Operators
and
Matrices
Volume 18, Number 4 (2024), 941–966
doi:10.7153/oam-2024-18-57

UNCERTAINTY PRINCIPLES FOR THE WHITTAKER WIGNER TRANSFORM

HATEM MEJJAOLI

(*Communicated by D. Han*)

Abstract. The Whittaker Wigner transform (WWT) is a novel addition to the class of Wigner transforms, which has gained a respectable status in the realm of time-frequency signal analysis within a short span of time. Knowing the fact that the study of the time-frequency analysis is both theoretically interesting and practically useful, the aim of this paper is to explore a class of quantitative uncertainty principles associated with the WWT, including the Heisenberg's uncertainty, Benedick's UP, Donoho-Stark's UP, Benedick-Amrein-Berthier UP and the uncertainty principle for orthonormal sequences.

1. Introduction

The Fourier transform stands out as a significant discovery in mathematical sciences, playing a crucial role in modern scientific and technological advancements. In signal processing, extensive research has utilized the Fourier transform to analyze stationary signals or processes with statistically invariant properties over time. However, many signals exhibit non-stationary characteristics, requiring a time-frequency analysis for a comprehensive representation. The Wigner transform (WT), also known as the short-time Fourier transform (STFT), marked a breakthrough in time-frequency analysis. This method involves decomposing non-transient signals using time and frequency-shifted basis functions, termed Wigner window functions. The WT, with its clear resemblance to the classical Fourier transform, has garnered considerable attention in the past few decades. Its applications span various fields, including harmonic analysis, signal and image processing, pseudo-differential operators, sampling theory, quantum mechanics, geophysics, astrophysics, and medicine (cf. [7,16,18,19,26,31,32,45]) and others.

Although, Fourier transforms have many success stories and fascinated the mathematical, physical and engineering communities over decades, however they still face numerous shortcomings. One of the significant disadvantages of the Fourier transforms is that it does not give any information about the occurrence of the frequency component at a particular time; they only enable us to analyse the signals either in time domain or the frequency domain, but not simultaneously in both domains [10,36]. An appropriate redressable of these limitations was given by Gabor [16] in the form of windowed

Mathematics subject classification (2020): 33E30, 43A32, 81S07, 81S30.
Keywords and phrases: Index Whittaker transform, Whittaker Wigner transform, uncertainty principles.

Paper OaM-18-57

Fourier transform by using a Gaussian distribution function as a window function with the aim of constructing efficient time-frequency localized expansions of finite energy signals $f \in L^2(\mathbb{R})$ as

$$\mathscr{V}_h(f)(\xi,b) = \int_{\mathbb{R}} f(x)\overline{h(x-b)}e^{-i\xi x}\,dx, \quad \xi, b \in \mathbb{R}.$$

With the aid of this transform, one can analyze the spectral contents of non-transient signals in localized neighbourhoods of time. This astonishing feature of the Wigner transform provides the local characteristics of the Fourier transform with time resolution equal to the size of the window. Soon after its inception in quantum mechanics, the Wigner transform profound influenced diverse branches of science and engineering including harmonic analysis, signal and image processing, wave propagation, quantum optics, geophysics, astrophysics and many more [27]. Apart from applications, the theoretical skeleton of Wigner transform has likewise been extensively studied and investigated on other groups including the locally compact Abelian and non-abelian groups [12, 13, 15], hypergroups [8], Gelfand pairs [44] and so on. For more about Wigner transforms and their applications, we allude to [14, 20, 21].

The index Whittaker transform is one of the most powerful tools for a treatment of the integral transforms of the index Whittaker convolution type, their convolutions, and generating operators. Many fundamental results of this transform are already known. The harmonic analysis associated with this transform has been developed by Sousa and al in [41, 42]. We pin down that this transform has a rich structure, and recently has been gaining a lot of attention (see, e.g., [1, 2, 5, 23, 24, 33–35, 40–43, 47]).

As the harmonic analysis associated with the index Whittaker transform has been extensively investigated and has witnessed a remarkable development, it is natural to study several aspects of the time-frequency analysis associated with the Whittaker Wigner transform (WWT) introduced in [40].

In this paper, we attempt to investigate few versions of the quantitative uncertainty principles for the proposed transform. In the classical setting, the notion of the quantitative uncertainty principles for the Wigner transform was first introduced by Wilczok [45]. Next, this subject has been extended for the generalized Wigner transforms (see [6, 9, 18, 19, 26–30] and others) and for several classes of groups of the form $K \ltimes \mathbb{R}^d$ (see [3]).

The objectives of this study are mentioned below:

- To study the reproducing kernel associated with the WWT.
- To derive several versions of the Heisenberg's uncertainty principle via different techniques.
- To study the concentration-based uncertainty principles.

The remainder of this paper is arranged as follows. In §2 we recall the main results about the harmonic analysis associated with the index Whittaker transform and we review some results on the WWT. §3 is devoted to study various quantitative uncertainty principles for the proposed transform. In particular, we present many variants of Heisenberg's inequalities, uncertainty principle for orthonormal sequence, Donoho-Stark's uncertainty principle, Benedicks-type uncertainty principle and Benedick-Amrein-Berthier's uncertainty principle.

2. Preliminaries

This section gives an introduction to the harmonic analysis associated with the index Whittaker transform. Main references are [41, 42].

2.1. Index Whittaker transform

Here, we shall take a survey of the index Whittaker transform together with the fundamental properties. To facilitate the narrative, we set some notations as under:

- $C_b(\mathbb{R}_+)$ the space of bounded continuous functions on $\mathbb{R}_+$.
- $C_c(\mathbb{R}_+)$ the space of continuous functions on $\mathbb{R}_+$ with compact support.
- For $p \in [1,\infty]$, the conjugate exponent shall be denoted by p'.
- $L^p_{\gamma_k}(\mathbb{R}_+)$, $1 \leqslant p \leqslant \infty$, the space of measurable functions f on $\mathbb{R}_+$ satisfying

$$\|f\|_{L^p_{\gamma_k}(\mathbb{R}_+)} := \left(\int_{\mathbb{R}_+} |f(x)|^p d\gamma_k(x)\right)^{1/p} < \infty, \quad \text{if} \quad 1 \leqslant p < \infty$$
$$\|f\|_{L^\infty_{\gamma_k}(\mathbb{R}_+)} := ess \sup_{x\in\mathbb{R}_+} |f(x)| < \infty,$$

where

$$d\gamma_k(x) := A_k(x)dx = x^{1-4k}e^{-\frac{1}{2x^2}}dx. \tag{2.1}$$

For $p = 2$, we provide this space with the scalar product

$$\langle f,g\rangle_{L^2_{\gamma_k}(\mathbb{R}_+)} := \int_{\mathbb{R}_+} f(x)\overline{g(x)}d\gamma_k(x).$$

- $L^p_{\nu_k}(\mathbb{R}_+)$, $1 \leqslant p \leqslant \infty$, the space of measurable functions f on $\mathbb{R}_+$ satisfying

$$\|f\|_{L^p_{\nu_k}(\mathbb{R}_+)} := \left(\int_{\mathbb{R}_+} |f(x)|^p d\nu_k(x)\right)^{1/p} < \infty, \quad \text{if } 1 \leqslant p < \infty$$
$$\|f\|_{L^\infty_{\nu_k}(\mathbb{R}_+)} := ess \sup_{x\in\mathbb{R}_+} |f(x)| < \infty,$$

where

$$\begin{aligned} d\nu_k(\lambda) &:= C_k(\lambda)d\lambda \\ &= 2^{1-2k}\pi^{-2}\sinh(-2\pi i\Delta_\lambda)\left|\Gamma\left(\tfrac{1}{2}-k+\Delta_\lambda\right)\right|^2\chi_\Lambda(\lambda)d\lambda, \end{aligned} \tag{2.2}$$

here, χ_Λ is the characteristic function of the interval $\Lambda := ((1/2-k)^2, \infty)$ and

$$\Delta_\lambda := \sqrt{\Big(\frac{1}{2}-k\Big)^2 - \lambda}. \tag{2.3}$$

For $k < \frac{1}{2}$, and $f \in L^1_{\gamma_k}(\mathbb{R}_+)$, the index Whittaker transform is defined by

$$\mathscr{F}^W_k(f)(\lambda) := \int_{\mathbb{R}_+} f(x)\mathbf{W}_{k,\Delta_\lambda}(x)d\gamma_k(x), \quad \text{for all } \lambda \in \mathbb{R}_+, \tag{2.4}$$

where $\mathbf{W}_{k,\Delta_\lambda}$ is the confluent Whittaker-type function defined by

$$\mathbf{W}_{k,l}(x) := 2^k x^k e^{\frac{1}{4x^2}} \mathscr{W}_{k,l}\Big(\frac{1}{2x^2}\Big) = (2x^2)^{-\frac{1}{2}+k+l}\Psi\Big(\frac{1}{2}-k-l, 1-2l; \frac{1}{2x^2}\Big). \tag{2.5}$$

Here $\mathscr{W}_{k,l}(z)$ denote the Whittaker function of the second kind, $\Psi(a,b;z)$ is the confluent hypergeometric function of the second kind, $k < 1/2$, and $l \in \mathbb{C}$ are parameters.

Next, we give some properties of the Whittaker kernel.

PROPOSITION 2.1. (i) *For $l \in \mathbb{C}$ and $k < \frac{1}{2}$, the function $\mathbf{W}_{k,l}$ is the solution of the differential equation*

$$\delta_k u = \Big(\Big(\frac{1}{2}-k\Big)^2 - l^2\Big)u, \quad u(0) = 1,$$

where δ_k is the operator defined by

$$\delta_k := \frac{1}{4}\Big[x^2\frac{d^2}{dx^2} + (x^{-1} + (3-4k)x)\frac{d}{dx}\Big]. \tag{2.6}$$

(ii) *For $x \in \mathbb{R}_+$, we have*

$$\mathbf{W}_{k,\frac{1}{2}-k}(x) = 1. \tag{2.7}$$

(iii) *The Whittaker kernel $\mathbf{W}_{k,l}$ admits the integral representation*

$$\mathbf{W}_{k,l}(x) = \int_0^\infty \cosh(ls)\eta^k_x(s)ds, \quad k,\ l \in \mathbb{C}, \quad x > 0,$$

where

$$\eta^k_x(s) := (2\pi)^{-\frac{1}{2}}x^{-1+2k}\exp\Big(\frac{1}{2x^2} - \frac{1}{4x^2}\cosh^2\Big(\frac{s}{2}\Big)\Big)D_{2k}\Big(\frac{1}{x}\cosh\Big(\frac{s}{2}\Big)\Big),$$

being $D_\mu(z)$ the parabolic cylinder function given by

$$D_\mu(z) := \frac{z^\mu e^{-\frac{z^2}{4}}}{\Gamma(\frac{1}{2}(1-\mu))}\int_0^\infty e^{-s}s^{-\frac{1}{2}(1+\mu)}\Big(1+\frac{2s}{z^2}\Big)^{\frac{\mu}{2}}ds, \quad Re\, z > 0, \quad Re\,\mu < 1. \tag{2.8}$$

(iv) *For all $x \in \mathbb{R}_+$ we have*

$$\Big|\mathbf{W}_{k,l}(x)\Big| \leqslant 1, \quad |Re\, l| \leqslant \frac{1}{2} - k. \tag{2.9}$$

(v) *Let $k,\ l \in \mathbb{C}$. The Whittaker kernel $\mathbf{W}_{k,l}$ has the following product formula:*

$$\forall x,y \geqslant 0, \quad \mathbf{W}_{k,l}(x)\mathbf{W}_{k,l}(y) = \int_0^\infty \mathbf{W}_{k,l}(t)q_k(x,y,t)d\gamma_k(t), \tag{2.10}$$

where

$$q_k(x,y,t) = \frac{(xyt)^{-1+2k}}{(2\pi)^{\frac{1}{2}}}\exp\Big(\frac{1}{2x^2}+\frac{1}{2y^2}+\frac{1}{2t^2}-\Big(\frac{x^2+y^2+t^2}{4xyt}\Big)^2\Big)D_{2k}\Big(\frac{x^2+y^2+t^2}{2xyt}\Big). \tag{2.11}$$

REMARK 2.1. (1) The operator δ_k has the form of the Sturm-Liouville operator

$$\delta_k = \frac{1}{4}\Big[x^2\frac{d^2}{dx^2}+\frac{[x^2A_k(x)]'}{A_k(x)}\frac{d}{dx}\Big],$$

where A_k is the function given by (2.1).

(2) The index Whittaker transform $\mathscr{F}_k^W$ is bounded on the space $L^1_{\gamma_k}(\mathbb{R}_+)$, and for all f in $L^1_{\gamma_k}(\mathbb{R}_+)$, we have

$$||\mathscr{F}_k(f)||_{L^\infty_{\nu_k}(\mathbb{R}_+)} \leqslant ||f||_{L^1_{\gamma_k}(\mathbb{R}_+)}. \tag{2.12}$$

The authors in [41,42] have proved the following.

PROPOSITION 2.2. (i) Plancherel's theorem for $\mathscr{F}_k^W$. *The index Whittaker transform $f \mapsto \mathscr{F}_k^W(f)$ is an isometric isomorphism from $L^2_{\gamma_k}(\mathbb{R}_+)$ into $L^2_{\nu_k}(\mathbb{R}_+)$ and we have*

$$\int_{\mathbb{R}_+}|f(x)|^2d\gamma_k(x) = \int_{\mathbb{R}_+}|\mathscr{F}_k^W(f)(\lambda)|^2d\nu_k(\lambda). \tag{2.13}$$

(ii) Parseval's formula for $\mathscr{F}_k^W$. *For all f,g in $L^2_{\gamma_k}(\mathbb{R}_+)$ we have*

$$\int_{\mathbb{R}_+} f(x)\overline{g(x)}d\gamma_k(x) = \int_{\mathbb{R}_+}\mathscr{F}_k^W(f)(\lambda)\overline{\mathscr{F}_k^W(g)(\lambda)}d\nu_k(\lambda). \tag{2.14}$$

(iii) Inversion formula for $\mathscr{F}_k^W$. *Let $f \in L^1_{\gamma_k}(\mathbb{R}_+)$ such that $\mathscr{F}_k^W(f) \in L^1_{\nu_k}(\mathbb{R}_+)$, we have*

$$f(x) = \int_{(\frac{1}{2}-k)^2}^\infty \mathscr{F}_k^W(f)(\lambda)\mathbf{W}_{k,\Delta_\lambda}(x)d\nu_k(\lambda), \quad x \in \mathbb{R}_+. \tag{2.15}$$

PROPOSITION 2.3. *Let f be in $L^p_{\gamma_k}(\mathbb{R}_+)$, $p \in [1,2]$. Then $\mathscr{F}_k^W(f)$ belongs to $L^{p'}_{\nu_k}(\mathbb{R}_+)$ and we have*

$$\big\|\mathscr{F}_k^W(f)\big\|_{L^{p'}_{\nu_k}(\mathbb{R}_+)} \leqslant \|f\|_{L^p_{\gamma_k}(\mathbb{R}_+)}.$$

2.2. Whittaker translation operator

Recently the authors in [41] have given the following explicit formula for the generalized translation operators.

THEOREM 2.1. *Let $x \in \mathbb{R}_+$ and let $f \in C_b(\mathbb{R}_+)$. For $k < \frac{1}{2}$, the Whittaker translation operator τ_x^k is given by*

$$\tau_x^k f(y) = \int_{\mathbb{R}_+} f(z) d\zeta_{x,y}^k(z), \tag{2.16}$$

here

$$d\zeta_{x,y}^k(z) := \begin{cases} q_k(x,y,z)d\gamma_k(z), & \text{if} \quad xy \neq 0, \\ d\delta_x(z), & \text{if} \quad y = 0, \\ d\delta_y(z), & \text{if} \quad x = 0, \end{cases}$$

where $q_k(x,y,.)$ is defined by (2.11).

REMARK 2.2. (i) For all $x,y \in \mathbb{R}_+$, we have the product formula

$$\tau_x^k \mathbf{W}_{k,l}(y) = \mathbf{W}_{k,l}(x)\mathbf{W}_{k,l}(y). \tag{2.17}$$

(ii) For all $x,y \in \mathbb{R}_+$, we have

$$\int_{\mathbb{R}_+} q_k(x,y,z)d\gamma_k(z) = 1. \tag{2.18}$$

(iii) For all $x,y,z \in \mathbb{R}_+$, we have

$$q_k(x,y,z) = q_k(y,x,z) = q_k(z,x,z). \tag{2.19}$$

(iv) For any $x,y,z \in \mathbb{R}_+$, we have

$$q_k(x,y,z) > 0. \tag{2.20}$$

Now, let us go back to the properties of the generalized translation operator.

THEOREM 2.2. ([41]) *Let $k < \frac{1}{2}$, then*

(i) *For all $f \in L^1_{\text{loc}}(d\gamma_k)$ and for all $x,y \in \mathbb{R}_+$, we have*

$$\tau_x^k f(y) = \tau_y^k f(x) \quad \textit{and} \quad \tau_0^k f = f.$$

(ii) *For all $1 \leqslant p \leqslant \infty$ and $f \in L^p_{\gamma_k}(\mathbb{R}_+)$,*

$$\|\tau_x^k f\|_{L^p_{\gamma_k}(\mathbb{R}_+)} \leqslant \|f\|_{L^p_{\gamma_k}(\mathbb{R}_+)}. \tag{2.21}$$

(iii) *If* $f \in L^1_{\gamma_k}(\mathbb{R}_+)$, *and* $x \in \mathbb{R}_+$, *then*

$$\forall \lambda \in \mathbb{R}_+, \quad \mathscr{F}_k(\tau_x^k f)(\lambda) = \mathbf{W}_{k,\Delta_\lambda}(x)\mathscr{F}_k f(\lambda). \tag{2.22}$$

(iv) *Let* $f \in L^1_{\gamma_k}(\mathbb{R}_+)$ *be nonnegative. Then we have*

$$\forall x \in \mathbb{R}_+, \quad \tau_x^k f \geqslant 0.$$

(v) *For every* $f \in L^1_{\gamma_k}(\mathbb{R}_+)$ *we have*

$$\int_{\mathbb{R}_+} \tau_x^k f(y) d\gamma_k(y) = \int_{\mathbb{R}_+} f(y) d\gamma_k(y). \tag{2.23}$$

2.3. Whittaker Wigner transform

This subsection is devoted to give the main results on the Whittaker Wigner transform. The main reference is [40].

NOTATION. We denote by

- $\mathbb{R}^2_+ := \mathbb{R}_+ \times \mathbb{R}_+$.
- $d\mu_k(x,\lambda) := d\nu_k(x) d\gamma_k(\lambda)$, for all $(x,\lambda) \in \mathbb{R}^2_+$.
- $L^p_{\mu_k}(\mathbb{R}^2_+)$, $1 \leqslant p \leqslant \infty$, the space of measurable functions f on $\mathbb{R}^2_+$ satisfying

$$\|f\|_{L^p_{\mu_k}(\mathbb{R}^2_+)} := \Big(\int_{\mathbb{R}^2_+} |f(x,\lambda)|^p d\mu_k(x,\lambda) \Big)^{1/p} < \infty, \quad 1 \leqslant p < \infty,$$
$$\|f\|_{L^\infty_{\mu_k}(\mathbb{R}^2_+)} := \operatorname*{ess\,sup}_{(x,\lambda)\in\mathbb{R}^2_+} |f(x,\lambda)| < \infty, \quad p = \infty.$$

DEFINITION 2.1. For any function h in $L^2_{\nu_k}(\mathbb{R}_+)$ and any $a \in \mathbb{R}_+$, we define the modulation of h by a as :

$$h_a := \mathscr{F}_k^W\Big(\sqrt{\tau_a^k(|(\mathscr{F}_k^W)^{-1}(h)|^2)}\Big), \tag{2.24}$$

where τ_a^k, $a \in \mathbb{R}_+$, are the Whittaker translation operators.

REMARK 2.3. For h in $L^2_{\nu_k}(\mathbb{R}_+)$, we have

$$\|h_a\|_{L^2_{\nu_k}(\mathbb{R}_+)} = \|h\|_{L^2_{\nu_k}(\mathbb{R}_+)}. \tag{2.25}$$

DEFINITION 2.2. The generalized Whittaker convolution product $f \star_k g$ of two functions f,g in $L^2_{\nu_k}(\mathbb{R}_+)$, is defined by

$$\begin{aligned}\forall\, \xi \in \mathbb{R}_+, \quad f \star_k g(\xi) &:= \int_{\mathbb{R}_+} (\mathscr{F}_k^W)^{-1}(f)(x)(\mathscr{F}_k^W)^{-1}(g)(x)\mathbf{W}_{k,\Delta_\xi}(x) d\gamma_k(x) \\ &= \mathscr{F}_k^W((\mathscr{F}_k^W)^{-1}(f)(\mathscr{F}_k^W)^{-1}(g))(\xi).\end{aligned} \tag{2.26}$$

This convolution is commutative, associative and satisfies the following properties.

PROPOSITION 2.4. *(i) If* $f \in L^2_{\nu_k}(\mathbb{R}_+)$ *and* $g \in L^1_{\nu_k}(\mathbb{R}_+)$, *then* $f *_k g$ *belongs to* $L^2_{\nu_k}(\mathbb{R}_+)$, *moreover*

$$(\mathscr{F}_k^W)^{-1}(f \star_k g) = (\mathscr{F}_k^W)^{-1}(f)(\mathscr{F}_k^W)^{-1}(g). \tag{2.27}$$

(ii) If $f, g \in L^2_{\nu_k}(\mathbb{R}_+)$, *then* $f \star_k g$ *belongs to* $L^2_{\nu_k}(\mathbb{R}_+)$ *if and only if* $(\mathscr{F}_k^W)^{-1}(f)$ $(\mathscr{F}_k^W)^{-1}(g)$ *belongs to* $L^2_{\gamma_k}(\mathbb{R}_+)$, *and*

$$(\mathscr{F}_k^W)^{-1}(f \star_k g) = (\mathscr{F}_k^W)^{-1}(f)(\mathscr{F}_k^W)^{-1}(g) \quad \text{in } L^2_{\nu_k}(\mathbb{R}_+). \tag{2.28}$$

(iii) When $f, g \in L^2_{\nu_k}(\mathbb{R}_+)$, *then*

$$\int_{\mathbb{R}_+} |f *_k g(\xi)|^2 d\nu_k(\xi), = \int_{\mathbb{R}_+} |(\mathscr{F}_k^W)^{-1}(f *_k g)(x)|^2 d\gamma_k(x), \tag{2.29}$$

where both sides are finite or infinite.

DEFINITION 2.3. Let h be in $L^2_{\nu_k}(\mathbb{R}_+)$. For a function f in $L^2_{\nu_k}(\mathbb{R}_+)$, the Whittaker Wigner transform is defined by

$$\mathscr{W}_h^k(f)(y,a) := f \star_k h_a(y), \tag{2.30}$$

where $\star_k$ is the generalized Whittaker convolution product defined by (2.26).

PROPOSITION 2.5. *For* f, h *in* $L^2_{\nu_k}(\mathbb{R}_+)$ *we have*

$$\|\mathscr{W}_h^k(f)\|_{L^\infty_{\mu_k}(\mathbb{R}^2_+)} \leqslant \|f\|_{L^2_{\nu_k}(\mathbb{R}_+)} \|h\|_{L^2_{\nu_k}(\mathbb{R}_+)}. \tag{2.31}$$

PROPOSITION 2.6. (Plancherel's formula) *Let* h *be in* $L^2_{\nu_k}(\mathbb{R}_+)$. *Then, for all* f *in* $L^2_{\nu_k}(\mathbb{R}_+)$, *we have*

$$||\mathscr{W}_h^k(f)||_{L^2_{\mu_k}(\mathbb{R}^2_+)} = \|h\|_{L^2_{\nu_k}(\mathbb{R}_+)} \|f\|_{L^2_{\nu_k}(\mathbb{R}_+)}. \tag{2.32}$$

As in the classical case, the Whittaker Wigner transform preserves the orthogonality relation. However, we have the following result.

COROLLARY 2.1. *Let* h *be in* $L^2_{\nu_k}(\mathbb{R}_+)$. *Then, for all* f, g *in* $L^2_{\nu_k}(\mathbb{R}_+)$, *we have*

$$\int_{\mathbb{R}^2_+} \mathscr{W}_h^k(f)(y,a)\overline{\mathscr{W}_h^k(g)(y,a)} d\mu_k(y,a) = ||h||^2_{L^2_{\nu_k}(\mathbb{R}_+)} \int_{\mathbb{R}_+} f(x)\overline{g(x)} d\nu_k(x). \tag{2.33}$$

We close this subsection by giving the following new results.

PROPOSITION 2.7. *Let f be in $L^2_{\nu_k}(\mathbb{R}_+)$. For all $p \in [2,\infty)$, we have*

$$\|\mathscr{W}_h^k(f)\|_{L^p_{\mu_k}(\mathbb{R}^2)} \leqslant \|f\|_{L^2_{\nu_k}(\mathbb{R}_+)}\|h\|_{L^2_{\nu_k}(\mathbb{R}_+)}. \tag{2.34}$$

Proof. Using Proposition 2.5 and Proposition 2.6, the result follows by applying the Riesz-Thorin interpolation theorem. □

PROPOSITION 2.8. *Let h be in $L^2_{\nu_k}(\mathbb{R}_+)$. Then, $\mathscr{W}_h^k(L^2_{\nu_k}(\mathbb{R}_+))$ is a reproducing kernel Hilbert space in $L^2_{\nu_k}(\mathbb{R}_+)$ with kernel function*

$$\mathscr{K}_h(x',a';x,a) := \frac{1}{\|h\|^2_{L^2_{\nu_k}(\mathbb{R}_+)}}\int_{\mathbb{R}_+} h_{x',a'}(y)\overline{h_{x,a}(y)}d\nu_k(y), \tag{2.35}$$

where for any $t,s \in \mathbb{R}_+$, $h_{t,s}$ is defined by

$$(\mathscr{F}_k^W)^{-1}(h_{t,s})(\xi) := (\mathscr{F}_k^W)^{-1}(h_s)(\xi)\mathbf{W}_{k,\Delta_\xi}(t).$$

Moreover, the kernel is pointwise bounded:

$$|\mathscr{K}_h(x',a';x,a)| \leqslant 1;\ \forall (x',a'),\, (x,a) \in \mathbb{R}^2_+. \tag{2.36}$$

Proof. Let $f \in L^2_{\nu_k}(\mathbb{R}_+)$. We have

$$\mathscr{W}_h^k(f)(x,a) = \int_{\mathbb{R}_+} f(y)\overline{h_{x,a}(y)}d\nu_k(y),\ (x,a) \in \mathbb{R}^2_+.$$

Using Parseval's relation (2.33), we obtain

$$\mathscr{W}_h^k(f)(x,a) = \frac{1}{\|h\|^2_{L^2_{\nu_k}(\mathbb{R}_+)}}\int_{\mathbb{R}^{2d}} \mathscr{W}_h^k(f)(x',a')\overline{\mathscr{W}_h^k(h_{x,a})(x',a')}d\mu_k(x',a').$$

On the other hand, using Proposition 2.4, one can easily see that for every $(x,a),(x',a')$ $\in \mathbb{R}^2_+$ the function

$$x' \mapsto \frac{1}{\|h\|^2_{L^2_{\nu_k}(\mathbb{R}_+)}}\mathscr{W}_h^k(h_{x,a})(x',a') = \frac{1}{\|h\|^2_{L^2_{\nu_k}(\mathbb{R}_+)}}\int_{\mathbb{R}_+} h_{x',a'}(y)\overline{h_{x,a}(y)}d\nu_k(y)$$

belongs to $L^2_{\nu_k}(\mathbb{R}_+)$. Therefore, the result is obtained. □

3. Uncertainty principles for the Whittaker Wigner transform

In this section we will interest to several types of quantitative uncertainty principles.

3.1. Generalized Heisenberg uncertainty principles for $\mathscr{W}_h^k$

In order to prove a concentration result of the Whittaker Wigner transform, we need the following notations:

$P_h : L^2_{\mu_k}(\mathbb{R}^2_+) \to L^2_{\mu_k}(\mathbb{R}^2_+)$ the orthogonal projection from $L^2_{\mu_k}(\mathbb{R}^2_+)$ onto $\mathscr{W}_h^k(L^2_{\nu_k}(\mathbb{R}_+))$.

$P_U : L^2_{\mu_k}(\mathbb{R}^2_+) \to L^2_{\mu_k}(\mathbb{R}^2_+)$ the orthogonal projection from $L^2_{\mu_k}(\mathbb{R}^2_+)$ onto the subspace of function supported in the subset $U \subset \mathbb{R}^2_+$ with

$$0 < \mu_k(U) := \int_U d\mu_k(x,a) < \infty.$$

We put

$$||P_U P_h|| := \sup\Big\{ ||P_U P_h v||_{L^2_{\mu_k}(\mathbb{R}^2_+)} : \ v \in L^2_{\mu_k}(\mathbb{R}^2_+),\ ||v||_{L^2_{\mu_k}(\mathbb{R}^2_+)} = 1 \Big\}.$$

In the following we will prove the concentration of $\mathscr{W}_h^k(f)$ in small sets.

PROPOSITION 3.1. *Let h be in $L^2_{\nu_k}(\mathbb{R}_+)$ and $U \subset \mathbb{R}^2_+$ with*

$$0 < \mu_k(U) < 1.$$

Then, for all $f \in L^2_{\nu_k}(\mathbb{R}_+)$ we have

$$\|\mathscr{W}_h^k(f) - \chi_U \mathscr{W}_h^k(f)\|_{L^2_{\mu_k}(\mathbb{R}^2_+)} \geqslant \sqrt{(1-\mu_k(U))}\|h\|_{L^2_{\nu_k}(\mathbb{R}_+)}\|f\|_{L^2_{\nu_k}(\mathbb{R}_+)}, \tag{3.1}$$

where χ_U denotes the characteristic function of U.

Proof. From Plancherel's formula (2.32) we have

$$||h||^2_{L^2_{\nu_k}(\mathbb{R}_+)}||f||^2_{L^2_{\nu_k}(\mathbb{R}_+)} = \|\mathscr{W}_h^k(f)\|^2_{L^2_{\mu_k}(\mathbb{R}^2_+)} = \|\mathscr{W}_h^k(f)\|^2_{L^2_{\mu_k}(U)} + \|\mathscr{W}_h^k(f)\|^2_{L^2_{\mu_k}(U^c)}. \tag{3.2}$$

On the other hand from the relation (2.31), we have

$$\int_U |\mathscr{W}_h^k(f)(x,a)|^2 d\mu_k(x,a) \leqslant \|\mathscr{W}_h^k(f)\|^2_{L^\infty_{\mu_k}(\mathbb{R}^2_+)}\mu_k(U) \leqslant \mu_k(U)\|f\|^2_{L^2_{\nu_k}(\mathbb{R}_+)}\|h\|^2_{L^2_{\nu_k}(\mathbb{R}_+)}. \tag{3.3}$$

Thus the result follows immediately by integrating (3.2) in (3.3). □

REMARK 3.1. We assume that $0 < \mu_k(U) < 1$. If $\mathscr{W}_h^k(f)$ is supported in U, then $f = 0$.

PROPOSITION 3.2. *Let h be in $L^2_{\nu_k}(\mathbb{R}_+)$ such that $\|h\|_{L^2_{\nu_k}(\mathbb{R}_+)} = 1$. Let $s > 0$. Then the following uncertainty inequalities hold.*
(1) Heisenberg-type uncertainty inequalities for $\mathscr{W}_h^k$:
There exists a positive constant $C_1(k,s)$ such that, for all f in $L^2_{\nu_k}(\mathbb{R}_+)$, we have

$$\Big|\Big| \, ||(x,a)||^s \mathscr{W}_h^k(f) \Big|\Big|_{L^2_{\mu_k}(\mathbb{R}^2_+)} \geqslant C_1(k,s)\|f\|_{L^2_{\nu_k}(\mathbb{R}_+)}. \tag{3.4}$$

(2) Faris Local uncertainty inequality for $\mathscr{W}_h^k$:
There exists a positive constant $C_2(k,s)$ such that, for all f in $L^2_{\nu_k}(\mathbb{R}_+)$, and every subset $U \subset \mathbb{R}^2_+$ such that $0 < \mu_k(U) < \infty$, we have

$$||\mathscr{W}_h^k(f)||_{L^2_{\mu_k}(U)} \leqslant C_2(k,s)\sqrt{\mu_k(U)} \Big|\Big| \, ||(x,a)||^s \mathscr{W}_h^k(f) \Big|\Big|_{L^2_{\mu_k}(\mathbb{R}^2_+)}. \tag{3.5}$$

Proof. (1) Let $r > 0$ such that $0 < \mu_k(B(0,r)) < 1$ where $B(0,r)$ is the open ball of $\mathbb{R}^2_+$ defined by

$$B(0,r) = \Big\{(x,a) \in \mathbb{R}^2_+ : ||(x,a)|| < r\Big\}.$$

By applying the relation (3.1) with $U = B(0,r)$ we obtain

$$\begin{aligned}
&(1-\mu_k(B(0,r)))||h||^2_{L^2_{\nu_k}(\mathbb{R}_+)}||f||^2_{L^2_{\nu_k}(\mathbb{R}_+)} \\
&\leqslant \int_{B(0,r)^c} |\mathscr{W}_h^k(f)(x,a)|^2 d\mu_k(x,a) \\
&\leqslant \frac{1}{r^{2s}} \int_{||(x,a)||\geqslant r} ||(x,a)||^{2s} |\mathscr{W}_h^k(f)(x,a)|^2 d\mu_k(x,a) \\
&\leqslant \frac{1}{r^{2s}} \Big|\Big| \, ||(x,a)||^s \mathscr{W}_h^k(f) \Big|\Big|^2_{L^2_{\mu_k}(\mathbb{R}^2_+)}.
\end{aligned}$$

Thus we obtain the relation (3.4) with $C_1(k,s) := r^s\sqrt{1-\mu_k(B(0,r))}$.
(2) Using the fact that

$$||\mathscr{W}_h^k(f)||_{L^2_{\mu_k}(U)} \leqslant \sqrt{\mu_k(U)}||\mathscr{W}_h^k(f)||_{L^\infty_{\mu_k}(\mathbb{R}^2_+)},$$

and the fact that

$$||\mathscr{W}_h^k(f)||_{L^\infty_{\mu_k}(\mathbb{R}^2_+)} \leqslant ||f||_{L^2_{\nu_k}(\mathbb{R}_+)},$$

then we get

$$||\mathscr{W}_h^k(f)||_{L^2_{\mu_k}(U)} \leqslant \sqrt{\mu_k(U)}||f||_{L^2_{\nu_k}(\mathbb{R}_+)}.$$

Finally, we obtain the result from (3.4). □

We close this subsection by studying the localization of the k-entropy of the Whittaker Wigner transform. Indeed, we begin by the following definition:

Let ρ be a probability density function on $\mathbb{R}^2_+$, such that

$$\int_{\mathbb{R}^2_+} \rho(y,a)d\mu_k(y,a) = 1.$$

Following Shannon [37], the k-entropy of a probability density function ρ on $\mathbb{R}^2_+$ is defined by

$$E_k(\rho) := -\int_{\mathbb{R}^2_+} \ln(\rho(y,a))\rho(y,a)d\mu_k(y,a).$$

Henceforth, we extend the definition of the k-entropy of a nonnegative measurable function ρ on $\mathbb{R}^2_+$ whenever the previous integral on the right hand side is well defined.

The aim of this part is given by the following result.

PROPOSITION 3.3. (Heisenberg's uncertainty principles via the k-Entropy) *For all $f \in L^2_{\nu_k}(\mathbb{R}_+)$, we have*

$$E_k(|\mathscr{W}_h^k(f)|^2) \geqslant -2\|f\|^2_{L^2_{\nu_k}(\mathbb{R}_+)}\|h\|^2_{L^2_{\nu_k}(\mathbb{R}_+)} \ln\left(\|f\|_{L^2_{\nu_k}(\mathbb{R}_+)}\|h\|_{L^2_{\nu_k}(\mathbb{R}_+)}\right). \tag{3.6}$$

Proof. Assume that $\|f\|_{L^2_{\nu_k}(\mathbb{R}_+)}\|h\|_{L^2_{\nu_k}(\mathbb{R}_+)} = 1$. By (2.31),

$$|\mathscr{W}_h^k(f)(y,a)| \leqslant \|f\|_{L^2_{\nu_k}(\mathbb{R}_+)}\|h\|_{L^2_{\nu_k}(\mathbb{R}_+)} = 1. \tag{3.7}$$

In particular $E_k(|\mathscr{W}_h^k(f)|^2) \geqslant 0$. Next, let us drop the above assumption, and let

$$\phi := \frac{f}{\|f\|_{L^2_{\nu_k}(\mathbb{R}_+)}} \quad \text{and} \quad \psi := \frac{h}{\|h\|_{L^2_{\nu_k}(\mathbb{R}_+)}}.$$

Then, $\phi, \psi \in L^2_{\nu_k}(\mathbb{R}_+)$ and $\|\phi\|_{L^2_{\nu_k}(\mathbb{R}_+)}\|\psi\|_{L^2_{\nu_k}(\mathbb{R}_+)} = 1$.

Therefore, $E_k(|\mathscr{W}_\psi^k(\phi)|^2) \geqslant 0$. Moreover,

$$\mathscr{W}_\psi^k(\phi) = \frac{1}{\|f\|_{L^2_{\nu_k}(\mathbb{R}_+)}\|h\|_{L^2_{\nu_k}(\mathbb{R}_+)}}\mathscr{W}_h^k(f),$$

so, we obtain

$$E_k(|\mathscr{W}_\psi^k(\phi)|^2) = \frac{}{\|f\|^2_{L^2_{\nu_k}(\mathbb{R}_+)}\|h\|^2_{L^2_{\nu_k}(\mathbb{R}_+)}}E_k(|\mathscr{W}_h^k(f)|^2) + 2\ln(\|f\|_{L^2_{\nu_k}(\mathbb{R}_+)}\|h\|_{L^2_{\nu_k}(\mathbb{R}_+)}).$$

Using the fact that $E_k(|\mathscr{W}_\psi^k(\phi)|^2) \geqslant 0$, we deduce that

$$E_k(|\mathscr{W}_h^k(f)|^2) \geqslant -2\|f\|^2_{L^2_{\nu_k}(\mathbb{R}_+)}\|h\|^2_{L^2_{\nu_k}(\mathbb{R}_+)} \ln\left(\|f\|_{L^2_{\nu_k}(\mathbb{R}_+)}\|h\|_{L^2_{\nu_k}(\mathbb{R}_+)}\right). \quad \square$$

3.2. Uncertainty principle for orthonormal sequences

In this subsection we will assume that h is a fixed function in $L^2_{\nu_k}(\mathbb{R}_+)$ such that $||h||_{L^2_{\nu_k}(\mathbb{R}_+)} = 1$.

We denote by $B(L^2_{\nu_k}(\mathbb{R}_+))$ the space of bounded operators from $L^2_{\nu_k}(\mathbb{R}_+)$ into itself.

DEFINITION 3.1. (i) The singular values $(s_n(A))_{n\in\mathbb{N}}$ of a compact operator A in $B(L^2_{\nu_k}(\mathbb{R}_+))$ are the eigenvalues of the positive self-adjoint operator $|A| = \sqrt{A^*A}$.

(ii) The Schatten class S_1 is the space of all compact operators whose singular values lie in $l^1(\mathbb{N})$. The space S_1 is equipped with the norm

$$||A||_{S_1} := \sum_{n=1}^{\infty} s_n(A). \tag{3.8}$$

REMARK 3.2. We note that S_2 is the space of Hilbert-Schmidt operators, whereas S_1 is the space of trace class operators.

DEFINITION 3.2. The trace of an operator A in S_1 is defined by

$$tr(A) = \sum_{n=1}^{\infty} \langle Av_n, v_n\rangle_{L^2_{\nu_k}(\mathbb{R}_+)} \tag{3.9}$$

where $(v_n)_n$ is any orthonormal basis of $L^2_{\nu_k}(\mathbb{R}_+)$.

REMARK 3.3. If A is positive, then

$$tr(A) = ||A||_{S_1}. \tag{3.10}$$

Moreover, a compact operator A on the Hilbert space $L^2_{\nu_k}(\mathbb{R}_+)$ is Hilbert-Schmidt, if the positive operator A^*A is in the space of trace class S_1. Then

$$||A||^2_{HS} := ||A||^2_{S_2} = ||A^*A||_{S_1} = tr(A^*A) = \sum_{n=1}^{\infty} ||Av_n||^2_{L^2_{\nu_k}(\mathbb{R}_+)} \tag{3.11}$$

for any orthonormal basis $(v_n)_n$ of $L^2_{\nu_k}(\mathbb{R}_+)$.

DEFINITION 3.3. Let $0 < \varepsilon < 1$ and $U \subset \mathbb{R}^2_+$ be a measurable subset. For $f \in L^2_{\nu_k}(\mathbb{R}_+)$, we say that $\mathscr{W}^k_h(f)$ is ε-concentrated on U if

$$\left\|\mathscr{W}^k_h(f)\right\|_{L^2_{\mu_k}(U^c)} \leqslant \varepsilon \left\|\mathscr{W}^k_h(f)\right\|_{L^2_{\mu_k}(\mathbb{R}^2_+)},$$

where U^c is the complement of U in $\mathbb{R}^2_+$.

PROPOSITION 3.4. *Let $(\varphi_n)_{n\in\mathbb{N}}$ be an orthonormal sequence in $L^2_{\nu_k}(\mathbb{R}_+)$ and U be a measurable subset of $\mathbb{R}^2_+$ such that $\mu_k(U) < \infty$. For every nonempty finite subset $\mathscr{E} \subset \mathbb{N}$, we have*

$$\sum_{n\in\mathscr{E}} \left(1 - \left\|\mathbb{1}_{U^c}\mathscr{W}_h^k(\varphi_n)\right\|_{L^2_{\mu_k}(\mathbb{R}^2_+)}\right) \leqslant \mu_k(U).$$

Proof. Since $(\varphi_n)_{n\in\mathbb{N}}$ is an orthonormal sequence in $L^2_{\nu_k}(\mathbb{R}_+)$, by (2.32) we deduce that $\left(\mathscr{W}_h^k(\varphi_n)\right)_{n\in\mathbb{N}}$ is an orthonormal sequence in $L^2_{\mu_k}(\mathbb{R}^2_+)$. Moreover, since the operator $P_U P_h$ is of Hilbert-Schmidt type, then, by (3.11) and (3.9), it is easy to see that

$$\begin{aligned}\sum_{n\in\mathscr{E}} \langle P_U\mathscr{W}_h^k(\varphi_n), \mathscr{W}_h^k(\varphi_n)\rangle_{L^2_{\mu_k}(\mathbb{R}^2_+)} &= \sum_{n\in\mathscr{E}} \langle P_h P_U P_h\mathscr{W}_h^k(\varphi_n), \mathscr{W}_h^k(\varphi_n)\rangle_{L^2_{\mu_k}(\mathbb{R}^2_+)} \\ &\leqslant \operatorname{tr}(P_h P_U P_h) = \|P_U P_h\|^2_{HS}.\end{aligned}$$

Further, proceeding as in [26], we prove that

$$\|P_U P_h\|_{HS} \leqslant \sqrt{\mu_k(U)}.$$

Thus,

$$\sum_{n\in\mathscr{E}} \langle P_U\mathscr{W}_h^k(\varphi_n), \mathscr{W}_h^k(\varphi_n)\rangle_{L^2_{\mu_k}(\mathbb{R}^2_+)} \leqslant \mu_k(U). \tag{3.12}$$

On the other hand, by Cauchy-Schwarz's inequality we have for every $n \in \mathscr{E}$,

$$\begin{aligned}\langle P_U\mathscr{W}_h^k(\varphi_n), \mathscr{W}_h^k(\varphi_n)\rangle_{L^2_{\mu_k}(\mathbb{R}^2_+)} &= 1 - \langle P_{U^c}\mathscr{W}_h^k(\varphi_n), \mathscr{W}_h^k(\varphi_n)\rangle_{L^2_{\mu_k}(\mathbb{R}^2_+)} \\ &\geqslant 1 - \|\mathbb{1}_{U^c}\mathscr{W}_h^k(\varphi_n)\|_{L^2_{\mu_k}(\mathbb{R}^2_+)}.\end{aligned}$$

In particular, by relation (3.12), we infer

$$\sum_{n\in\mathscr{E}} \left(1 - \|\mathbb{1}_{U^c}\mathscr{W}_h^k(\varphi_n)\|_{L^2_{\mu_k}(\mathbb{R}^2_+)}\right) \leqslant \sum_{n\in\mathscr{E}} \langle P_U\mathscr{W}_h^k(\varphi_n), \mathscr{W}_h^k(\varphi_n)\rangle_{L^2_{\mu_k}(\mathbb{R}^2_+)} \leqslant \mu_k(U). \quad \square$$

Next, we shall use Proposition 3.4 to prove that if the WWT of an orthonormal sequence is ε-concentrated on a given centered ball in $\mathbb{R}^2_+$, then a such sequence is necessary finite.

PROPOSITION 3.5. *Let ε and δ be two positive real numbers such that $0 < \varepsilon < 1$. Let $\mathscr{E} \subset \mathbb{N}$ be a nonempty subset and $(\varphi_n)_{n\in\mathscr{E}}$ be an orthonormal sequence in $L^2_{\nu_k}(\mathbb{R}_+)$. If, for every $n \in \mathscr{E}$, $\mathscr{W}_h^k(\varphi_n)$ is ε-concentrated on the ball $B(0,\delta) := \{(y,\nu) \in \mathbb{R}^2_+ : ||(y,\nu)|| < \delta\}$, then the set $\mathscr{E}$ is finite and*

$$\operatorname{Card}(\mathscr{E}) \leqslant \frac{\mu_k(B(0,\delta))}{1-\varepsilon}. \tag{3.13}$$

Proof. Let $\mathscr{M} \subset \mathscr{E}$ be a nonempty finite subset, then by Proposition 3.4, we deduce that

$$\sum_{n\in\mathscr{M}} \left(1 - \|\mathbb{1}_{B^c(0,\delta)}\mathscr{W}_h^k(\varphi_n)\|_{L^2_{\mu_k}(\mathbb{R}^2_+)}\right) \leqslant \mu_k(B(0,\delta)). \tag{3.14}$$

However, for every $n \in \mathscr{M}$, we have

$$\|\mathbb{1}_{B^c(0,\delta)}\mathscr{W}_h^k(\varphi_n)\|_{L^2_{\mu_k}(\mathbb{R}^2_+)} \leqslant \varepsilon. \tag{3.15}$$

Hence, by combining relations (3.14) and (3.15), we derive that

$$\mathrm{Card}(\mathscr{M}) \leqslant \frac{\mu_k(B(0,\delta))}{1-\varepsilon},$$

which means that $\mathscr{E}$ is finite and satisfies relation (3.13). □

For a positive real number p, the generalized p^{th} time-frequency dispersion of $\mathscr{W}_h^k(f)$ is defined by

$$\rho_p(\mathscr{W}_h^k(f)) = \left(\int_{\mathbb{R}^2_+} ||(y,\nu)||^p \left|\mathscr{W}_h^k(f)(y,\nu)\right|^2 d\mu_k(y,\nu)\right)^{\frac{1}{p}}.$$

COROLLARY 3.1. *Let A and p be two positive real numbers. Let $\mathscr{E} \subset \mathbb{N}$ be a nonempty subset and $(\varphi_n)_{n\in\mathscr{E}}$ be an orthonormal sequence in $L^2_{\nu_k}(\mathbb{R}_+)$. Assume that for every $n \in \mathscr{E}$,*

$$\rho_p(\mathscr{W}_h^k(\varphi_n)) \leqslant A.$$

Then $\mathscr{E}$ is finite and

$$\mathrm{Card}(\mathscr{E}) \leqslant 2\mu_k(B(0,A2^{\frac{2}{p}})).$$

Proof. Since $\rho_p(\mathscr{W}_h^k(\varphi_n)) \leqslant A$ for every $n \in \mathscr{E}$, it follows

$$\int_{B^c(0,A2^{\frac{2}{p}})} |\mathscr{W}_h^k(\varphi_n)(y,\nu)|^2 d\mu_k(y,\nu) \leqslant \frac{1}{\left(A2^{\frac{2}{p}}\right)^p}\rho_p^p(\mathscr{W}_h^k(\varphi_n)) \leqslant \frac{1}{4}. \tag{3.16}$$

The inequality (3.16) means that for every $n \in \mathscr{E}$, $\mathscr{W}_h^k(\varphi_n)$ is $\frac{1}{2}$-concentrated in the ball $B(0,A2^{\frac{2}{p}})$. According to Proposition 3.5, we deduce that $\mathscr{E}$ is finite and

$$\mathrm{Card}(\mathscr{E}) \leqslant 2\mu_k(B(0,A2^{\frac{2}{p}})). \quad \square$$

3.3. L^p-Donoho-Stark's uncertainty principle for the WWT transform

We shall investigate the case where f and $\mathscr{W}_h^k(f)$ are close to zero outside measurable sets. Here the notion of "close to zero" is formulated as follows.

DEFINITION 3.4. Let $0 \leqslant \varepsilon < 1$ and let E be a measurable set of $\mathbb{R}_+$. We say that $f \in L^p_{\nu_k}(\mathbb{R}_+)$, $1 \leqslant p \leqslant 2$, is ε-concentrated on E in $L^p_{\nu_k}(\mathbb{R}_+)$-norm if there is a measurable function g vanishing outside E such that

$$||f-g||_{L^p_{\nu_k}(\mathbb{R}_+)} \leqslant \varepsilon \|f\|_{L^p_{\nu_k}(\mathbb{R}_+)}.$$

REMARK 3.4. Let E be a measurable set of $\mathbb{R}_+$. We introduce a projection operator P_E as

$$P_E f(t) = \begin{cases} f(t), & \text{if} \quad t \in E, \\ 0, & \text{if} \quad t \notin E. \end{cases}$$

Let $0 \leqslant \varepsilon_E < 1$. Then f is ε_E-concentrated on E in $L^p_{\nu_k}(\mathbb{R}_+)$-norm if and only if

$$||f-P_E f||_{L^p_{\nu_k}(\mathbb{R}_+)} \leqslant \varepsilon_E \|f\|_{L^p_{\nu_k}(\mathbb{R}_+)}.$$

DEFINITION 3.5. Let T be a subset of $\mathbb{R}^2_+$ and $h \in L^2_{\nu_k}(\mathbb{R}_+)$. We define a projection operator Q_T as

$$Q_T f = (\mathscr{W}_h^k)^{-1}\Big(P_T(\mathscr{W}_h^k(f))\Big). \tag{3.17}$$

Let $0 \leqslant \varepsilon_T < 1$. We say that $\mathscr{W}_h^k(f)$ is ε_T-concentrated on T in $L^{p'}_{\mu_k}(\mathbb{R}^2_+)$-norm, $1 \leqslant p \leqslant 2$, if and only if

$$\|\mathscr{W}_h^k(f) - \mathscr{W}_h^k(Q_T f)\|_{L^{p'}_{\mu_k}(\mathbb{R}^2_+)} \leqslant \varepsilon_T \|\mathscr{W}_h^k(f)\|_{L^{p'}_{\mu_k}(\mathbb{R}^2_+)}. \tag{3.18}$$

PROPOSITION 3.6. *Let $T \subset \mathbb{R}^2_+$ be a measurable subset, $f \in L^2_{\nu_k}(\mathbb{R}_+)$ and $h \in L^2_{\nu_k}(\mathbb{R}_+)$ be nonzero functions. Then, for every $p > 2$ and $\varepsilon > 0$, if $\mathscr{W}_h^k(f)$ is ε-concentrated in T with respect to the norm $\|\ \|_{L^2_{\mu_k}(\mathbb{R}^2_+)}$, then*

$$\mu_k(T) \geqslant (1-\varepsilon^2)^{\frac{p}{p-2}}. \tag{3.19}$$

Proof. As $\mathscr{W}_h^k(f)$ is ε-concentrated in T with respect to the norm $\|\ \|_{L^2_{\mu_k}(\mathbb{R}^2_+)}$, we have

$$\|\chi_{T^c}\mathscr{W}_h^k(f)\|_{L^2_{\mu_k}(\mathbb{R}^2_+)} \leqslant \varepsilon \|f\|_{L^2_{\nu_k}(\mathbb{R}_+)}\|h\|_{L^2_{\nu_k}(\mathbb{R}_+)}.$$

Moreover using Plancherel's formula (2.32) we deduce that

$$\|\chi_T\mathscr{W}_h^k(f)\|^2_{L^2_{\mu_k}(\mathbb{R}^2_+)} \geqslant \|f\|^2_{L^2_{\nu_k}(\mathbb{R}_+)}\|h\|^2_{L^2_{\nu_k}(\mathbb{R}_+)}(1-\varepsilon^2).$$

Applying Hölder's inequality for the conjugate exponent $\frac{p}{2}$ and $\frac{p}{p-2}$, we get

$$\|\chi_T \mathscr{W}_h^k(f)\|^2_{L^2_{\mu_k}(\mathbb{R}^2_+)} \leqslant \|\mathscr{W}_h^k(f)\|^2_{L^p(\mathbb{R}^2_+)}(\mu_k(T))^{\frac{p-2}{p}}.$$

Now, using Relation (2.34) we obtain

$$\|\chi_T \mathscr{W}_h^k(f)\|^2_{L^2_{\mu_k}(\mathbb{R}^2_+)} \leqslant (\|h\|_{L^2_{\nu_k}(\mathbb{R}_+)}\|f\|_{L^2_{\nu_k}(\mathbb{R}_+)})^2(\mu_k(T))^{\frac{p-2}{p}}.$$

Finally,

$$\mu_k(T) \geqslant (1-\varepsilon^2)^{\frac{p}{p-2}}. \quad \square$$

PROPOSITION 3.7. *We assume that* $0 \neq h \in L^2_{\nu_k}(\mathbb{R}_+)$. *Let* $f \in L^1_{\nu_k}(\mathbb{R}_+) \cap L^2_{\nu_k}(\mathbb{R}_+)$ *such that* $\|\mathscr{W}_h^k(f)\|_{L^2_{\mu_k}(\mathbb{R}^2_+)} = 1$. *If* f *is* ε_E*-concentrated on* E *in* $L^1_{\nu_k}(\mathbb{R}_+)$*-norm and* $\mathscr{W}_h^k(f)$ *is* ε_T*-concentrated on* T *in* $L^2_{\mu_k}(\mathbb{R}^2_+)$*-norm, then*

$$\nu_k(E) \geqslant (1-\varepsilon_E)^2\|h\|^2_{L^2_{\nu_k}(\mathbb{R}_+)}\|f\|^2_{L^1_{\nu_k}(\mathbb{R}_+)} \quad \textit{and} \quad \|h\|^2_{L^2_{\nu_k}(\mathbb{R}_+)}\mu_k(T)\|f\|^2_{L^2_{\nu_k}(\mathbb{R}_+)} \geqslant 1-\varepsilon_T^2.$$

In particular,

$$\nu_k(E)\mu_k(T)\|f\|^2_{L^2_{\nu_k}(\mathbb{R}_+)} \geqslant (1-\varepsilon_E)^2(1-\varepsilon_T^2)\|f\|^2_{L^1_{\nu_k}(\mathbb{R}_+)}.$$

Proof. By the orthogonality of the projection operator P_T, the fact that

$$\|h\|_{L^2_{\nu_k}(\mathbb{R}_+)}\|f\|_{L^2_{\nu_k}(\mathbb{R}_+)} = \|\mathscr{W}_h^k(f)\|_{L^2_{\mu_k}(\mathbb{R}^2_+)} = 1$$

and $\mathscr{W}_h^k(f)$ is ε_T-concentrated on T in L^2-norm, it follows that

$$\|P_T(\mathscr{W}_h^k(f))\|^2_{L^2_{\mu_k}(\mathbb{R}^2_+)} = \|\mathscr{W}_h^k(f)\|^2_{L^2_{\mu_k}(\mathbb{R}^2_+)} - \|\mathscr{W}_h^k(f) - P_T(\mathscr{W}_h^k(f))\|^2_{L^2_{\mu_k}(\mathbb{R}^2_+)} \geqslant 1-\varepsilon_T^2,$$

and thus,

$$\begin{aligned} 1-\varepsilon_T^2 &\leqslant \int_T |\mathscr{W}_h^k(f)(y,\nu)|^2 d\mu_k(y,\nu) \\ &\leqslant \mu_k(T)||\mathscr{W}_h^k(f)||^2_{L^\infty_{\mu_k}(\mathbb{R}^2_+)} \leqslant \|h\|^2_{L^2_{\nu_k}(\mathbb{R}_+)}\mu_k(T)||f||^2_{L^2_{\nu_k}(\mathbb{R}_+)}. \end{aligned}$$

Similarly, f is ε_E-concentrated on E in $L^1_{\nu_k}(\mathbb{R}_+)$-norm,

$$(1-\varepsilon_E)\|f\|_{L^1_{\nu_k}(\mathbb{R}_+)} \leqslant \int_E |f(x)|d\nu_k(x) \leqslant \frac{\sqrt{\nu_k(E)}}{\|h\|_{L^2_{\nu_k}(\mathbb{R}_+)}}.$$

Here we used the Cauchy-Schwarz inequality and the fact that $\|f\|_{L^2_{\nu_k}(\mathbb{R}_+)}\|h\|_{L^2_{\nu_k}(\mathbb{R}_+)} = 1$. $\square$

3.4. Benedicks-type uncertainty principle for $\mathscr{W}_h^k$

We begin this subsection by recalling some concepts necessary for future use.

DEFINITION 3.6. Let U,V be two measurable subsets of $\mathbb{R}_+$. Then:

(1) We say that the pair (U,V) is weakly annihilating, if $supp\, f \subset U$ and $supp\,(\mathscr{F}_k^W(f)) \subset V$ implies $f=0$.

(2) We say that the pair (U,V) is strongly annihilating, if there exists a positive constant $C_k(U,V)$ such that for every function f in $L^2_{\nu_k}(\mathbb{R}_+)$, with $supp\,(\mathscr{F}_k^W(f)) \subset V$, we have

$$||f||^2_{L^2_{\nu_k}(\mathbb{R}_+)} \leqslant C_k(U,V)||f||^2_{L^2_{\gamma_k}(U^c)}. \tag{3.20}$$

Here $A^c := \mathbb{R}_+\backslash A$ is the complement of A. The constant $C_k(U,V)$ will be called the annihilation constant of (U,V).

We proceed as in [4], we prove the following Benedicks-type uncertainty principle for the index Whittaker transform.

PROPOSITION 3.8. *Let U,V be two measurable subsets of $\mathbb{R}_+$ with*

$$\gamma_k(U) := \int_U d\gamma_k(x) < \infty \quad \text{and} \quad \nu_k(V) := \int_V d\nu_k(x) < \infty.$$

Then the pair (U,V) is a strongly annihilating pair.

THEOREM 3.1. *Let h be in $L^2_{\nu_k}(\mathbb{R}_+)$ and $U \subset \mathbb{R}^2_+$ with $0 < \mu_k(U) < \infty$.*

If $P_h(L^2_{\mu_k}(\mathbb{R}^2_+)) \cap P_U(L^2_{\mu_k}(\mathbb{R}^2_+)) = \{0\}$. Then, there exists a positive constant $C := C(h,U)$ such that for all $f \in L^2_{\nu_k}(\mathbb{R}_+)$, we have

$$\|\mathscr{W}_h^k(f) - \chi_U \mathscr{W}_h^k(f)\|_{L^2_{\mu_k}(\mathbb{R}^2_+)} \geqslant C\|f\|_{L^2_{\nu_k}(\mathbb{R}_+)}. \tag{3.21}$$

For the proof of this theorem, we need the following lemma.

LEMMA 3.1. ([45]) *Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two closed subspaces of a Hilbert space $\mathscr{H}$ satisfying $\mathscr{H}_1 \bigcap \mathscr{H}_2 = \{0\}$. Let $P_{\mathscr{H}_1}$ and $P_{\mathscr{H}_2}$ denote the corresponding orthogonal projections, and assume the product $P_{\mathscr{H}_1}P_{\mathscr{H}_2}$ to be a compact operator. Then, there exists a constant $C > 0$ such that for $f \in \mathscr{H}$*

$$||P_{\mathscr{H}_1^\perp} f||_{\mathscr{H}} + ||P_{\mathscr{H}_2^\perp} f||_{\mathscr{H}} \geqslant C||f||_{\mathscr{H}}. \tag{3.22}$$

Proof of Theorem 3.1. Defining $\mathscr{H}_1$ and $\mathscr{H}_2$ by

$$\mathscr{H}_1 := P_U(L^2_{\mu_k}(\mathbb{R}^2_+)), \quad \mathscr{H}_2 := P_h(L^2_{\mu_k}(\mathbb{R}^2_+)).$$

We proceed as in [26], we prove that

$$||P_U P_h||_{HS} := \Big(\int_{\mathbb{R}_+^2 \times \mathbb{R}_+^2} |\chi_U(x,a)|^2 |\mathscr{K}_h(x',a';x,a)|^2 d\mu_k(x',a') d\mu_k(x,a) \Big)^{\frac{1}{2}}$$
$$\leqslant \sqrt{\mu_k(U)} \|h\|_{L^2_{\nu_k}(\mathbb{R}_+)} < \infty.$$

Hence, $P_U P_h$ is a Hilbert-Schmidt operator and, therefore, compact. Now, Lemma 3.1 implies the existence of a constant $C > 0$ such that (3.22) holds for $P_{\mathscr{H}_1} := P_U$ and $P_{\mathscr{H}_2} := P_h$. Since

$$P_{\mathscr{H}_2^\perp}(\mathscr{W}_h^k(f)) = (Id - P_h)\mathscr{W}_h^k(f) = 0,$$

this leads to (3.21). □

DEFINITION 3.7. Let h be in $L^2_{\nu_k}(\mathbb{R}_+)$ and $U \subset \mathbb{R}_+^2$ such that $0 < \mu_k(U) < \infty$.

(1) We say that U is weakly annihilating, if any function $f \in L^2_{\nu_k}(\mathbb{R}_+)$ vanishes when its Whittaker Wigner transform $\mathscr{W}_h^k(f)$ with respect to the window h is supported in U.

(2) We say that U is strongly annihilating, if there exists a constant $C_k(U) > 0$ such that for every function $f \in L^2_{\nu_k}(\mathbb{R}_+)$,

$$C_k(U) \|\mathscr{W}_h^k(f) - \chi_U \mathscr{W}_h^k(f)\|_{L^2_{\mu_k}(\mathbb{R}_+^2)} \geqslant \|f\|_{L^2_{\nu_k}(\mathbb{R}_+)} \|h\|_{L^2_{\nu_k}(\mathbb{R}_+)}. \tag{3.23}$$

The constant $C_k(U)$ will be called the annihilation constant of U.

REMARK 3.5. (1) It is clear that, every strongly annihilating set is also a weakly.

(2) From Proposition 3.1, we see that any set $U \subset \mathbb{R}_+^2$ with $0 < \mu_k(U) < 1$, is strongly annihilating.

(3) As the operator $P_U P_h$ is Hilbert-Schmidt hence is compact, then from [22] we have if U is weakly annihilating, it is also strongly annihilating.

(4) If $||P_U P_h|| < 1$, then for all $f \in L^2_{\nu_k}(\mathbb{R}_+)$

$$\frac{1}{\sqrt{1 - ||P_U P_h||^2}} \|\mathscr{W}_h^k(f) - \chi_U \mathscr{W}_h^k(f)\|_{L^2_{\mu_k}(\mathbb{R}_+^2)} \geqslant \|f\|_{L^2_{\nu_k}(\mathbb{R}_+)} \|h\|_{L^2_{\nu_k}(\mathbb{R}_+)}. \tag{3.24}$$

(5) Following the result established in a general context in [22] p. 88, we have if U is strongly annihilating, then $||P_U P_h|| < 1$.

In the next, we give Benedicks-type uncertainty principle for the Whittaker Wigner transform.

THEOREM 3.2. *Let* $0 \neq h$ *be in* $L^2_{\nu_k}(\mathbb{R}_+)$ *such that*

$$\mu_k(\{(\mathscr{F}_k^W)^{-1}(h) \neq 0\}) < \infty. \tag{3.25}$$

Then for any subset $U \subset \mathbb{R}^2_+$ *such that for almost* $a \in \mathbb{R}_+$,

$$\int_U \chi(x,a)d\nu_k(x) < \infty,$$

we have

$$P_h(L^2_{\mu_k}(\mathbb{R}^2_+)) \cap P_U(L^2_{\mu_k}(\mathbb{R}^2_+)) = \{0\}. \tag{3.26}$$

Proof. Let F be a non-trivial function in $P_h(L^2_{\mu_k}(\mathbb{R}^2_+)) \cap P_U(L^2_{\mu_k}(\mathbb{R}^2_+))$, then there exists a function $f \in L^2_{\nu_k}(\mathbb{R}_+)$ such that $F = \mathscr{W}^k_h(f)$ and $supp F \subset U$.

Let $a \in \mathbb{R}_+$, and let ψ_a be the function defined on $\mathbb{R}_+$ by

$$\psi_a(y) = (\mathscr{F}^W_k)^{-1}(f)(y)\sqrt{\tau^k_a|(\mathscr{F}^W_k)^{-1}(h)|^2(y)}. \tag{3.27}$$

Then for all $(x,a) \in U$

$$F(x,a) = \mathscr{F}^W_k(\psi_a)(x).$$

Thus

$$supp\, \mathscr{F}^W_k(\psi_a) \subset \{x \in \mathbb{R}_+ : (x,a) \in U\}.$$

Moreover, since for almost $a \in \mathbb{R}_+$, $\int_U \chi(x,a)d\nu_k(x) < \infty$, we have $\nu_k(supp\, \mathscr{F}^W_k(\psi_a)) < \infty$. On the other hand, involving (3.27) and the hypothesis (3.25), we derive that $\gamma_k(supp\, \psi_a) < \infty$ Thus, using Proposition 3.8, we deduce that for every $a \in \mathbb{R}_+$, $\psi_a = 0$, which implies that $F = 0$. □

Consequently, we obtain the following improvement.

COROLLARY 3.2. *Let* $0 \neq h$ *be in* $L^2_{\nu_k}(\mathbb{R}_+)$ *such that*

$$\mu_k(\{(\mathscr{F}^W_k)^{-1}(h) \neq 0\}) < \infty.$$

Let $U \subset \mathbb{R}^2_+$ *such that* $\mu_k(U) < \infty$, *then, there exists a positive constant* $C := \mathscr{C}_k(h,U)$ *such that for all* $f \in L^2_{\nu_k}(\mathbb{R}_+)$, *we have*

$$\|\mathscr{W}^k_h(f) - \chi_U \mathscr{W}^k_h(f)\|_{L^2_{\mu_k}(\mathbb{R}^2_+)} \geqslant C\|f\|_{L^2_{\nu_k}(\mathbb{R}_+)}\|h\|_{L^2_{\nu_k}(\mathbb{R}_+)}. \tag{3.28}$$

Now we will derive a sufficient condition by means of which one can recover a signal F belongs to $L^2_{\mu_k}(\mathbb{R}^2_+)$ from the knowledge of a truncated version of it, following the Donoho-Stark criterion [11].

Let h be in $L^2_{\nu_k}(\mathbb{R}_+)$. A signal $F \in L^2_{\mu_k}(\mathbb{R}^2_+)$ is transmitted to a receiver who knows that $F \in \mathscr{W}^k_h(L^2_{\nu_k}(\mathbb{R}_+))$. Suppose that the observation of F is corrupted by a noise $n \in L^2_{\mu_k}(\mathbb{R}^2_+)$ (which is nonetheless assumed to be small) and unregistered values on $U \in \mathbb{R}^2_+$. Thus, the observable function s satisfies

$$s(x,a) = \begin{cases} F(x,a) + n(x,a) & \text{if} \quad (x,a) \in U^c \\ 0 & \text{if} \quad (x,a) \in U. \end{cases} \tag{3.29}$$

Here we have assumed without loss of generality that $n = 0$ on U. Equivalently,

$$s = (Id - P_U)F + n. \tag{3.30}$$

We say that F can be stably reconstructed from s, if there exists a linear operator

$$L_{U,h} : L^2_{\mu_k}(\mathbb{R}^2_+) \to L^2_{\mu_k}(\mathbb{R}^2_+)$$

and a constant $C_k(U,h)$ such that

$$||F - L_{U,h}(s)||_{L^2_{\mu_k}(\mathbb{R}^2_+)} \leqslant C_k(U,h)||n||_{L^2_{\mu_k}(\mathbb{R}^2_+)}. \tag{3.31}$$

THEOREM 3.3. *We assume that the windows h satisfies (3.25), and $U \subset \mathbb{R}^2_+$ such that $\mu_k(U) < \infty$. Then F can be stably reconstructed from s. Moreover, the constant $C_k(U,h)$ in (3.31) is not larger than $(1 - ||P_U P_h||)^{-1}$.*

Proof. We apply the same arguments that used in [11, 19]. From Corollary 3.2, U is strongly annihilating, then from Remark 3.5 we have $||P_U P_h|| < 1$. Therefore $I - P_U P_h$ is invertible. Let

$$L_{U,h} = (Id - P_U P_h)^{-1}.$$

As $F \in \mathscr{W}^k_h(L^2_{\nu_k}(\mathbb{R}_+))$, then $(I - P_U)F = (I - P_U P_g)F$. Thus by simple calculations we see that

$$F - L_{U,h}s = -L_{U,h}n.$$

So that

$$\begin{aligned} ||F - L_{U,h}s||_{L^2_{\mu_k}(\mathbb{R}^2_+)} &= ||L_{U,h}n||_{L^2_{\mu_k}(\mathbb{R}^2_+)} \leqslant ||(Id - P_U P_h)^{-1}||\, ||n||_{L^2_{\mu_k}(\mathbb{R}^2_+)} \\ &\leqslant (1 - ||P_U P_h||)^{-1}\, ||n||_{L^2_{\mu_k}(\mathbb{R}^2_+)}, \end{aligned}$$

which allows to conclude. □

REMARK 3.6. We assume that h is a fixed function in $L^2_{\nu_k}(\mathbb{R}_+)$ such that $||h||_{L^2_{\nu_k}(\mathbb{R}_+)} = 1$.

(1) As $||P_U P_h|| \leqslant ||P_U P_h||_{HS} \leqslant \sqrt{\mu_k(U)}$, we deduce that if $0 < \mu_k(U) < 1$, any F can be stably reconstructed from s and the constant $C(U,h)$ in (3.31) is not larger than $(1 - \sqrt{\mu_k(U)})^{-1}$.

(2) (An algorithm for computing $L_{U,h}s$)
The identity

$$L_{U,h} = (Id - P_U P_h)^{-1} = \sum_{j=0}^{\infty} (P_U P_h)^j$$

suggest an algorithm for computing $L_{U,h}s$. Using the similar method given in [11], we give an algorithm for computing $L_{U,h}s$. Indeed, put

$$F^{(n)} = \sum_{j=0}^{n} (P_U P_h)^j s,$$

then $F^{(n)} \to L_{U,h}(s)$ as $n \to \infty$. Now

$$\begin{aligned} F^{(0)} &= s \\ F^{(1)} &= s + P_U P_h F^{(0)} \\ F^{(2)} &= s + P_U P_h F^{(1)} \\ &\dots \end{aligned} \tag{3.32}$$

and so on. The iteration converges at a geometric rate to the fixed point

$$F = s + P_U P_h F.$$

Algorithms of type (3.32), have been applied to a host of problems in signal recovery see [11], and others.

We close this subsection by investigate the Benedicks-Amrein-Berthier uncertainty principle for the WWT. Indeed, we proceed as in [17], we prove the following version of the Benedicks-Amrein-Berthier uncertainty principle for the Whittaker Fourier transform which states that if E_1 and E_2 are two subsets of $\mathbb{R}_+$ such that $0 < \nu_k(E_1)\gamma_k(E_2) < 1$, then for any $g \in L^2_{\nu_k}(\mathbb{R}_+)$

$$\begin{aligned} &\int_{\mathbb{R}_+} |g(t)|^2 d\nu_k(t) \\ &\leqslant C_k(E_1,E_2)\Big\{ \int_{\mathbb{R}_+\setminus E_1} |g(t)|^2 d\nu_k(t) + \int_{\mathbb{R}_+\setminus E_2} |(\mathscr{F}_k^W)^{-1}(g)(\xi)|^2 d\gamma_k(\xi) \Big\}, \end{aligned} \tag{3.33}$$

where $C_k(E_1,E_2) := (1-\sqrt{\nu_k(E_1)\gamma_k(E_2)})^{-1}$.

In the follow, we interest to establish the Benedick-Amrein-Berthier's uncertainty principle for the WWT by employing the inequality (3.33).

THEOREM 3.4. *Let $f \in L^2_{\nu_k}(\mathbb{R}_+)$, we have*

$$\begin{aligned} &\int_{\mathbb{R}_+\setminus E_1} \int_{\mathbb{R}_+} \left|\mathscr{W}_h^k(f)(y,a)\right|^2 d\mu_k(y,a) \\ &+ ||h||^2_{L^2_{\nu_k}(\mathbb{R}_+)} \int_{\mathbb{R}_+\setminus E_2} |(\mathscr{F}_k^W)^{-1}(f)(\xi)|^2 d\gamma_k(\xi) \\ &\geqslant \frac{||h||^2_{L^2_{\nu_k}(\mathbb{R}_+)} ||f||^2_{L^2_{\nu_k}(\mathbb{R}_+)}}{C_k(E_1,E_2)} \end{aligned} \tag{3.34}$$

where $C_k(E_1,E_2)$ is the constant given in relation (3.33).

Proof. Let $a \in \mathbb{R}_+$, we apply the function $\mathscr{W}_h^k(f)(.,a)$ in (3.33), we obtain

$$\begin{aligned} &\int_{\mathbb{R}_+} \left|\mathscr{W}_h^k(f)(y,a)\right|^2 d\nu_k(y) \\ &\leqslant C_k(E_1,E_2)\Big\{ \int_{\mathbb{R}_+\setminus E_1} \left|\mathscr{W}_h^k(f)(y,a)\right|^2 d\nu_k(y) \\ &\quad + \int_{\mathbb{R}_+\setminus E_2} \left|(\mathscr{F}_k^W)^{-1}\left[\mathscr{W}_h^k(f)(.,a)\right](\xi)\right|^2 d\gamma_k(\xi) \Big\}. \end{aligned} \tag{3.35}$$

By integrating (3.35) with respect to the measure $d\gamma_k(a)$, we obtain

$$\begin{aligned}&\int_{\mathbb{R}_+}\int_{\mathbb{R}_+}\left|\mathscr{W}_h^k(f)(y,a)\right|^2 d\mu_k(y,a)\\ &\leqslant C_k(E_1,E_2)\Big\{\int_{\mathbb{R}_+\setminus E_1}\int_{\mathbb{R}_+}\left|\mathscr{W}_h^k(f)(y,a)\right|^2 d\mu_k(y,a)\\ &\quad+\int_{\mathbb{R}_+\setminus E_2}\int_{\mathbb{R}_+}|(\mathscr{F}_k^W)^{-1}\Big[\mathscr{W}_h^k(f)](y,a)\Big](\xi)\Big|^2 d\gamma_k(a)d\gamma_k(\xi)\Big\}.\end{aligned}$$

Using (2.30) and (2.29), together with Plancherel's formula (2.32), the above inequality becomes

$$\begin{aligned}&\int_{\mathbb{R}_+\setminus E_1}\int_{\mathbb{R}_+}\left|\mathscr{W}_h^k(f)(y,a)\right|^2 d\mu_k(y,a)\\ &+\int_{\mathbb{R}_+\setminus E_2}\int_{\mathbb{R}_+}|(\mathscr{F}_k^W)^{-1}(f)(\xi)\tau_a^k|(\mathscr{F}_k^W)^{-1}(h)|^2(\xi)|^2 d\gamma_k(a)d\gamma_(\xi)\\ &\geqslant\frac{||h||^2_{L^2_{\nu_k}(\mathbb{R}_+)}||f||^2_{L^2_{\nu_k}(\mathbb{R}_+)}}{C_k(E_1,E_2)}\end{aligned}$$

which further implies

$$\begin{aligned}&\int_{\mathbb{R}_+\setminus E_1}\int_{\mathbb{R}_+}\left|\mathscr{W}_h^k(f)(y,a)\right|^2 d\mu_k(y,a)\\ &+\int_{\mathbb{R}_+\setminus E_2}|(\mathscr{F}_k^W)^{-1}(f)(\xi)|^2\Big\{\int_{\mathbb{R}_+}\tau_a^k|(\mathscr{F}_k^W)^{-1}(h)|^2(\xi)d\gamma_k(a)\Big\}d\gamma_k(\xi)\\ &\geqslant\frac{||h||^2_{L^2_{\nu_k}(\mathbb{R}_+)}||f||^2_{L^2_{\nu_k}(\mathbb{R}_+)}}{C_k(E_1,E_2)}.\end{aligned}$$

Thus using the fact that $h\in L^2_{\nu_k}(\mathbb{R}_+)$, (2.23) and (2.13), we get

$$\begin{aligned}&\int_{\mathbb{R}_+\setminus E_1}\int_{\mathbb{R}_+}\left|\mathscr{W}_h^k(f)(y,a)\right|^2 d\mu_k(y,a)\\ &+||h||^2_{L^2_{\nu_k}(\mathbb{R}_+)}\int_{\mathbb{R}_+\setminus E_2}|(\mathscr{F}_k^W)^{-1}(f)(\xi)|^2 d\gamma_k(\xi)\\ &\geqslant\frac{||h||^2_{L^2_{\nu_k}(\mathbb{R}_+)}||f||^2_{L^2_{\nu_k}(\mathbb{R}_+)}}{C_k(E_1,E_2)}\end{aligned}$$

which is the desired Benedick-Amrein-Berthier's uncertainty principle for the Whittaker window transforms. □

4. Conclusion and perspectives

In the present paper, we have successfully studied some of quantitative uncertainty principles for the generalized Wigner transform associated with the index Whittaker transform. Indeed, we examined several quantitative uncertainty principles for the

WWT, such as Heisenberg's type inequalities, Shannon's uncertainty principle, Faris-Price type uncertainty principle and local uncertainty principles. The obtained results have a novelty and contribution to the literature. It is our hope that this work motivate the researchers to study the qualitative uncertainty principles as Hardy's, Morgan's, Beurling's and Miyachi's uncertainty principles associated with the WWT. Finally, we indicate that in the future work, we will study some applications of the theories of the reproducing kernel Hilbert spaces and the time-frequency analysis for the generalized Wigner transform.

Acknowledgements. The author is deeply indebted to the referees for providing constructive comments and helps in improving the contents of this article. The author dedicates this paper to the memory of the Professor Belhassen Kaabi.

Data Availability. Data sharing not applicable to this article as no data sets were generated or analysed during the current study.

Conflict of Interest. The authors declare that they have no conflicts of interest.

REFERENCES

[1] F. A. AL-MUSALLAM, *A Whittaker transform over a half line*, Integral Transforms Spec. Funct. **12** (3) (2001), 201–212.
[2] F. A. AL-MUSALLAM, V. K. TUAN, *A finite and an infinite Whittaker integral transform*, Comput. Math. Appl. **46** (12) (2003), 1847–1859.
[3] A. BENSAL, A. KUMAR, *Heisenberg uncertainty inequality for Gabor transform*, Journal of Mathematical Inequalities **10** (3) (2016), 737–749.
[4] N. BEN SALEM, S. GHOBBER, *On support properties of functions and their Jacobi transform*, Indian J. Pure Appl. Math. **46** (6) (2015), 879–892.
[5] P. A. BECKER, *On the integration of products of Whittaker functions with respect to the second index*, J. Math. Phys. **45** (2) (2004), 761–773.
[6] M. Y. BHAT, A. H. DAR, *Donoho-Stark's and Hardy's uncertainty principles for the short-time quaternion offset linear canonical transform*, Filomat **37**: 14 (2023), 4467–448.
[7] P. BREMAUD, *Mathematical Principles of Signal Processing, Fourier and Wavelet Analysis*, Springer, New York, 2002.
[8] W. CZAJA, G. GIGANTE, *Continuous Gabor transform for strong hypergroups*, J. Fourier Anal. Appl. **9** (2003), 321–339.
[9] A. H. DAR, M. Y. BHAT, *Wigner distribution and associated uncertainty principles in the framework of octonion linear canonical transform*, Optik **272** (2023), 170213, https://doi.org/10.1016/j.ijleo.2022.170213.
[10] L. DEBNATH, D. BHATTA, *Integral Transforms and Their Applications*, Third Edition, Chapman and Hall, CRC Press, Boca Raton, Florida, 2015.
[11] D. L. DONOHO, P. B. STARK, *Uncertainty principles and signal recovery*, SIAM J. Appl. Math. **49** (1989), 906–931.
[12] A. G. FARASHAHI, *Continuous partial Gabor transform for semi-direct product of locally compact groups*, Bull. Malaysian Math. Soc. **38** (2015), 779–803.
[13] A. G. FARASHAHI, R. KAMYABI-GOL, *Continuous Gabor transform for a class of non-Abelian groups*, Bull. Belg. Math. Soc. **19** (4) (2012), 683–701.
[14] H. G. FEICHTINGER, T. STROHMER, *Advances in Gabor Analysis*, Birkhäuser, Boston, 2003.
[15] H. G. FEICHTINGER, W. KOZEK, F. LUEF, *Gabor analysis over finite abelian groups*, Appl. Comput. Harmon. Anal. **26** (2) (2009), 230–248.
[16] D. GABOR, *Theory of communication. part 1: The analysis of information*, Journal of the Institution of Electrical Engineers-Part III: Radio and Communication Engineering **93** (26) (1946), 429–441.

[17] S. GHOBBER, P. JAMING, *Uncertainty principles for integral orperators*, Studia Math., **220** (2014), 197–220.

[18] S. GHOBBER, S. OMRI, *Time-frequency concentration of the windowed Hankel transform*, Integ. Transf. Spec. Funct. **25** (6) (2014), 481–496.

[19] S. GHOBBER, *Dunkl-Gabor transform and time-frequency concentration*, Czechoslov. Math. J. **65** (1) (2015), 255–270.

[20] K. GRÖCHENIG, *Aspects of Wigner analysis on locally compact abelian groups*, Wigner analysis and algorithms, 211–231, Appl. Numer. Harmon. Anal., Birkhäuser Boston, Boston, MA, 1998.

[21] K. GRÖCHENIG, *Foundation of Time–Frequency Analysis*, Birkhäuser, Boston, 2001.

[22] V. HAVIN, B. JÖRICKE, *The Uncertainty Principle in Harmonic Analysis*, Ergebnisse der Mathematik und ihrer Grenzgebiete **3**, Folge. 28, Springer, Berlin, 1994.

[23] J. MAAN, A. PRASAD, *Abelian theorems in the framework of the distributional index Whittaker transform*, Math. Commun. **27** (1) (2022), 1–9.

[24] J. MAAN, A. PRASAD, *Index Whittaker transform for Boehmians*, Indian J. Pure Appl. Math. **55** (2) (2023), https://doi.org/10.1007/s13226-023-00381-7.

[25] E. MALINNIKOVA, *Orthonormal sequences in $L^2(\mathbb{R}^d)$ and time frequency localization*, J. Fourier Anal. Appl. **16** (6) (2010), 983–1006.

[26] H. MEJJAOLI, N. SRAIEB, *Uncertainty Principles for the Continuous Dunkl Gabor Transform and the Dunkl Continuous Wavelet Transform*, Mediterr. J. Math. **5** (2008), 443–466.

[27] H. MEJJAOLI, N. SRAIEB, *Gabor transform in quantum calculus and applications*, Fractional Calculus and Applied Analysis **12** (3) (2009), 319–336.

[28] H. MEJJAOLI, *Practical inversion formulas for the Dunkl-Gabor transform on $\mathbb{R}^d$*, Integ. Transf. and Special Funct. **23** (12) (2012), 875–890.

[29] H. MEJJAOLI, A.-O. AHMED SALEM, *Weinstein Gabor Transform and Applications*, Advances in Pure Mathematics **2** (2012), 203–210.

[30] H. MEJJAOLI, M. JELASSI, Y. OTHMANI, *Multivariable Bessel Gabor transform and applications*, Operators and Matrices **9** (3) (2015), 637–657.

[31] H. MEJJAOLI, *k-Hankel Gabor transform on $\mathbb{R}^d$ and its applications to the reproducing kernel theory*, Complex Anal. Oper. Theory **15** (14) (2021), 1–54.

[32] H. MEJJAOLI, *Time-frequency analysis associated with k-Hankel Gabor transform on $\mathbb{R}^d$*, J. Pseudo-Differ. Oper. Appl. **12** (41) (2021), 1–58, https://doi.org/10.1007/s11868-021-00399-7.

[33] H. MEJJAOLI, *Toeplitz operators associated with the Whittaker Gabor transform and applications*, J. Appl. Anal. (2024), https://doi.org/10.1515/jaa-2024-0091.

[34] H. MEJJAOLI, *Time-frequency analysis associated for the index Whittaker transform and applications*, In revision.

[35] A. PRASAD, J. MAAN, S. K. VERMA, *Wavelet transforms associated with the index Whittaker transform*, Math. Methods Appl. Sci. **43** (13) (2021), 10734–10752.

[36] F. A. SHAH, A. Y. TANTARY, *Wavelet Transforms: Kith and Kin*, Chapman and Hall/CRC, Boca Raton, 2022.

[37] C. E. SHANNON, *A mathematical theory of communication*, ACM SIGMOBILE Mobile Computing and Communications Review **5** (1) (2001), 3–55.

[38] D. SLEPIAN, H. O. POLLAK, *Prolate spheroidal wave functions, Fourier analysis and uncertainty I.*, Bell. System Tech. J. **40** (1961), 43–63.

[39] D. SLEPIAN, *Prolate spheroidal wave functions, Fourier analysis and uncertainty IV: Extensions to many dimensions, generalized prolate spheroidal functions*, Bell. System Tech. J. **43** (1964), 3009–3057.

[40] F. SOLTANI, *Reconstruction and best approximate inversion formulas for the modified Whittaker-Stockwell transform*, To appear in The Ramanujan Journal.

[41] R. SOUSA, M. GUERRA, S. YAKUBOVICH, *Lévy processes with respect to the index Whittaker convolution*, Trans. Amer. Math. Soc. **374** (2021), 2383–2419.

[42] R. SOUSA, M. GUERRA, S. YAKUBOVICH, *On the product formula and convolution associated with the index Whittaker transform*, J. Math. Anal. Appl. **475** (1) (2019), 939–965.

[43] H. M. SRIVASTAVA, Y. U. V. VASILÉV, S. B. YAKUBOVICH, *A class of index transforms with Whittaker's function as the kernel*, Quart. J. Math. Oxford **49** (2) (1998), 375–394.

[44] K. TRIMÈCHE, *Generalized Wavelets and Hypergroups*, Gordon and Breach Science Publishers, 1997.

[45] E. WILCZOK, *New uncertainty principles for the continuous Wigner transform and the continuous Gabor transform*, Doc. Math., J., DMV (electronic) **5** (2000), 201–226.
[46] M. W. WONG, *Wavelet transforms and localization operators*, **136**, Springer Science & Business Media, 2002.
[47] S. B. YAKUBOVICH, *Index transforms*, World Scientific Publishing Company, Singapore, 1996.

(Received April 19, 2024)

Hatem Mejjaoli
Taibah University, College of Sciences
Department of Mathematics
PO BOX 30002 Al Madinah AL Munawarah, Saudi Arabia
e-mail: hmejjaoli@gmail.com