduction

Positive definite (or positive semi-definite) matrices have similar properties with positive (or nonnegative) numbers, especially about inequalities, see [6, 7, 13, 16]. One of the fundamental inequalities is the following: For any two positive definite matrices with the same size, we have (e.g. [7, p. 511])

$$\det(A+B) \geqslant \det(A) + \det(B). \tag{1.1}$$

There are many generalizations of (1.1). In 1970, E. V. Haynsworth [4] made the first improvement of (1.1) by using the Schur complement method. Soon after that, D. J. Hartfiel [3] obtained a quantitative improvement of Haynsworth's result. He proved

THEOREM 1.1. [3] *Let A, B be positive definite $n \times n$ matrices. Then*

$$\det(A+B) \geqslant \det(A) + \det(B) + (2^n - 2)\sqrt{\det(AB)}. \tag{1.2}$$

In 2014, M. Lin [10] generalized (1.1) to three positive definite matrices.

THEOREM 1.2. [10, Theorem 1.1] *Let A, B, C be positive definite $n \times n$ matrices. Then*

$$\begin{aligned} &\det(A+B+C) + \det(A) + \det(B) + \det(C) \\ &\geqslant \det(A+B) + \det(B+C) + \det(A+C). \end{aligned} \tag{1.3}$$

Recently, Y. Hong and F. Qi [5] obtained a sharper lower bound for Theorem 1.2, which is also a generalization of Theorem 1.1.

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THEOREM 1.3. [5, Theorem 3] *Let A,B,C be positive definite $n\times n$ matrices. Then*

$$\begin{aligned}&\det(A+B+C)+\det(A)+\det(B)+\det(C)\\ \geqslant&\det(A+B)+\det(B+C)+\det(A+C)+(3^n+3-3\cdot 2^n)\left[\det(ABC)\right]^{\frac{1}{3}}.\end{aligned} \tag{1.4}$$

Please see [2, 8, 11, 12] for other forms of generalization about (1.1)–(1.4).

By using tensor product, W. Berndt and S. Sra [1] extended (1.1) and (1.3) to multiple version, which is called the Hlawka type inequality.

THEOREM 1.4. [1, Conjecture 3.1 and Corollary 3.4] *Let $A_1,A_2,\cdots,A_m$ be positive definite $n\times n$ matrices. For each $k=1,2,\cdots,m$, define*

$$s_k(A_1,A_2,\cdots,A_m):=\sum_{1\leqslant i_1<i_2<\cdots<i_k\leqslant m}\det\left(A_{i_1}+A_{i_2}+\cdots+A_{i_k}\right).$$

Then

$$\sum_{k=1}^{m}(-1)^{m-k}s_k(A_1,A_2,\cdots,A_m)\geqslant 0. \tag{1.5}$$

At the same time, W. Berndt and S. Sra [1] also got another generalization to multiple version, which is called the Popoviciu type inequality.

THEOREM 1.5. [1, Theorem 4.3] *Let $A_1,A_2,\cdots,A_m$ $(m\geqslant 3)$ be positive definite $n\times n$ matrices. Then*

$$\det\left(\sum_{j=1}^{m}A_j\right)+(m-2)\sum_{j=1}^{m}\det(A_j)\geqslant\sum_{1\leqslant i<j\leqslant m}\det(A_i+A_j). \tag{1.6}$$

More recently, we [14] proved a quantitative improvement of Theorem 1.5 in the following form.

THEOREM 1.6. [14, Theorem 7] *Let $A_1,A_2,\cdots,A_m$ $(m\geqslant 3)$ be positive definite $n\times n$ matrices. Then*

$$\begin{aligned}&\det\left(\sum_{j=1}^{m}A_j\right)+(m-2)\sum_{j=1}^{m}\det(A_j)\\ \geqslant&\sum_{1\leqslant i<j\leqslant m}\det(A_i+A_j)+\left[\left(m^n-m-(2^{n-1}-1)(m-1)m\right)\right]\left[\det(A_1A_2\cdots A_m)\right]^{\frac{1}{m}}.\end{aligned} \tag{1.7}$$

We also revisited Theorem 1.6 and got an improvement for $m=4$.

THEOREM 1.7. [15, Theorem 1.5] *Let A_1,A_2,A_3,A_4 be positive definite $n\times n$ matrices. Then*

$$\begin{aligned} &\det\left(\sum_{j=1}^{4}A_j\right)+2\sum_{j=1}^{4}\det(A_j)-\sum_{1\leqslant i<j\leqslant 4}\det(A_i+A_j)\\ &\geqslant(3^n-3\cdot 2^n+3)\left[(\det(A_1A_2A_3))^{\frac{1}{3}}+(\det(A_1A_2A_4))^{\frac{1}{3}}\right.\\ &\quad\left.+(\det(A_1A_3A_4))^{\frac{1}{3}}+(\det(A_2A_3A_4))^{\frac{1}{3}}\right]\\ &\quad+\left(4^n-4\cdot 3^n+3\cdot 2^{n+1}-4\right)(\det(A_1A_2A_3A_4))^{\frac{1}{4}}. \end{aligned} \tag{1.8}$$

To generalize Theorem 1.1 and Theorem 1.3, we propose the following conjecture.

CONJECTURE 1.8. *Let $A_1,A_2,\cdots,A_m$ be positive definite $n\times n$ matrices. For each $k=1,2,\cdots,m$, define*

$$s_k(A_1,A_2,\cdots,A_m):=\sum_{1\leqslant i_1<i_2<\cdots<i_k\leqslant m}\det\left(A_{i_1}+A_{i_2}+\cdots+A_{i_k}\right).$$

Then

$$\sum_{k=1}^{m}(-1)^{m-k}s_k(A_1,A_2,\cdots,A_m)\geqslant\left(\sum_{k=1}^{m}(-1)^{m-k}C_m^k k^n\right)[\det(A_1A_2\cdots A_m)]^{\frac{1}{m}}, \tag{1.9}$$

where $C_m^k=\dfrac{m!}{(m-k)!k!}$.

REMARK 1.9. It is easy to check that (1.9) reduces to (1.2) when $m=2$ and to (1.4) when $m=3$.

The main goal of this paper is to give an elementary proof of Theorem 1.4.

2. An elementary proof of Theorem 1.4

Firstly, we establish several lemmas needed in the proof of Theorem 1.4.

LEMMA 2.1. *Let B be an $n\times n$ matrix. For $j=1,2,\cdots,n$, let B_j be the submatrix of B by deleting the j-th row and j-th column. Then*

$$\frac{d}{dt}\left[\det(B+tI_n)\right]=\sum_{j=1}^{n}\det(B_j+tI_{n-1}).$$

Proof. Let $B+tI_n=(b_1(t),b_2(t),\cdots,b_n(t))$ and $e_1=(1,0,\cdots,0)^T$, $e_2=(0,1,\cdots,0)^T,\cdots,$ $e_n=(0,0,\cdots,1)^T$, then by the product rule of multilinear functions (e.g. [9, p. 125]), we have

$$\begin{aligned}\frac{d}{dt}\left[\det\left(B+tI_n\right)\right] &= \frac{d}{dt}\left[\det\left(\left(b_1(t),b_2(t),\cdots,b_n(t)\right)\right]\right. \\ &= \sum_{j=1}^{n}\det\left(\left(b_1(t),b_2(t),\cdots,b_j'(t),\cdots,b_n(t)\right)\right. \\ &= \sum_{j=1}^{n}\det\left(\left(b_1(t),b_2(t),\cdots,e_j,\cdots,b_n(t)\right)\right. \\ &= \sum_{j=1}^{n}\det\left(B_j+tI_{n-1}\right). \quad \square\end{aligned}$$

LEMMA 2.2. *For any positive integer m, we have*

$$\sum_{k=1}^{m}(-1)^{m-k}kC_m^k = 0.$$

Proof. By the definition of the combinatorial number, we have

$$\begin{aligned}\sum_{k=1}^{m}(-1)^{m-k}kC_m^k &= \sum_{k=1}^{m}(-1)^{m-k}k\frac{m!}{(m-k)!k!} \\ &= m\sum_{k=1}^{m}(-1)^{m-k}\frac{(m-1)!}{[(m-1)-(k-1)!](k-1)!} \\ &= m\sum_{k=1}^{m}(-1)^{m-k}C_{m-1}^{k-1} \\ &= m\sum_{k=0}^{m-1}(-1)^{m-1-k}C_{m-1}^{k}.\end{aligned}$$

Moreover, by Binomial Theorem, we get

$$\sum_{k=0}^{m-1}(-1)^{m-1-k}C_{m-1}^{k} = [1+(-1)]^{m-1} = 0. \quad \square$$

Now we are in a position to prove Theorem 1.4.

Proof of Theorem 1.4. The proof of Theorem 1.4 is by induction on m and n.
Step 1:

- For $m=2$ and any n, it follows from the fundamental inequality (1.1).
- For $n=1$ and any positive integer m, we have

$$\begin{aligned}\sum_{k=1}^{m}(-1)^{m-k}s_k\left(A_1,A_2,\cdots,A_m\right) &= \sum_{k=1}^{m}(-1)^{m-k}\frac{kC_m^k}{m}\left(A_1+A_2+\cdots+A_m\right) \\ &= \frac{A_1+A_2+\cdots+A_m}{m}\cdot\sum_{k=1}^{m}(-1)^{m-k}\frac{kC_m^k}{m} \\ &= 0,\end{aligned}$$

where the last step follows from Lemma 2.2.

Step 2: Suppose that the inequality holds for $m, n-1$ and $m+1, n-1$. Now, suppose $A_1, A_2, \cdots, A_{m+1}$ are positive definite $n \times n$ matrices. First, notice

$$\begin{aligned}
&\left[\sum_{k=1}^{m+1}(-1)^{m+1-k}s_k\left(A_1,A_2,\cdots,A_{m+1}\right)\right]\cdot\det\left(A_{m+1}^{-1}\right)\\
&=\sum_{k=1}^{m+1}(-1)^{m+1-k}\left[s_k\left(A_1,A_2,\cdots,A_{m+1}\right)\cdot\det\left(A_{m+1}^{-1}\right)\right]\\
&=\sum_{k=1}^{m+1}(-1)^{m+1-k}s_k\left(\widehat{A}_1,\widehat{A}_2,\cdots,\widehat{A}_m,I_n\right),
\end{aligned}$$

where $\widehat{A}_i = A_{m+1}^{-\frac{1}{2}} A_i A_{m+1}^{-\frac{1}{2}}$ for $1 \leqslant i \leqslant m$. Next, for any $t \in [0, +\infty)$ define

$$\varphi(t)=\sum_{k=1}^{m+1}(-1)^{m+1-k}s_k\left(\widehat{A}_1,\widehat{A}_2,\cdots,\widehat{A}_m,\ t\cdot I_n\right),$$

then we only need to prove $\varphi(1) \geqslant 0$. In fact, by the definition of s_k, we have

$$\begin{aligned}
\varphi(0)&=\sum_{k=1}^{m+1}(-1)^{m+1-k}s_k\left(\widehat{A}_1,\widehat{A}_2,\cdots,\widehat{A}_m,\ O_n\right)\\
&=s_{m+1}\left(\widehat{A}_1,\widehat{A}_2,\cdots,\widehat{A}_m,\ O_n\right)\\
&\quad+\sum_{k=2}^{m}(-1)^{m+1-k}s_k\left(\widehat{A}_1,\widehat{A}_2,\cdots,\widehat{A}_m,\ O_n\right)\\
&\quad+(-1)^m s_1\left(\widehat{A}_1,\widehat{A}_2,\cdots,\widehat{A}_m,\ O_n\right)\\
&=s_m\left(\widehat{A}_1,\widehat{A}_2,\cdots,\widehat{A}_m\right)\\
&\quad+\sum_{k=2}^{m}(-1)^{m+1-k}\left[s_k\left(\widehat{A}_1,\widehat{A}_2,\cdots,\widehat{A}_m\right)+s_{k-1}\left(\widehat{A}_1,\widehat{A}_2,\cdots,\widehat{A}_m\right)\right]\\
&\quad+(-1)^m s_1\left(\widehat{A}_1,\widehat{A}_2,\cdots,\widehat{A}_m\right)\\
&=0.
\end{aligned}$$

CLAIM. For any $t \geqslant 0$, we have $\varphi'(t) \geqslant 0$.
Then φ is increasing on $[0, +\infty)$. So

$$\varphi(1) \geqslant \varphi(0) = 0,$$

the proof of the theorem is completed if the claim is true. □

Here is the proof of the claim:

Proof of the Claim. Let $\widehat{A}_{i,j}(1 \leqslant i \leqslant m)$ be the submatrix of $\widehat{A}_i$ by deleting the j-th row and j-th column. In order to compute $\varphi'(t)$, we consider the following three cases.

- *Case* 1: $k=1$.

$$\begin{aligned}
&\left[s_1\left(\widehat{A}_1,\widehat{A}_2,\cdots,\widehat{A}_m,\ t\cdot I_n\right)\right]' \\
&=\left[\sum_{i=1}^{m}\det\left(\widehat{A}_i\right)+\det(t\cdot I_n)\right]' \\
&=[\det(t\cdot I_n)]' \\
&=\sum_{j=1}^{n}\det(t\cdot I_{n-1}) \quad \text{(by Lemma 2.1)} \\
&=\sum_{j=1}^{n}\left[s_1\left(\widehat{A}_{1,j},\widehat{A}_{2,j},\cdots,\widehat{A}_{m,j},\ t\cdot I_{n-1}\right)-s_1\left(\widehat{A}_{1,j},\widehat{A}_{2,j},\cdots,\widehat{A}_{m,j}\right)\right].
\end{aligned} \tag{2.1}$$

- *Case* 2: $2 \leqslant k \leqslant m$. Since

$$\begin{aligned}
&s_k\left(\widehat{A}_1,\widehat{A}_2,\cdots,\widehat{A}_m,\ t\cdot I_n\right) \\
&= \sum_{1\leqslant i_1<i_2<\cdots<i_k\leqslant m}\det\left(\widehat{A}_{i_1}+\widehat{A}_{i_2}+\cdots+\widehat{A}_{i_k}\right) \\
&\quad+\sum_{1\leqslant i_1<i_2<\cdots<i_{k-1}\leqslant m}\det\left(\widehat{A}_{i_1}+\widehat{A}_{i_2}+\cdots+\widehat{A}_{i_{k-1}}+t\cdot I_n\right),
\end{aligned}$$

by Lemma 2.1, we have

$$\begin{aligned}
&\left[s_k\left(\widehat{A}_1,\widehat{A}_2,\cdots,\widehat{A}_m,\ t\cdot I_n\right)\right]' \\
&=\sum_{1\leqslant i_1<i_2<\cdots<i_{k-1}\leqslant m}\left[\det\left(\widehat{A}_{i_1}+\widehat{A}_{i_2}+\cdots+\widehat{A}_{i_{k-1}}+t\cdot I_n\right)\right]' \\
&=\sum_{1\leqslant i_1<i_2<\cdots<i_{k-1}\leqslant m}\sum_{j=1}^{n}\det\left(\widehat{A}_{i_1,j}+\widehat{A}_{i_2,j}+\cdots+\widehat{A}_{i_{k-1},j}+t\cdot I_{n-1}\right) \\
&=\sum_{j=1}^{n}\sum_{1\leqslant i_1<i_2<\cdots<i_{k-1}\leqslant m}\det\left(\widehat{A}_{i_1,j}+\widehat{A}_{i_2,j}+\cdots+\widehat{A}_{i_{k-1},j}+t\cdot I_{n-1}\right) \\
&=\sum_{j=1}^{n}\left[s_k\left(\widehat{A}_{1,j},\widehat{A}_{2,j},\cdots,\widehat{A}_{m,j},\ t\cdot I_{n-1}\right)-s_k\left(\widehat{A}_{1,j},\widehat{A}_{2,j},\cdots,\widehat{A}_{m,j}\right)\right].
\end{aligned} \tag{2.2}$$

- *Case* 3: $k=m+1$.

$$\begin{aligned}\left[s_{m+1}\left(\widehat{A}_1,\widehat{A}_2,\cdots,\widehat{A}_m,\ t\cdot I_n\right)\right]' &= \left[\det\left(\widehat{A}_1+\widehat{A}_2+\cdots+\widehat{A}_m+t\cdot I_n\right)\right]' \\ &= \sum_{j=1}^{n}\det\left(\widehat{A}_{1,j}+\widehat{A}_{2,j}+\cdots+\widehat{A}_{m,j}+t\cdot I_{n-1}\right) \\ &= \sum_{j=1}^{n} s_{m+1}\left(\widehat{A}_{1,j},\widehat{A}_{2,j},\cdots,\widehat{A}_{m,j},\ t\cdot I_{n-1}\right).\end{aligned} \tag{2.3}$$

Now, from (2.1)–(2.3) and Lemma 2.1, we get

$$\begin{aligned}\varphi'(t) &= \sum_{k=1}^{m+1}(-1)^{m+1-k}\left[s_k\left(\widehat{A}_1,\widehat{A}_2,\cdots,\widehat{A}_m,\ t\cdot I_n\right)\right]' \\ &= \sum_{j=1}^{n} s_{m+1}\left(\widehat{A}_{1,j},\widehat{A}_{2,j},\cdots,\widehat{A}_{m,j},\ t\cdot I_{n-1}\right) \\ &\quad + \sum_{j=1}^{n}\sum_{k=1}^{m}(-1)^{m+1-k}\left[s_k\left(\widehat{A}_{1,j},\widehat{A}_{2,j},\cdots,\widehat{A}_{m,j},\ t\cdot I_{n-1}\right) - s_k\left(\widehat{A}_{1,j},\widehat{A}_{2,j},\cdots,\widehat{A}_{m,j}\right)\right] \\ &= \sum_{j=1}^{n}\left[\sum_{k=1}^{m+1}(-1)^{m+1-k}s_k\left(\widehat{A}_{1,j},\widehat{A}_{2,j},\cdots,\widehat{A}_{m,j},\ t\cdot I_{n-1}\right)\right. \\ &\qquad\qquad \left. + \sum_{k=1}^{m}(-1)^{m-k}s_k\left(\widehat{A}_{1,j},\widehat{A}_{2,j},\cdots,\widehat{A}_{m,j}\right)\right] \\ &\geqslant 0,\end{aligned}$$

where the inequality follows from the assumption of induction. The proof is finished. □

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