Operators and Matrices
Volume 18, Number 4 (2024), 925–939 doi:10.7153/oam-2024-18-56

EUCLIDEAN OPERATOR RADIUS AND NUMERICAL RADIUS INEQUALITIES

SUVENDU JANA, PINTU BHUNIA AND KALLOL PAUL

(*Communicated by I. M. Spitkovsky*)

Abstract. Let T be a bounded linear operator on a complex Hilbert space $\mathscr{H}$. We obtain various lower and upper bounds for the numerical radius of T by developing the Euclidean operator radius bounds of a pair of operators, which are stronger than the existing ones. In particular, we develop an inequality that improves on the inequality

$$w(T) \geqslant \frac{1}{2}\|T\| + \frac{1}{4}\left|\|Re(T)\| - \frac{1}{2}\|T\|\right| + \frac{1}{4}\left|\|Im(T)\| - \frac{1}{2}\|T\|\right|.$$

Various equality conditions of the existing numerical radius inequalities are also provided. Further, we study the numerical radius inequalities of 2×2 off-diagonal operator matrices. Applying the numerical radius bounds of operator matrices, we develop upper bounds of $w(T)$ by using t-Aluthge transform. In particular, we improve the well known inequality

$$w(T) \leqslant \frac{1}{2}\|T\| + \frac{1}{2}w(\widetilde{T}),$$

where $\widetilde{T} = |T|^{1/2}U|T|^{1/2}$ is the Aluthge transform of T and $T = U|T|$ is the polar decomposition of T.

1. Introduction

Let $\mathbb{B}(\mathscr{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert $(\mathscr{H}, \langle .,. \rangle)$. For $T \in \mathbb{B}(\mathscr{H})$, T^* denotes the adjoint of T and $|T| = (T^*T)^{\frac{1}{2}}$. The Cartesian decomposition of T is $T = Re(T) + iIm(T)$, where $Re(T) = \frac{1}{2}(T+T^*)$ and $Im(T) = \frac{1}{2i}(T - T^*)$. For $0 \leqslant t \leqslant 1$, the t-Aluthge transform of $T \in \mathbb{B}(\mathscr{H})$ is defined as $\widetilde{T_t} = |T|^t U |T|^{1-t}$, where $T = U|T|$ is the polar decomposition of T and U is the partial isometry. In particular, for $t = \frac{1}{2}$, $\widetilde{T} = \widetilde{T_{\frac{1}{2}}} = |T|^{1/2}U|T|^{1/2}$ is the Aluthge transform of T. The numerical radius of T, denoted by $w(T)$, is defined as $w(T) = \sup\{|\langle Tx, x\rangle| : x \in \mathscr{H}, \|x\| = 1\}$. The numerical radius $w(\cdot)$ is a norm on $\mathbb{B}(\mathscr{H})$ and satisfies

$$\frac{1}{2}\|T\| \leqslant w(T) \leqslant \|T\|. \tag{1.1}$$

Mathematics subject classification (2020): Primary 47A12; Secondary 15A60, 47A30, 47A50.
Keywords and phrases: Euclidean operator radius, numerical radius, operator norm, Cartesian decomposition, bounded linear operator.
Dr. Pintu Bhunia would like to thank SERB, Govt. of India for the financial support in the form of National Post Doctoral Fellowship (N-PDF, File No. PDF/2022/000325) under the mentorship of Prof. Apoorva Khare.

Paper OaM-18-56

For further information on the numerical radius and related inequalities improving (1.1), we refer to [1, 2, 5, 6, 8, 9, 12, 20, 22, 26]. Based on the importance of the concept of numerical radius, various generalizations have been studied for the last few years. Such a generalization is the Euclidean operator radius, see [24]. For $B,C \in \mathbb{B}(\mathscr{H})$, the Euclidean operator radius of B and C, denoted by $w_e(B,C)$, is defined as

$$w_e(B,C) = \sup\left\{ \sqrt{|\langle Bx,x\rangle|^2 + |\langle Cx,x\rangle|^2} : x \in \mathscr{H}, \|x\| = 1 \right\}.$$

The Euclidean operator radius $w_e(.,.)$, defines a norm on $\mathbb{B}^2(\mathscr{H})(=\mathbb{B}(\mathscr{H})\times\mathbb{B}(\mathscr{H}))$, which satisfies the inequality (see [24])

$$\frac{1}{8}\|B^*B + C^*C\| \leqslant w_e^2(B,C) \leqslant \|B^*B + C^*C\|. \tag{1.2}$$

Here the constants $\frac{1}{8}$ and 1 are best possible. In [13, Th. 1], Dragomir proved that

$$\frac{1}{2}w\left(B^2 + C^2\right) \leqslant w_e^2(B,C) \tag{1.3}$$

and the constant $\frac{1}{2}$ is best possible. See [16, 21, 25] for more generalizations on the Euclidean operator radius and related results.

In [16], authors studied improvements of the inequalities (1.2) and (1.3). In this article we continue the study in that direction. We obtain lower and upper bounds for the Euclidean operator radius of a pair of bounded linear operators B and C, which improve on the earlier related bounds. From the Euclidean operator radius bounds we develop various lower and upper bounds for the numerical radius of a bounded linear operator T, which improve (1.1) and the inequality $\frac{1}{4}\|T^*T + TT^*\| \leqslant w^2(T) \leqslant \frac{1}{2}\|T^*T + TT^*\|$, given in [19]. We study equality conditions of the existing numerical radius inequalities of a bounded linear operator T. Further, we obtain numerical radius bounds for the 2×2 off-diagonal operator matrices, which generalize and improve on the existing ones. Applying the numerical radius bounds of 2×2 off-diagonal operator matrices, we obtain an upper bound for the numerical radius of a bounded linear operator T by using t-Aluthge transform, which improves and generalizes the bound $w(T) \leqslant \frac{1}{2}\|T\| + \frac{1}{2}w(\widetilde{T})$, given in [27].

2. Main results

We begin with the following proposition that gives lower bounds for the Euclidean operator radius $w_e(B,C)$.

PROPOSITION 2.1. *Let $B,C \in \mathbb{B}(\mathscr{H})$. Then the following inequalities hold:*
(i) $w_e(B,C) \geqslant \max\{w(B), w(C)\}$.
(ii) $w_e(B,C) \geqslant \frac{1}{\sqrt{2}}w\left(B + e^{i\theta}C\right)$ *for all* $\theta \in \mathbb{R}$.
(iii) $w_e(B,C) \geqslant \sqrt{\frac{1}{2}w(B^2 + e^{i\theta}C^2) + \frac{1}{2}|w^2(B) - w^2(C)|}$ *for all* $\theta \in \mathbb{R}$.
(iv) $w_e(B,C) \geqslant \sqrt{\frac{1}{2}w(BC + CB)}$.

Proof. (i) Follows trivially from the definition.
(ii) We have

$$\begin{aligned} w_e(B,C) &= \sup_{\|x\|=1} \sqrt{|\langle Bx,x\rangle|^2+|\langle Cx,x\rangle|^2} \\ &\geqslant \sup_{\|x\|=1} \sqrt{\frac{1}{2}\left(|\langle Bx,x\rangle|+|\langle Cx,x\rangle|\right)^2} \\ &\geqslant \sup_{\|x\|=1} \sqrt{\frac{1}{2}\left(|\langle Bx,x\rangle+e^{i\theta}\langle Cx,x\rangle|\right)^2} \\ &= \frac{1}{\sqrt{2}} w\left(B+e^{i\theta}C\right). \end{aligned}$$

(iii) From (i), we have

$$\begin{aligned} w_e^2(B,C) &\geqslant \max\left\{w^2(B),w^2(C)\right\} \\ &= \frac{1}{2}(w^2(B)+w^2(C))+\frac{1}{2}|w^2(B)-w^2(C)| \\ &\geqslant \frac{1}{2}\left(w(B^2)+w(C^2)\right)+\frac{1}{2}|w^2(B)-w^2(C)| \\ &\geqslant \frac{1}{2}w(B^2+e^{i\theta}C^2)+\frac{1}{2}|w^2(B)-w^2(C)|. \end{aligned}$$

(iv) From (ii), we have $w_e(B,C) \geqslant \frac{1}{\sqrt{2}}w(B+C)$ and $w_e(B,C) \geqslant \frac{1}{\sqrt{2}}w(B-C)$. Thus,

$$\begin{aligned} 2w_e^2(B,C) &\geqslant \frac{1}{2}w^2(B+C)+\frac{1}{2}w^2(B-C) \\ &\geqslant \frac{1}{2}w\left((B+C)^2\right)+\frac{1}{2}w\left((B-C)^2\right) \\ &\geqslant \frac{1}{2}w\left((B+C)^2-(B-C)^2\right) \\ &= w(BC+CB). \end{aligned}$$

This completes the proof. □

Proposition 2.1 (iii) generalizes and improves the inequality

$$w_e(B,C) \geqslant \sqrt{\frac{1}{2}w(B^2+C^2)},$$

proved in [13, Th. 1]. Now, by using Proposition 2.1 we prove the following theorem.

THEOREM 2.2. *Let $B,C \in \mathbb{B}(\mathscr{H})$. Then*

$$\sqrt{\frac{1}{4}w(B^2+C^2)+\frac{1}{4}(w^2(B)+w^2(C))+\frac{1}{2}|w^2(B)-w^2(C)|} \leqslant w_e(B,C).$$

Proof. Take $t_1 = \max\left\{w^2(B), \frac{1}{2}w\left(B^2+C^2\right)\right\}$, $t_2 = \max\left\{w^2(C), \frac{1}{2}w\left(B^2+C^2\right)\right\}$, $m_1 = \left|w^2(B) - \frac{1}{2}w\left(B^2+C^2\right)\right|$ and $m_2 = \left|w^2(C) - \frac{1}{2}w\left(B^2+C^2\right)\right|$. From the inequalities (i) and (iii) of Proposition 2.1, we have

$$
\begin{aligned}
w_e^2(B,C) &\geqslant \max\{t_1,t_2\} \\
&= \frac{1}{2}(t_1+t_2) + \frac{1}{2}|t_1-t_2| \\
&= \frac{1}{4}\left(w^2(B)+w^2(C)\right) + \frac{1}{4}w(B^2+C^2) + \frac{1}{4}(m_1+m_2) + \frac{1}{2}|t_1-t_2| \\
&\geqslant \frac{1}{4}\left(w(B^2)+w(C^2)\right) + \frac{1}{4}w(B^2+C^2) + \frac{1}{4}(m_1+m_2) + \frac{1}{2}|t_1-t_2| \\
&\geqslant \frac{1}{4}w(B^2+C^2) + \frac{1}{4}w(B^2+C^2) + \frac{1}{4}(m_1+m_2) + \frac{1}{2}|t_1-t_2| \\
&= \frac{1}{2}w(B^2+C^2) + \frac{1}{4}(m_1+m_2) + \frac{1}{2}|t_1-t_2| \\
&= \frac{1}{4}w(B^2+C^2) + \frac{1}{4}\left(w^2(B)+w^2(C)\right) + \frac{1}{2}\left|w^2(B)-w^2(C)\right|,
\end{aligned}
$$

as desired. □

REMARK 2.3. (i) The inequality obtained in Theorem 2.2 is a refinement of the inequality $\sqrt{\frac{1}{2}w\left(B^2+C^2\right)} \leqslant w_e(B,C)$, given in [13, Th. 1].

(ii) If $w_e(B,C) = \sqrt{\frac{1}{2}w\left(B^2+C^2\right)}$, then from Theorem 2.2 it follows that $w(B) = w(C) = \sqrt{\frac{1}{2}w\left(B^2+C^2\right)}$. However, the converse, in general, may not hold. As for example, considering a non-zero normal operator $B=C$, we get $w(B) = w(C) = \sqrt{\frac{1}{2}w\left(B^2+C^2\right)}$, but $w_e(B,C) = \sqrt{2}w(B) \neq w(B) = \sqrt{\frac{1}{2}w\left(B^2+C^2\right)}$.

(iii) If $w_e(B,C) = \sqrt{\frac{1}{2}w\left(B^2+C^2\right) + \frac{1}{2}\left|w^2(B)-w^2(C)\right|}$ then from Theorem 2.2 it follows that $w\left(B^2+C^2\right) = w^2(B)+w^2(C)$ and $w_e(B,C) = \max\{w(B),w(C)\}$. The converse of the result is also valid.

As an immediate consequence of Theorem 2.2 we have the following result.

COROLLARY 2.4. *Let $B,C \in \mathbb{B}(\mathscr{H})$ be normal, then*

$$
\begin{aligned}
w_e(B,C) &\geqslant \sqrt{\frac{1}{4}\|B^2+C^2\| + \frac{1}{4}\left(\|B\|^2+\|C\|^2\right) + \frac{1}{2}\left|\|B\|^2-\|C\|^2\right|} \\
&= \sqrt{\frac{1}{2}\|B^2+C^2\| + \frac{1}{4}(p_1+p_2) + \frac{1}{2}|s_1-s_2|},
\end{aligned}
$$

where $s_1 = \max\left\{\|B\|^2, \frac{1}{2}\|B^2+C^2\|\right\}$, $s_2 = \max\left\{\|C\|^2, \frac{1}{2}\|B^2+C^2\|\right\}$, $p_1 = \left|\|B\|^2 - \frac{1}{2}\|B^2+C^2\|\right|$ *and* $p_2 = \left|\|C\|^2 - \frac{1}{2}\|B^2+C^2\|\right|$.

Corollary 2.4 is better than the first inequality in (1.2) when B and C are self-adjoint operators. From Corollary 2.4 we obtain the following numerical radius bound for a bounded linear operator T.

COROLLARY 2.5. *Let $T \in \mathbb{B}(\mathscr{H})$. Then*

$$\sqrt{\frac{1}{8}\|T^*T+TT^*\|+\frac{1}{4}(\|Re(T)\|^2+\|Im(T)\|^2)+\frac{1}{2}\,|\|Re(T)\|^2-\|Im(T)\|^2|} \leqslant w(T),$$

Proof. Considering $B = Re(T)$ and $C = Im(T)$ in Corollary 2.4 we obtain that

$$\begin{aligned} w(T) &\geqslant \sqrt{\frac{1}{8}\|T^*T+TT^*\|+\frac{1}{4}(\|Re(T)\|^2+\|Im(T)\|^2)+\frac{1}{2}\,|\|Re(T)\|^2-\|Im(T)\|^2|} \\ &= \sqrt{\frac{1}{4}\|T^*T+TT^*\|+\frac{1}{4}(\alpha+\beta)+\frac{1}{2}\big|\gamma-\delta\big|}, \end{aligned}$$

where $\alpha = \big|\|Re(T)\|^2 - \frac{1}{4}\|T^*T+TT^*\|\big|$, $\beta = \big|\|Im(T)\|^2 - \frac{1}{4}\|T^*T+TT^*\|\big|$, $\gamma = \max\left\{\|Re(T)\|^2, \frac{1}{4}\|TT^*+T^*T\|\right\}$ and $\delta = \max\left\{\|Im(T)\|^2, \frac{1}{4}\|TT^*+T^*T\|\right\}$. □

REMARK 2.6. (i) Clearly, the bound obtained in Corollary 2.5 is stronger than the bound

$$\sqrt{\frac{1}{4}\|T^*T+TT^*\|+\frac{1}{2}\big|\|Re(T)\|^2-\|Im(T)\|^2\big|} \leqslant w(T), \tag{2.1}$$

obtained in [7, Th. 2.9].

(ii) From Corollary 2.5 it follows that, if

$$\sqrt{\frac{1}{4}\|T^*T+TT^*\|+\frac{1}{2}\big|\|Re(T)\|^2-\|Im(T)\|^2\big|} = w(T),$$

then $\frac{1}{2}\|T^*T+TT^*\| = \|Re(T)\|^2+\|Im(T)\|^2$ and $w(T) = \max\{\|Re(T)\|, \|Im(T)\|\}$. The converse also holds.

Using Corollary 2.4 we also obtain the following lower bound for the numerical radius.

COROLLARY 2.7. *Let $T \in \mathbb{B}(\mathscr{H})$, then*

$$\sqrt{\frac{1}{8}\|T^*T+TT^*\|+\frac{1}{8}(\|Re(T)+Im(T)\|^2+\|Re(T)-Im(T)\|^2)+\frac{\beta}{4}} \leqslant w(T),$$

where $\beta = \big|\|Re(T)+Im(T)\|^2 - \|Re(T)-Im(T)\|^2\big|$.

Proof. Considering $B = \frac{Re(T)+Im(T)}{\sqrt{2}}$ and $C = \frac{Re(T)-Im(T)}{\sqrt{2}}$ in Corollary 2.4, we have

$$\begin{aligned} w(T) &\geqslant \sqrt{\frac{1}{8}\|T^*T+TT^*\| + \frac{1}{8}(\|Re(T)+Im(T)\|^2 + \|Re(T)-Im(T)\|^2) + \frac{\beta}{4}} \\ &= \sqrt{\frac{1}{4}\|T^*T+TT^*\| + \frac{1}{4}(\gamma+\delta) + \frac{1}{2}|\xi-\eta|}, \end{aligned}$$

where

$$\begin{aligned} \gamma &= \left| \frac{\|Re(T)+Im(T)\|^2}{2} - \frac{1}{4}\|T^*T+TT^*\| \right|, \\ \delta &= \left| \frac{\|Re(T)-Im(T)\|^2}{2} - \frac{1}{4}\|T^*T+TT^*\| \right|, \\ \xi &= \max\left\{ \frac{\|Re(T)+Im(T)\|^2}{2}, \frac{1}{4}\|TT^*+T^*T\| \right\}, \\ \eta &= \max\left\{ \frac{\|Re(T)-Im(T)\|^2}{2}, \frac{1}{4}\|TT^*+T^*T\| \right\}. \quad \square \end{aligned}$$

REMARK 2.8. (i) In [10, Th. 2.3] the authors obtained the inequality

$$\sqrt{\frac{1}{4}\|T^*T+TT^*\| + \frac{1}{4}\left|\|Re(T)+Im(T)\|^2 - \|Re(T)-Im(T)\|^2\right|} \leqslant w(T).$$

It is easy to conclude that the inequality obtained in Corollary 2.7 is a refinement of the above inequalty.

(ii) From the inequality in Corollary 2.7, it follows that if

$$\sqrt{\frac{1}{4}\|T^*T+TT^*\| + \frac{1}{4}\left|\|Re(T)+Im(T)\|^2 - \|Re(T)-Im(T)\|^2\right|} = w(T)$$

then $\|T^*T+TT^*\| = \|Re(T)+Im(T)\|^2 + \|Re(T)-Im(T)\|^2$ and

$$w(T) = \max\left\{ \frac{\|Re(T)+Im(T)\|}{\sqrt{2}}, \frac{\|Re(T)-Im(T)\|}{\sqrt{2}} \right\}.$$

The converse also holds.

Next, we obtain an upper bound for the Euclidean operator radius $w_e(B,C)$.

THEOREM 2.9. *If $B, C \in \mathbb{B}(\mathscr{H})$ then for all $t \in [0,1]$,*

$$\begin{aligned} & w_e(B,C) \\ \leqslant & \left\|t^2B^*B + (1-t)^2C^*C\right\|^{\frac{1}{2}} + \frac{1}{\sqrt{2}}\left\{w^2((1-t)B+tC) + w^2((1-t)B-tC)\right\}^{\frac{1}{2}}. \end{aligned}$$

In particular, for $t = \frac{1}{2}$

$$w_e(B,C) \leqslant \frac{1}{2}\|B^*B+C^*C\|^{\frac{1}{2}} + \frac{1}{2\sqrt{2}}\left\{w^2(B+C) + w^2(B-C)\right\}^{\frac{1}{2}}. \tag{2.2}$$

Proof. Take $x \in \mathscr{H}$ with $\|x\| = 1$. We have

$$\begin{aligned}
&\left(|\langle Bx,x\rangle|^2 + |\langle Cx,x\rangle|^2\right)^{\frac{1}{2}} \\
&= \left(|t\langle Bx,x\rangle + (1-t)\langle Bx,x\rangle|^2 + |(1-t)\langle Cx,x\rangle + t\langle Cx,x\rangle|^2\right)^{\frac{1}{2}} \\
&\leqslant \left(t^2|\langle Bx,x\rangle|^2 + (1-t)^2|\langle Cx,x\rangle|^2\right)^{\frac{1}{2}} + \left((1-t)^2|\langle Bx,x\rangle|^2 + t^2|\langle Cx,x\rangle|^2\right)^{\frac{1}{2}} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Minkowski inequality)} \\
&\leqslant \left(t^2\|Bx\|^2 + (1-t)^2\|Cx\|^2\right)^{\frac{1}{2}} \\
&\quad + \left(\frac{1}{2}|\langle((1-t)B+tC)x,x\rangle|^2 + \frac{1}{2}|\langle((1-t)B-tC)x,x\rangle|^2\right)^{\frac{1}{2}} \\
&\leqslant \left\|t^2B^*B + (1-t)^2C^*C\right\|^{\frac{1}{2}} + \left\{\frac{1}{2}w^2((1-t)B+tC) + \frac{1}{2}w^2((1-t)B-tC)\right\}^{\frac{1}{2}}.
\end{aligned}$$

Taking supremum over all $x \in \mathscr{H}$ with $\|x\| = 1$, we get the first inequality. In particular, considering $t = \frac{1}{2}$ we get the second inequality. □

Our next result reads as follows.

THEOREM 2.10. *Let $T \in \mathbb{B}(\mathscr{H})$, then the following inequalities hold:*

$$(i) \quad \sqrt{\frac{1}{4}\|T^*T + TT^*\| + \alpha} \leqslant w(T) \leqslant \sqrt{\frac{1}{4}\|T^*T + TT^*\| + \beta},$$

where $\alpha = \frac{1}{2}\left|\|Re(T)\|^2 - \|Im(T)\|^2\right|$, $\beta = \frac{1}{2}\left(\|Re(T)\|^2 + \|Im(T)\|^2\right)$.

$$(ii) \quad \sqrt{\frac{1}{4}\|T^*T + TT^*\| + \gamma} \leqslant w(T) \leqslant \sqrt{\frac{1}{4}\|T^*T + TT^*\| + \delta},$$

where

$$\begin{aligned}
\gamma &= \frac{1}{4}\left|\|Re(T) + Im(T)\|^2 - \|Re(T) - Im(T)\|^2\right|, \\
\delta &= \frac{1}{4}\left(\|Re(T) + Im(T)\|^2 + \|Re(T) - Im(T)\|^2\right).
\end{aligned}$$

Proof. (i) First inequality follows from [7, Th. 2.9] and the second inequality follows from the inequality (2.2) by considering $B = T$ and $C = T^*$.

(ii) First inequality follows from [10, Th. 2.3] and the second inequality follows from the inequality (2.2) by considering $B = Re(T)$ and $C = Im(T)$. □

To present our next result we need the following Hermite-Hadamard inequality, see [23, p. 137]. For a convex function $f : J \to \mathbb{R}$ and $a, b \in J$, we have

$$f\left(\frac{a+b}{2}\right) \leqslant \int_0^1 f(ta + (1-t)b)dt \leqslant \frac{f(a)+f(b)}{2}. \tag{2.3}$$

Also, we need the following lemmas.

LEMMA 2.11. [3, (4.24)] *Let $A \in \mathbb{B}(\mathscr{H})$ be self-adjoint with spectrum contained in the interval J and let $x \in \mathscr{H}$ with $\|x\| = 1$. If f is a convex function on J, then*

$$f(\langle Ax, x\rangle) \leqslant \langle f(A)x, x\rangle.$$

LEMMA 2.12. [17] (Generalized Cauchy-Schwarz inequality) *If $A \in \mathbb{B}(\mathscr{H})$, then*

$$|\langle Ax, y\rangle|^2 \leqslant \langle |A|^{2\alpha}x, x\rangle\langle |A^*|^{2(1-\alpha)}y, y\rangle,$$

for all $x, y \in \mathscr{H}$ and for all $\alpha \in [0,1]$.

Now we are in a position to prove our result.

THEOREM 2.13. *Let $B, C \in \mathbb{B}(\mathscr{H})$. If $f : [0,\infty) \to [0,\infty)$ is an increasing operator convex function, then*

$$\begin{aligned} f\left(w_e^2(B,C)\right) &\leqslant \left\| \int_0^1 f\left(t(B^*B + C^*C) + (1-t)(BB^* + CC^*)\right) dt \right\| \\ &\leqslant \frac{1}{2}\left\| f(B^*B + C^*C) + f(BB^* + CC^*) \right\|. \end{aligned}$$

Proof. Take $x \in \mathscr{H}$ with $\|x\| = 1$. We have

$$\begin{aligned} & f(|\langle Bx, x\rangle|^2 + |\langle Cx, x\rangle|^2) \\ &\leqslant f(\langle |B|x, x\rangle\langle |B^*|x, x\rangle + \langle |C|x, x\rangle\langle |C^*|x, x\rangle) \text{ (by Lemma 2.12)} \\ &\leqslant f\left(\{\langle |B|x, x\rangle^2 + \langle |C|x, x\rangle^2\}^{\frac{1}{2}}\{\langle |B^*|x, x\rangle^2 + \langle |C^*|x, x\rangle^2\}^{\frac{1}{2}}\right) \\ &\leqslant f\left(\frac{1}{2}\langle (B^*B + C^*C)x, x\rangle + \frac{1}{2}\langle (BB^* + CC^*)x, x\rangle\right) \\ &\leqslant \int_0^1 f\left(\left\langle \left(t(B^*B + C^*C) + (1-t)(BB^* + CC^*)\right)x, x\right\rangle\right) dt \text{ (by (2.3))}. \end{aligned}$$

Now,

$$\begin{aligned} & f\left(\left\langle \left(t(B^*B + C^*C) + (1-t)(BB^* + CC^*)\right)x, x\right\rangle\right) \\ &\leqslant \left\langle f\left(t(B^*B + C^*C) + (1-t)(BB^* + CC^*)\right)x, x\right\rangle \text{ (by Lemma 2.11)} \\ &\leqslant t\langle f(B^*B + C^*C)x, x\rangle + (1-t)\langle f(BB^* + CC^*)x, x\rangle, \end{aligned}$$

where the last inequality follows form operator convexity of f. Therefore,

$$\begin{aligned} & \int_0^1 f\left(\left\langle \left(t(B^*B + C^*C) + (1-t)(BB^* + CC^*)\right)x, x\right\rangle\right) dt \\ &\leqslant \langle \int_0^1 f\left(t(B^*B + C^*C) + (1-t)(BB^* + CC^*)\right) dt\, x, x\rangle \\ &\leqslant \frac{1}{2}\left(\langle f(B^*B + C^*C)x, x\rangle + \langle f(BB^* + CC^*)x, x\rangle\right). \end{aligned}$$

Taking the supremum over $x \in \mathscr{H}$ with $\|x\| = 1$, we get

$$f\left(w_e^2(B,C)\right) \leqslant \left\| \int_0^1 f\left(t(B^*B + C^*C) + (1-t)(BB^* + CC^*)\right) dt \right\| \leqslant \frac{1}{2} \|f(B^*B + C^*C) + f(BB^* + CC^*)\|.$$

Thus, we complete the proof. □

Since for $1 \leqslant r \leqslant 2$ the function $f(x) = x^r$, $x \geqslant 0$ is an increasing operator convex function, we have

$$w_e^{2r}(B,C) \leqslant \left\| \int_0^1 (t(B^*B + C^*C) + (1-t)(BB^* + CC^*))^r \, dt \right\| \tag{2.4}$$

$$\leqslant \frac{1}{2} \|(B^*B + C^*C)^r + (BB^* + CC^*)^r\|. \tag{2.5}$$

In particular, for $r = 1$,

$$w_e^2(B,C) \leqslant \left\| \int_0^1 (t(B^*B + C^*C) + (1-t)(BB^* + CC^*)) \, dt \right\| \leqslant \frac{1}{2} \|(B^*B + C^*C) + (BB^* + CC^*)\|. \tag{2.6}$$

The above inequality can also be derived from

$$w_e^2(B,C) \leqslant \left\|\alpha(|B|^2 + |C|^2) + (1-\alpha)(|B^*|^2 + |C^*|^2)\right\|, 0 \leqslant \alpha \leqslant 1,$$

proved by Moslehian et al. [21, Prop. 3.9]. Now, if we take $B = C = T$ in (2.4) we obtain the following numerical radius inequality.

COROLLARY 2.14. *Let $T \in \mathbb{B}(\mathscr{H})$, then*

$$w^2(T) \leqslant \left\| \int_0^1 \left(tT^*T + (1-t)TT^*\right)^r dt \right\|^{1/r} \leqslant \left\| \frac{(T^*T)^r + (TT^*)^r}{2} \right\|^{1/r},$$

for $1 \leqslant r \leqslant 2$.

Next, in the following theorem we develop a lower bound for the numerical radius of a bounded linear operator T.

THEOREM 2.15. *Let $T \in \mathbb{B}(\mathscr{H})$, then*

$$\frac{1}{4}\|T\| + \frac{1}{4}(\|Re(T)\| + \|Im(T)\|) + \frac{1}{2}\left|\|Re(T)\| - \|Im(T)\|\right| \leqslant w(T).$$

Proof. From the Cartesian decomposition of T, it is easy to verify that $\|Re(T)\| \leqslant w(T)$, $\|Im(T)\| \leqslant w(T)$ and $\frac{1}{2}\|T\| \leqslant w(T)$. Take $r_1 = \left|\|Re(T)\| - \frac{1}{2}\|T\|\right|$, $r_2 =$

$\big|\|Im(T)\| - \frac{1}{2}\|T\|\big|$, $q_1 = \max\left\{\|Re(T)\|, \frac{1}{2}\|T\|\right\}$ and $q_2 = \max\left\{\|Im(T)\|, \frac{1}{2}\|T\|\right\}$. We have

$$\begin{aligned}
w(T) &\geqslant \max\{q_1, q_2\} \\
&= \frac{1}{2}(q_1+q_2) + \frac{1}{2}|q_1-q_2| \\
&= \frac{1}{4}\|T\| + \frac{1}{4}(\|Re(T)\| + \|Im(T)\|) + \frac{1}{4}(r_1+r_2) + \frac{1}{2}|q_1-q_2| \\
&\geqslant \frac{1}{4}\|T\| + \frac{1}{4}\|Re(T) + iIm(T)\| + \frac{1}{4}(r_1+r_2) + \frac{1}{2}|q_1-q_2| \\
&= \frac{1}{2}\|T\| + \frac{1}{4}(r_1+r_2) + \frac{1}{2}|q_1-q_2| \\
&= \frac{1}{2}\|T\| + \frac{1}{4}\left|\|Re(T)\| - \frac{1}{2}\|T\|\right| + \frac{1}{4}\left|\|Im(T)\| - \frac{1}{2}\|T\|\right| + \frac{1}{2}|q_1-q_2| \\
&= \frac{1}{4}\|T\| + \frac{1}{4}(\|Re(T)\| + \|Im(T)\|) + \frac{1}{2}\big|\|Re(T)\| - \|Im(T)\|\big|,
\end{aligned}$$

as desired. □

REMARK 2.16. (i) It follows from [15] that

$$\frac{1}{2}\|T\| + \frac{1}{4}\left|\|Re(T)\| - \frac{1}{2}\|T\|\right| + \frac{1}{4}\left|\|Im(T)\| - \frac{1}{2}\|T\|\right| \leqslant w(T). \tag{2.7}$$

Clearly, the inequality in Theorem 2.15 refines the inequality (2.7).

(ii) It follows from Theorem 2.15 that if

$$\frac{1}{2}\|T\| + \frac{1}{4}\left|\|Re(T)\| - \frac{1}{2}\|T\|\right| + \frac{1}{4}\left|\|Im(T)\| - \frac{1}{2}\|T\|\right| = w(T)$$

then $\max\left\{\|Re(T)\|, \frac{1}{2}\|T\|\right\} = \max\left\{\|Im(T)\|, \frac{1}{2}\|T\|\right\}$. However, the converse may not be true.

(iii) For $T \in \mathbb{B}(\mathscr{H})$, Bhunia and Paul [7, Th. 2.1] proved that

$$\frac{1}{2}\|T\| + \frac{1}{2}\big|\|\Re(T)\| - \|\Im(T)\|\big| \leqslant w(T). \tag{2.8}$$

Clearly, the inequality in Theorem 2.15 refines (2.8).

(iv) It follows from Theorem 2.15 that if $w(T) = \frac{1}{2}\|T\| + \frac{1}{2}\big|\|\Re(T)\| - \|\Im(T)\|\big|$, then $\|T\| = \|Re(T)\| + \|Im(T)\|$ and $w(T) = \max\{\|Re(T)\|, \|Im(T)\|\}$. The converse is also true.

3. Numerical radius bounds of 2×2 operator matrices

Using the numerical radius inequalities obtained in Section 2, here we develop the numerical radius bounds of 2×2 off-diagonal operator matrices. Suppose $\mathscr{H}\oplus\mathscr{H}$ is the direct sum of two copies of $\mathscr{H}$, and $\begin{pmatrix} B & X \\ Y & C \end{pmatrix} \in \mathbb{B}(\mathscr{H}\oplus\mathscr{H})$ is a 2×2 operator matrix, defined by $\begin{pmatrix} B & X \\ Y & C \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} Bx+Xy \\ Yx+Cy \end{pmatrix}$, $\forall \begin{pmatrix} x \\ y \end{pmatrix} \in \mathscr{H}\oplus\mathscr{H}$. Considering $T = \begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix} \in \mathbb{B}(\mathscr{H}\oplus\mathscr{H})$ in Theorem 2.15, Corollary 2.5, Corollary 2.7 and Theorem 2.10 respectively, we get the following bounds for the numerical radius of the 2×2 off-diagonal operator matrix $\begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix}$.

THEOREM 3.1. *Let* $T = \begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix} \in \mathbb{B}(\mathscr{H}\oplus\mathscr{H})$, *then the following inequalities hold:*

$$(i)\ w(T) \geqslant \max\left\{\frac{\|X\|}{4},\frac{\|Y\|}{4}\right\} + \frac{1}{4}\left(\frac{\|X+Y^*\|}{2}+\frac{\|X-Y^*\|}{2}\right)+\frac{1}{2}\left|\frac{\|X+Y^*\|}{2}-\frac{\|X-Y^*\|}{2}\right|.$$

$$(ii)\ w^2(T) \geqslant \max\left\{\frac{\|X^*X+YY^*\|}{8},\frac{\|XX^*+Y^*Y\|}{8}\right\} + \frac{1}{4}\left(\frac{\|X+Y^*\|^2}{4}+\frac{\|X-Y^*\|^2}{4}\right)+\frac{1}{2}\left|\frac{\|X+Y^*\|^2}{4}-\frac{\|X-Y^*\|^2}{4}\right|.$$

$$(iii)\ w^2(T) \geqslant \max\left\{\frac{\|X^*X+YY^*\|}{8},\frac{\|XX^*+Y^*Y\|}{8}\right\} + \frac{1}{8}\left(\frac{\|(1-i)X+(1+i)Y^*\|^2}{4}+\frac{\|(1+i)X-(1-i)Y^*\|^2}{4}\right) + \frac{1}{4}\left|\frac{\|(1-i)X+(1+i)Y^*\|^2}{4}-\frac{\|(1+i)X-(1-i)Y^*\|^2}{4}\right|.$$

$$(iv)\ w^2(T) \leqslant \max\left\{\frac{\|X^*X+YY^*\|}{4},\frac{\|XX^*+Y^*Y\|}{4}\right\}+\frac{1}{2}\left|\frac{\|X+Y^*\|^2}{4}+\frac{\|X-Y^*\|^2}{4}\right|.$$

$$(v)\ w^2(T) \leqslant \max\left\{\frac{\|X^*X+YY^*\|}{4},\frac{\|XX^*+Y^*Y\|}{4}\right\} + \frac{1}{4}\left|\frac{\|(1-i)X+(1+i)Y^*\|^2}{4}+\frac{\|(1+i)X-(1-i)Y^*\|^2}{4}\right|.$$

REMARK 3.2. (i) We remark that the bound in Theorem 3.1 (i) is stronger than the bound in [4, Th. 2.7], namely,

$$w(T) \geqslant \max\left\{\frac{\|X\|}{2}, \frac{\|Y\|}{2}\right\} + \frac{1}{2}\left|\frac{\|X+Y^*\|}{2} - \frac{\|X-Y^*\|}{2}\right|.$$

(ii) It is easy to verify that the bound in Theorem 3.1 (ii) is stronger than the bound in [4, Th. 2.12], namely,

$$\begin{aligned} w^2(T) \geqslant \max&\left\{\frac{\|X^*X+YY^*\|}{4}, \frac{\|XX^*+Y^*Y\|}{4}\right\} \\ &+\frac{1}{2}\left|\frac{\|X+Y^*\|^2}{4} - \frac{\|X-Y^*\|^2}{4}\right|. \end{aligned}$$

Now, by applying the operator matrix technique we develop upper bounds for the numerical radius of a bounded linear operator T by using the t-Aluthge transform. First we give the following upper bound for the numerical radius $w\begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix}$, where $X,Y \in \mathbb{B}(\mathscr{H})$.

THEOREM 3.3. [11, Th. 2.5 and Cor. 2.6] *Let* $X,Y \in \mathbb{B}(\mathscr{H})$ *and* $T = \begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix} \in \mathbb{B}(\mathscr{H} \oplus \mathscr{H})$. *If* $S = |X|^2 + |Y^*|^2$ *and* $P = |X^*|^2 + |Y|^2$, *then*

$$w^2\begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix} = w^2(T) \leqslant \sqrt{\min\{\beta,\gamma\}},$$

where

$$\beta = \frac{1}{16}\|S\|^2 + \frac{1}{4}w^2(YX) + \frac{1}{8}w(YXS+SYX),$$

$$\gamma = \frac{1}{16}\|P\|^2 + \frac{1}{4}w^2(XY) + \frac{1}{8}w(XYP+PXY).$$

For $X,Y \in \mathbb{B}(\mathscr{H})$, we have the following inequalities:

$$w(XY) \leqslant w\begin{pmatrix} XY & 0 \\ 0 & YX \end{pmatrix} = w\left(\begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix}^2\right) \leqslant w^2\begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix}. \tag{3.1}$$

Now, by using (3.1) and Theorem 3.3, we prove the following result.

COROLLARY 3.4. *Let* $T \in \mathbb{B}(\mathscr{H})$. *If* $P_t = |T|^{2(1-t)} + |T|^{2t}$, $0 \leqslant t \leqslant 1$, *then*

$$\begin{aligned} w(T) &\leqslant \sqrt{\frac{1}{16}\|P_t\|^2 + \frac{1}{4}w^2(\widetilde{T}_t) + \frac{1}{8}w(\widetilde{T}_t P_t + P_t \widetilde{T}_t)} \qquad (3.2) \\ &\leqslant \frac{1}{4}\left\||T|^{2(1-t)} + |T|^{2t}\right\| + \frac{1}{2}w(\widetilde{T}_t). \end{aligned}$$

In particular, for $t = \frac{1}{2}$

$$\begin{aligned} w(T) &\leqslant \sqrt{\frac{1}{4}\|T\|^2 + \frac{1}{4}w^2(\widetilde{T}) + \frac{1}{4}w(\widetilde{T}|T| + |T|\widetilde{T})} \\ &\leqslant \frac{1}{2}\|T\| + \frac{1}{2}w(\widetilde{T}). \end{aligned} \tag{3.3}$$

Proof. Taking $X = U|T|^{1-t}$ and $Y = |T|^t$ in Theorem 3.3 (in the expression β) we obtain the inequality (3.2), and the next inequality follows from the inequality (see [14]) $w(XY + Y^*X) \leqslant 2\|Y\|w(X)$ for all $X, Y \in \mathbb{B}(\mathscr{H})$. The rest of the inequalitie follows by considering $t = \frac{1}{2}$. □

REMARK 3.5. (i) Let $T \in \mathbb{B}(\mathscr{H})$. Then, clearly the inequality (3.2) refines the bound

$$w(T) \leqslant \frac{1}{4}\left\| |T|^{2(1-t)} + |T|^{2t} \right\| + \frac{1}{2}w(\widetilde{T}_t),$$

obtained by Kittaneh et al. [18, Cor. 2.2].

(ii) We would like to remark that the inequality (3.3) is stronger than the inequality

$$w(T) \leqslant \frac{1}{2}\|T\| + \frac{1}{2}w(\widetilde{T}) \left(\leqslant \frac{1}{2}\|T\| + \frac{1}{2}\left\|T^2\right\|^{1/2} \right),$$

proved by Yamazaki [27, Th. 2.1].

Note that the inequality (3.3) is already proved in [10, Th. 2.6] but the approach here is different and simple.

Finally, we prove the following result.

COROLLARY 3.6. *Let* $T \in \mathbb{B}(\mathscr{H})$. *If* $Q_t = |T^*|^{2(1-t)} + |T|^{2t}$, $0 \leqslant t \leqslant 1$, *then*

$$\begin{aligned} w(T) &\leqslant \sqrt{\frac{1}{16}\|Q_t\|^2 + \frac{1}{4}w^2(T) + \frac{1}{8}w(TQ_t + Q_tT)} \\ &\leqslant \frac{1}{4}\||T^*|^{2(1-t)} + |T|^{2t}\| + \frac{1}{2}w(T) \\ &\leqslant \frac{1}{2}\left\||T^*|^{2(1-t)} + |T|^{2t}\right\|. \end{aligned} \tag{3.4}$$

Proof. Taking $X = U|T|^{1-t}$, $Y = |T|^t$ in Theorem 3.3 we obtain the inequality (3.4). The second inequality follows from $w(XY + Y^*X) \leqslant 2\|Y\|w(X)$ for all $X, Y \in \mathbb{B}(\mathscr{H})$ (see [14]). The last inequality follows trivially. □

REMARK 3.7. Note that the bound in (3.4) is sharper than the bound

$$w(T) \leqslant \frac{1}{4}\left\| |T^*|^{2(1-t)} + |T|^{2t} \right\| + \frac{1}{2}w(T),$$

proved by Kittaneh et al. [18].

REFERENCES

[1] A. ABU-OMAR AND F. KITTANEH, *A numerical radius inequality involving the generalized Aluthge transform*, Studia Math. **216** (2013), no. 1, 69–75.

[2] S. BAG, P. BHUNIA AND K. PAUL, *Bounds of numerical radius of bounded linear operators using t-Aluthge transform*, Math. Inequal. Appl. **23** (2020), no. 3, 991–1004.

[3] R. BHATIA, *Positive definite matrices*, Princeton University Press, Princeton, 2007.

[4] P. BHUNIA AND K. PAUL, *Numerical radius inequalities of* 2×2 *operator matrices*, Adv. Oper. Theory **8** (2023), no. 1, Paper No. 11, 17 pp.

[5] P. BHUNIA, S. S. DRAGOMIR, M. S. MOSLEHIAN AND K. PAUL, *Lectures on Numerical Radius Inequalities*, Infosys Sci. Found. Ser. Math. Sci. Springer, Cham, (2022), xii+209 pp.

[6] P. BHUNIA, S. JANA, M. S. MOSLEHIAN AND K. PAUL, *Improved inequality for the numerical radius via Cartesian decomposition*, Translation of Funktsional. Anal. i Prilozhen. **57** (2023), no. 1, 24–37, Funct. Anal. Appl. **57** (2023), no.1, 18-28.

[7] P. BHUNIA AND K. PAUL, *Development of inequalities and characterization of equality conditions for the numerical radius*, Linear Algebra Appl. **630** (2021), 306–315.

[8] P. BHUNIA AND K. PAUL, *Proper improvement of well-known numerical radius inequalities and their applications*, Results Math. **76** (2021), no. 4, Paper No. 177, 12 pp.

[9] P. BHUNIA AND K. PAUL, *New upper bounds for the numerical radius of Hilbert space operators*, Bull. Sci. Math. **167** (2021), Paper No. 102959, 11 pp.

[10] P. BHUNIA, S. JANA AND K. PAUL, *Refined inequalities for the numerical radius of Hilbert space operators*, Rocky Mountain J. Math. (2024), to appear.

[11] P. BHUNIA, S. BAG, K. PAUL, *Numerical radius inequalities of operator matrices with applications*, Linear Multilinear Algebra **69** (2021), no. 9, 1635–1644.

[12] P. BHUNIA, S. BAG AND K. PAUL, *Numerical radius inequalities and its applications in estimation of zeros of polynomials*, Linear Algebra Appl. **573** (2019), 166–177.

[13] S. S. DRAGOMIR, *Some inequalities for the Euclidean operator radius of two operators in Hilbert spaces*, Linear Algebra Appl. **419** (2006), 256–264.

[14] C. K. FONG, J. A. R. HOLBROOK, *Unitarily invariant operator norms*, Canad. J. Math. **35** (1983) 274–299.

[15] O. HIRZALLAH, F. KITTANEH AND K. SHEBRAWI, *Numerical Radius Inequalities for Certain* 2×2 *Operator Matrices*, Integr. Equ. Oper. Theory **71** (2011), 129–147.

[16] S. JANA, P. BHUNIA, K. PAUL, *Euclidean operator radius inequalities of a pair of bounded linear operators and their application*, Bull. Braz. Math. Soc. (N.S.) **54** (2023), no. 1, Paper No. 1, 14 pp.

[17] T. KATO, *Notes on some inequalities for linear operators*, Math. Ann. **125** (1952), 208–212.

[18] F. KITTANEH, H. R. MORADI, M. SABABHEH, *Sharper bounds for the numerical radius, Linear Multilinear Algebra*, (2023), https://doi.org/10.1080/03081087.2023.2177248.

[19] F. KITTANEH, *Numerical radius inequalities for Hilbert space operators*, Studia Math. **168** (2005), no. 1, 73–80.

[20] F. KITTANEH, *A numerical radius inequality and an estimate for the numerical radius of the Frobenius companion matrix*, Studia Math. **158** (2003), no. 1, 11–17.

[21] M. S. MOSLEHIAN, M. SATTARI AND K. SHEBRAWI, *Extensions of Euclidean operator radius inequalities*, Math. Scand. **120** (2017), no. 1, 129–144.

[22] M. E. OMIDVAR AND H. R. MORADI, *Better bounds on the numerical radii of Hilbert space operators*, Linear Algebra Appl. **604** (2020), 265–277.

[23] J. E. PEČARIĆ, F. PROSCHAN AND Y. L. TONG, *Convex functions, partial orderings, and statistical applications*, Mathematics in Science and Engineering, **187**, Academic Press, Inc., Boston, MA, 1992, xiv+467 pp, ISBN: 0-12-549250-226-02.

[24] G. POPESCU, *Unitary invariants in multivariable operator theory*, Mem. Amer. Math. Soc. **200** (2009), no. 941, vi+91 pp, ISBN: 978-0-8218-4396-3.

[25] S. SAHOO, N. C. ROUT AND M. SABABHEH, *Some extended numerical radius inequalities*, Linear Multilinear Algebra **69** (2021), no. 5, 907–920.

[26] P. Y. WU AND H.-L. GAU, *Numerical ranges of Hilbert space operators*, Encyclopedia of Mathematics and its Applications **179**, Cambridge University Press, Cambridge, 2021, xviii+483 pp, ISBN: 978-1-108-47906-6 47-02.

[27] T. YAMAZAKI, *On upper and lower bounds for the numerical radius and an equality condition*, Studia Math. **178** (2007), no. 1, 83–89.

(Received August 30, 2023)

Suvendu Jana
Department of Mathematics
Mahisadal Girls' College
Purba Medinipur 721628, West Bengal, India
e-mail: janasuva8@gmail.com

Pintu Bhunia
Department of Mathematics
Indian Institute of Science
Bengaluru 560012, Karnataka, India
e-mail: pintubhunia5206@gmail.com
pintubhunia@iisc.ac.in

Kallol Paul
Department of Mathematics
Jadavpur University
Kolkata 700032, West Bengal, India
e-mail: kalloldada@gmail.com