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THE SIGNLESS p–LAPLACIAN SPECTRAL RADIUS OF GRAPHS WITH GIVEN PARAMETER

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Abstract. This paper considers the spectral radius of signless p-Laplacian of a graph, which is a generalization of the quadratic form of the signless Laplacian matrix for $p=2$. Let $\mathscr{G}_{n,k}$, $\mathscr{G}_n^k$ and $\mathscr{G}_n^d$ be the set of connected graphs of order n with k cut vertices, k cut edges and fixed diameter d, respectively. The graphs maximizing the signless p-Laplacian spectral radius among $\mathscr{G}_{n,k}$, $\mathscr{G}_n^k$ and $\mathscr{G}_n^d$ are characterized.

1. Introduction

Throughout this paper, all graphs considered here are simple and connected. Let G be a connected graph with vertex set $V(G)$ and edge set $E(G)$, where $|V(G)|=n$ is the order of G and $|E(G)|=m$ is the size of G. For two vertices $u,v\in V(G)$, denote $u\sim v$ if $uv\in E(G)$. For a vertex $v\in V(G)$, define $N_G(v)=\{u\in V(G)\,|\,u\sim v\}$ to be the *neighborhood* of v, and $d_G(v)=|N_G(v)|$ is called the degree of v. In particular, denote the minimum degree, the maximum degree and the average degree of G by $\delta(G)$, $\Delta(G)$ and $\overline{d}(G)$, respectively. For two vertices $u,v\in V(G)$, the *distance* $d_G(u,v)$ of u and v is the length of a shortest path between u and v. If $x\in V(G)$, then $G-x$ denotes the graph obtained from G by deleting the vertex x and all its incident edges. If $xy\notin E(G)$, then $G+\{xy\}$ is a graph obtained from G by adding an edge xy. For $X\subseteq E(G)$, $G-X$ denotes the graph obtained from G by deleting all the edges in X. For $S\subseteq V(G)$, let $G[S]$ be the induced subgraph by S, and $E(S,V(G)\setminus S)$ be the set of all the edges with one end-vertex in S and the other in $V(G)\setminus S$. A vertex (resp. an edge) in a connected graph G is called a *cut vertex* (resp. *cut edge*) if its removal disconnects G. All other notations not given here are standard in [2], and if there is no danger of ambiguity, we delete the script G in the notations like $d_G(v)$ and write $d(v)=d_G(v)$ unless special statement.

Let A and D be the adjacency matrix and diagonal matrix of G, respectively. The Laplacian matrix L is defined as $L=D-A$, which is a linear operator

$$L:\mathbb{R}^V\to\mathbb{R}^V,\quad (Lf)(v)=\sum_{u\sim v}(f(v)-f(u)).$$

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And the signless Laplacian matrix is defined as $Q = D + A$, which is also a linear operator

$$Q : \mathbb{R}^V \to \mathbb{R}^V, \quad (Qf)(v) = \sum_{u \sim v} (f(v) + f(u)).$$

A natural non-linear generalization of Laplacian operator is p-Laplacian. For $p > 1$, the *p-Laplacian* of G is defined as

$$L_p : \mathbb{R}^V \to \mathbb{R}^V, \quad (L_p f)(v) = \sum_{u \sim v} \text{sign}(f(v) - f(u))|f(v) - f(u)|^{p-1},$$

where $\text{sign}(x)$ is the sign-function of a number x which is 1 if $x > 0$, -1 if $x < 0$ and 0 otherwise. A function f is called an eigenvector of L_p corresponding to the eigenvalue $\mu \in \mathbb{R}$ if it satisfies $(L_p f)(v) = \mu \text{sign}(f(v))|f(v)|^{p-1}$ for each $v \in V$. In the following, we always assume that q is the conjugate exponent of p, i.e., $(p-1)(q-1) = 1$.

Recently, the problem of eigenvalue estimates of p-Laplacian on Riemannian manifolds has been studied by many scholars for its significance in Riemannian geometry and graph theory. Takeuchi [10] considered the spectrum of the p-Laplacian on graphs, and proved a Cheeger type inequality and a Brooks type inequality for infinite graphs. Amghibech [1] presented several sharp upper and lower bounds for the largest p-Laplacian eigenvalues of graphs, respectively. Luo, Huang, Ding and Nie [8] proposed the full eigenvector analysis of p-Laplacian and obtained a natural global embedding for multi-class clustering problems. Wang and Huang [12] deduced an upper bound of the largest p-Laplacian eigenvalue.

Analogously, as a natural generalization of signless Laplacian, for $p > 1$, the *signless p-Laplacian* [1] is defined as the non-linear operator

$$Q_p : \mathbb{R}^V \to \mathbb{R}^V, \quad (Q_p f)(v) = \sum_{u \sim v} \text{sign}(f(u) + f(v))|f(u) + f(v)|^{p-1}.$$

And a function f is called an eigenvector of Q_p corresponding to the eigenvalue λ if $(Q_p f)(v) = \lambda \text{sign}(f(v))|f(v)|^{p-1}$ for each $v \in V$. The energy functional for signless p-Laplacian Q_p is defined to be

$$E_p f = \sum_{uv \in E} |f(u) + f(v)|^p,$$

and the l^p norm of a function f is $||f||_p = (\sum_{v \in V} |f(v)|^p)^{1/p}$. The largest eigenvalue of Q_p is

$$\lambda_p = \sup_{f \neq 0} \frac{E_p f}{||f||_p^p} = \sup_{f \neq 0} \frac{\sum_{uv \in E} |f(u) + f(v)|^p}{\sum_{v \in V} |f(v)|^p}.$$

In [3], Borba and Schwerdtfeger presented a Perron-Frobenius type property for λ_p and deduced several useful inequalities in terms of λ_p.

It is a hot topic to estimate the eigenvalues of graphs in spectral graph theory, and it is crucial to estimate the largest eigenvalue. There are a lot of works on the largest adjacency eigenvalues of graphs, see [9] for details. Wang and Huang [11] obtained the largest signless Laplacian spectral radius of connected graphs with given

diameter or cut vertices. Xue, Lin, Liu and Shu [13] determined the unique graph with maximum A_α-spectral radius among all connected graphs with fixed diameter. Lin, Huang and Xue [6] characterized the extremal graph with maximal A_α-spectral radius among graphs with fixed order and cut vertices. Liu, Lu and Tian [7] investigated the adjacency spectral radius of graphs with given cut edges. Motivated by these works, we consider the largest signless p-Laplacian eigenvalues of connected graphs with given number of cut vertices, cut edges or fixed diameter. We obtain the graphs maximizing the largest signless p-Laplacian eigenvalue λ_p among all connected graphs of order n with k cut vertices or k cut edges or diameter d, respectively.

2. Technical lemmas

In this part, we provide some useful lemmas. As a start, we need the Perron-Frobenius property for λ_p.

LEMMA 2.1. ([3]) *For $p>1$, let G be a connected graph and f be an eigenvector for λ_p. Then $\lambda_p>0$ and f is either strictly positive, i.e., $f(v)>0$ for all $v\in V(G)$, or strictly negative. As usual, the strictly positive one is called the Perron vector of Q_p.*

Moreover, Borba and Schwerdtfeger stated the following properties for the signless p-Laplacian spectral radius.

LEMMA 2.2. ([3]) *If two vertices u and v are not adjacent in a connected graph G, then $\lambda_p(G)<\lambda_p(G+\{uv\})$.*

LEMMA 2.3. ([3]) *Let $p>1$ and G be a connected graph. Then $2^{p-1}\overline{d}(G)\leqslant\lambda_p(G)\leqslant 2^{p-1}\Delta(G)$, where either equality holds if and only if G is regular.*

Finally, we need the following two graph transformations which do not decrease λ_p.

LEMMA 2.4. ([4]) *Let $G=(V,E)$ be a graph with Perron vector f. For a pair of vertices u and v and a non-empty subset $S\subseteq N(u)\setminus N(v)$, let $G'=G-\{us\mid s\in S\}+\{vs\mid s\in S\}$. If $f(u)\leqslant f(v)$, then $\lambda_p(G)<\lambda_p(G')$.*

LEMMA 2.5. ([4]) *Let $G=(V,E)$ be a graph with Perron vector f. For four vertices u_1,u_2,v_1,v_2 with $u_1\not\sim u_2$, $v_1\not\sim v_2$ and $u_i\sim v_i$ for $1\leqslant i\leqslant 2$, let $G'=G-\{u_1v_1,u_2v_2\}+\{u_1u_2,v_1v_2\}$. If $f(u_1)\geqslant f(v_2)$ and $f(u_2)\geqslant f(v_1)$, then $\lambda_p(G')\geqslant\lambda_p(G)$. Furthermore, if one of the two inequalities is strict, then $\lambda_p(G')>\lambda_p(G)$.*

The following result can be deduced from Claims 1-4 of Chen, Feng, Jin and Lu [4].

LEMMA 2.6. ([4]) *For $p>1$, given a connected graph G with $u\in V(G)$ and a path $P_s=x_1x_2\cdots x_{s-1}x_s$, denote by $G_s(u)$ the graph obtained by adding an edge between u and x_s. Assume that $a_i=\frac{f(x_{i+1})}{f(x_i)}$, where $1\leqslant i\leqslant s$ and $u=x_{s+1}$. Let f*

be a Perron vector of $G_s(u)$ corresponding to $\lambda_p(G_s(u))$. If $\lambda_p(G_s(u)) > 2^p$, then the following facts hold

(i) $a_i > \gamma$ for $1 \leqslant i \leqslant s$, where $\gamma > 1$ satisfying that $(1+\gamma)^{p-1} + (1+1/\gamma)^{p-1} = \lambda_p(G_s(u))$.

(ii) $a_{i+1} < a_i$ for $1 \leqslant i \leqslant s-1$.

Let G be a connected graph which is not a tree, $u, v \in V(G)$ with $d_G(u), d_G(v) \geqslant 2$. Assume that u and v are connected by a path $w_0(=v)w_1 \cdots w_{k-1}w_k(=u)$, where $d_G(w_i) = 2$ for $1 \leqslant i \leqslant k-1$. Given two paths $P_s = x_1x_2 \cdots x_{s-1}x_s$, $P_t = y_1y_2 \cdots y_{t-1}y_t$, let $G_{s,k,t}(u,v)$ be the graph obtained from G by joining one end vertex x_s of P_s to u and one end vertex y_t of P_t to v. Especially, notice that the case for $k = 0$ means that u and v are the same vertex and $k = 1$ means that u and v are adjacent. Inspired by Theorem 1.2 of Lin, Huang and Xue in [6], we deduce the following lemma.

LEMMA 2.7. *For $p > 1$, let $G_{s,k,t}(u,v)$ be defined as above. If $s - t \geqslant \max\{k+1, 2\}$, then*

$$\lambda_p(G_{s-1,k,t+1}(u,v)) > \lambda_p(G_{s,k,t}(u,v)).$$

Proof. Let $G_{s,k,t}(u,v)$ be defined as above and $\lambda_p = \lambda_p(G_{s,k,t}(u,v))$. By contradiction, assume that $\lambda_p \geqslant \lambda_p(G_{s-1,k,t+1}(u,v))$. Suppose that f is a Perron vector of $G_{s,k,t}(u,v)$ corresponding to λ_p. In short, denote $v = y_{t+1}$. Thus we have the following claim.

CLAIM 2.1. *$f(y_i) < f(x_{i+1})$ for $1 \leqslant i \leqslant t+1$.*

Proof. We prove this claim by induction. Firstly, it holds that $f(y_1) < f(x_2)$. Since otherwise, if $f(y_1) \geqslant f(x_2)$, then let $G' = G_{s,k,t}(u,v) - \{x_1x_2\} + \{x_1y_1\}$. Obviously, $G' \cong G_{s-1,k,t+1}(u,v)$. But by Lemma 2.4, we have $\lambda_p(G') > \lambda_p$, a contradiction. In the sequel, we assume that $f(y_{i-1}) < f(x_i)$. Now we need to prove that $f(y_i) < f(x_{i+1})$. If not, we have $f(y_i) \geqslant f(x_{i+1})$. Then let $G' = G_{s,k,t}(u,v) - \{x_ix_{i+1}, y_{i-1}y_i\} + \{x_iy_i, x_{i+1}y_{i-1}\}$. It is easy to see that $G' \cong G_{s-1,k,t+1}(u,v)$. Thus by Lemma 2.5, we have $\lambda_p(G') > \lambda_p$, a contradiction. This claim holds. □

In the sequel, we divide the remaining part of the proof into three parts.

Case 1: $k = 0$.

Notice that $N(v) \setminus \{y_t, x_s\} \neq \emptyset$ since $d_G(v) \geqslant 2$. Let $G' = G_{s,k,t}(u,v) - \{wv \mid w \in N(v) \setminus \{y_t, x_s\}\} + \{wx_{t+2} \mid w \in N(v) \setminus \{y_t, x_s\}\}$. Clearly, $G' \cong G_{s-1,k,t+1}(u,v)$. From Claim 2.1, we know that $f(v) = f(y_{t+1}) < f(x_{t+2})$. Thus by Lemma 2.4, we have $\lambda_p(G') > \lambda_p$, a contradiction.

Case 2: $k = s - t - 1$.

Notice that $N(v) \setminus \{y_t, w_1\} \neq \emptyset$ since $d_G(v) \geqslant 2$. Let $G' = G_{s,k,t}(u,v) - \{wv \mid w \in N(v) \setminus \{y_t, w_1\}\} + \{wx_{t+2} \mid w \in N(v) \setminus \{y_t, w_1\}\}$. Clearly, $G' \cong G_{s-1,k,t+1}(u,v)$. From Claim 2.1, we have $f(v) = f(y_{t+1}) < f(x_{t+2})$. Thus by Lemma 2.4, we have $\lambda_p(G') > \lambda_p$, a contradiction.

Case 3: $1 \leqslant k \leqslant s-t-2$.
First we show the following claim.

CLAIM 2.2. *$f(w_i) < f(x_{t+2+i})$ for $0 \leqslant i \leqslant k$.*

Proof. We prove it by induction. Firstly, we have $f(w_0) = f(y_{t+1}) < f(x_{t+2})$ according to Claim 2.1. Next we assume that $f(w_{i-1}) < f(x_{t+1+i})$. Now if $f(w_i) \geqslant f(x_{t+2+i})$, let $G' = G_{s,k,t}(u,v) - \{wv \mid w \in N(v) \setminus \{y_t, w_1\}\} - \{x_{t+2+i}x_{t+1+i}, w_{i-1}w_i\} + \{wx_{t+2} \mid w \in N(v) \setminus \{y_t, w_1\}\} + \{x_{t+1+i}w_i, x_{t+2+i}w_{i-1}\}$. It holds that $G' \cong G_{s-1,k,t+1}(u,v)$. Armed with $f(v) = f(y_{t+1}) < f(x_{t+2})$ in Claim 1, we deduce that $\lambda_p(G') > \lambda_p$ according to Lemmas 2.4 and 2.5, a contradiction. Thus $f(w_i) < f(x_{t+2+i})$ holds. We completes the proof. □

From Claim 2.2, it follows that $f(w_k) < f(x_{t+2+k})$. Moreover, according to the definition of $G_{s,k,t}(u,v)$, we have $\lambda_p > 2^p$ by Lemma 2.2. Thus we have $f(w_k) = f(u) > f(x_s) \geqslant f(x_{t+2+k})$ by Lemma 2.6 and $s \geqslant t+2+k$, a contradiction. The proof is completed. □

Notice that Chen, Feng, Jin and Lu [4] deduced the corresponding result of Lemma 2.7 for the case of $k = 0$.

3. Graphs with cut vertices

In this part, we determine the graph which attains the maximal signless p-Laplacian spectral radius among all graphs with fixed number of cut vertices. Let $\mathscr{G}_{n,k}$ be the set of all connected graphs with n vertices and k cut vertices. By $G_{n,k}$, we denote the graph which is obtained from K_{n-k} by adding pendent paths $P_{l_1}, \cdots, P_{l_{n-k}}$ of almost equal lengths (i.e., $|l_i - l_j| \leqslant 1$ for $1 \leqslant i, j \leqslant n-k$) to all vertices of K_{n-k}. Clearly, $G_{n,k} \in \mathscr{G}_{n,k}$. Next, we will show that $G_{n,k}$ is the maximal graph (graph with maximal signless p-Laplacian spectral radius) in the set $\mathscr{G}_{n,k}$.

THEOREM 3.1. *Among all connected graphs with n vertices and k cut vertices, the maximal signless p-Laplacian spectral radius is uniquely attained at $G_{n,k}$.*

Proof. Assume that G attains the maximal signless p-Laplacian spectral radius among all connected graphs with n vertices and k cut vertices. By Lemma 2.2, we have $\lambda_p(G) < \lambda_p(G+e)$ for $e \notin E(G)$. We deduce that each cut vertex of G connects exactly two blocks and each of these blocks are all cliques. Suppose that all the blocks of G are $K_{a_1}, K_{a_2}, \cdots, K_{a_{k+1}}$. Without loss of generality, we assume that $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_{k+1} \geqslant 2$. Obviously, $0 \leqslant k \leqslant n-2$. When $k = 0$, we have $G \cong K_n \cong G_{n,0}$. When $k = n-2$, we have $G \cong P_n \cong G_{n,n-2}$. When $1 \leqslant k \leqslant n-3$, we have the following claims.

CLAIM 1. *There does not exist a cut vertex of G connecting two cliques K_{a_i} and K_{a_j} with $a_i \geqslant 3$ and $a_j \geqslant 3$.*

Proof. By way of contradiction, assume that v_0 is the cut vertex of G belonging to both K_{a_i} and K_{a_j}. Choose $v_i \in V(K_{a_i})$, $v_j \in V(K_{a_j})$, $v_i, v_j \neq v_0$, without loss of

generality, suppose that $x_{v_i} \geqslant x_{v_j}$. Since $a_i, a_j \geqslant 3$, we have $N_{K_{a_j}}(v_j) \setminus \{v_0\} \neq \emptyset$. Let $G' = G - \{vv_j \mid v \in N_{K_{a_j}}(v_j) \setminus \{v_0\}\} + \{vv_i \mid v \in N_{K_{a_j}}(v_j) \setminus \{v_0\}\}$. Obviously, G and G' have the same number of cut vertices. Thus we have $\lambda_p(G') > \lambda_p(G)$ by Lemma 2.4, a contradiction. □

CLAIM 2. *$a_1 \geqslant 3$ and $a_2 = a_3 = \cdots = a_{k+1} = 2$.*

Proof. Suppose that $a_1 \geqslant a_2 \geqslant 3$. Hence, we have $a_1 = n+k-\sum_{i=2}^{k+1} a_i \leqslant n-k-1$. By Claim 1, we have $\Delta \leqslant a_1 \leqslant n-k-1$. Thus we have $\lambda_p(G) \leqslant 2^{p-1}\Delta \leqslant 2^{p-1}(n-k-1)$ by Lemma 2.3. However, by Lemma 2.2, we have $\lambda_p(G) \geqslant \lambda_p(G_{n,k}) > \lambda_p(K_{n-k}) = 2^{p-1}(n-k-1)$ since $G_{n,k}$ contains K_{n-k} as a proper subgraph, a contradiction. □

According to Claim 2, we know that G is a graph constructed by attaching some pendent paths to a clique K_{n-k}. Then by Lemma 2.7, we obtain that $G \cong G_n^k$. □

THEOREM 3.2. *Let $\tau = 2^{p-1}(n-k-1)$. If $k \leqslant \frac{n}{2}$, then*

$$\lambda_p(G_{n,k}) < \tau + \frac{\tau}{(\tau^{q-1}-1)^{p-1}}.$$

Proof. Since $k \leqslant \frac{n}{2}$, $G_{n,k}$ is the graph with $V(G_{n,k}) = \{v_1, v_2, \cdots, v_n\}$, $G_{n,k}[\{v_1, \cdots, v_{n-k}\}] = K_{n-k}$ and $v_{n-2k+i}v_{n-k+i} \in E(G_{n,k})$ for $i = 1, \cdots, k$. Let $\lambda_p = \lambda_p(G_{n,k})$ and f be the Perron vector corresponding to λ_p. By symmetry, $f(v_1) = \cdots = f(v_{n-2k})$, $f(v_{n-2k+1}) = \cdots = f(v_{n-k})$ and $f(v_{n-k+1}) = \cdots = f(v_n)$. For convenience, let $x = f(v_1) = \cdots = f(v_{n-2k})$, $y = f(v_{n-2k+1}) = \cdots = f(v_{n-k})$ and $z = f(v_{n-k+1}) = \cdots = f(v_n)$, respectively. Hence, from eigenfunctions we have that

$$\begin{cases} \lambda_p x^{p-1} = (n-2k-1)(x+x)^{p-1} + k(x+y)^{p-1}, \\ \lambda_p y^{p-1} = (n-2k)(x+y)^{p-1} + (k-1)(y+y)^{p-1} + (y+z)^{p-1}, \\ \lambda_p z^{p-1} = (y+z)^{p-1}, \end{cases}$$

this leads to

$$\begin{cases} \lambda_p = 2^{p-1}(n-2k-1) + k(1+\frac{y}{x})^{p-1}, \\ \lambda_p = (n-2k)(1+\frac{x}{y})^{p-1} + 2^{p-1}(k-1) + (1+\frac{z}{y})^{p-1}, \\ \lambda_p = (1+\frac{y}{z})^{p-1}. \end{cases}$$

From $(p-1)(q-1) = 1$, the first and the third equations, we have that

$$\begin{cases} \frac{y}{x} = \left(\frac{\lambda_p - 2^{p-1}(n-2k-1)}{k}\right)^{q-1} - 1, \\ \frac{y}{z} = \lambda_p^{q-1} - 1. \end{cases}$$

Combining with the second equation, we obtain

$$\lambda_p = \frac{n-2k}{\left[1-\left(\frac{k}{\lambda_p-2^{p-1}(n-2k-1)}\right)^{q-1}\right]^{p-1}} + 2^{p-1}(k-1) + \frac{1}{\left[1-\left(\frac{1}{\lambda_p}\right)^{q-1}\right]^{p-1}}.$$

Let

$$h(t) = \frac{n-2k}{\left[1-\left(\frac{k}{t-2^{p-1}(n-2k-1)}\right)^{q-1}\right]^{p-1}} + 2^{p-1}(k-1) + \frac{1}{\left[1-\left(\frac{1}{t}\right)^{q-1}\right]^{p-1}}.$$

It is easy to check that $h(t)$ is a monotone decreasing continuous function on $(2^{p-1}(n-2k-1)+k, +\infty)$.

Let $\tau = \lambda_p(K_{n-k}) = 2^{p-1}(n-k-1)$. Obviously, $\tau > 2^{p-1}(n-2k-1)+k$. Moreover, by Lemma 2.2, $\lambda_p > \tau$. Hence,

$$\lambda_p = h(\lambda_p) < h(\tau) = 2^{p-1}(n-2k) + 2^{p-1}(k-1) + \frac{\tau}{(\tau^{q-1}-1)^{p-1}} = \tau + \frac{\tau}{(\tau^{q-1}-1)^{p-1}},$$

and our proof is complete. □

THEOREM 3.3. *Let $n-k$ be a fixed constant and $\tau = 2^{p-1}(n-k-1)$. If $\frac{n}{2} < k \leqslant n-3$, then*

$$\lim_{n\to+\infty} \lambda_p(G_{n,k}) = \tau + \frac{\tau}{(\tau^{q-1}-1)^{p-1}}.$$

Proof. Since $\frac{n}{2} < k \leqslant n-3$, $G_{n,k}$ is the graph obtained from K_{n-k} by unifying each of its vertex v_i with one pendent vertex of the path P_{l_i}, where $|l_i - l_j| \leqslant 1$ and $1 \leqslant i, j \leqslant n-k$. For convenience, write $G_{n,k} = G^{n,k}_{l_1,\cdots,l_{n-k}}$. Since $n-k$ is a constant, the signless p-Laplacian spectral radius of $G_{n,k}$ is an increasing function of order n of $G_{n,k}$ and $\lambda_p = \lambda_p(G_{n,k}) \leqslant 2^{p-1}\Delta(G_{n,k})$ by Lemma 2.3, which implies that $\lim\limits_{n\to+\infty} \lambda_p = \lambda$ exists. Let $l = \max\{l_i \mid 1 \leqslant i \leqslant n-k\}$. Then $\lambda_p(G^{n,k}_{l_1,\cdots,l_{n-k}}) \leqslant \lambda_p(G^{n,k}_{l,\cdots,l})$ since $G^{n,k}_{l_1,\cdots,l_{n-k}}$ is a subgraph of $G^{n,k}_{l,\cdots,l}$. Now we consider the graph $G^{n,k}_{l,\cdots,l}$, where the vertices of P_l are labelled as $u_1, u_2, \cdots, u_l$ such that $u_l = v_i$ for some i. By symmetry, let $f(u_i)$ be the eigen-component of f on vertex u_i for $1 \leqslant i \leqslant l$, where f is the Perron vector of $G^{n,k}_{l,\cdots,l}$ corresponding to $\lambda_p(G^{n,k}_{l,\cdots,l})$. Then, from eigenequations, we have that

$$\begin{cases} \lambda_p(G^{n,k}_{l,\cdots,l}) = \left(1+\frac{f(u_2)}{f(u_1)}\right)^{p-1}, \\ \lambda_p(G^{n,k}_{l,\cdots,l}) = \left(1+\frac{f(u_1)}{f(u_2)}\right)^{p-1} + \left(1+\frac{f(u_3)}{f(u_2)}\right)^{p-1}, \\ \cdots \\ \lambda_p(G^{n,k}_{l,\cdots,l}) = \left(1+\frac{f(u_{l-2})}{f(u_{l-1})}\right)^{p-1} + \left(1+\frac{f(u_l)}{f(u_{l-1})}\right)^{p-1}, \\ \lambda_p(G^{n,k}_{l,\cdots,l}) = 2^{p-1}(n-k-1) + \left(1+\frac{f(u_{l-1})}{f(u_l)}\right)^{p-1}. \end{cases}$$

By Lemma 2.6, for $1 \leqslant j \leqslant l-1$, $\left\{\frac{f(u_{j+1})}{f(u_j)}\right\}$ is a decreasing sequence and $\frac{f(u_{j+1})}{f(u_j)} > \gamma$, where $\gamma > 1$ satisfying that $(1+\gamma)^{p-1}+(1+1/\gamma)^{p-1} = \lambda_p(G^{n,k}_{l,\cdots,l})$. Hence, $\lim\limits_{n\to+\infty} \frac{f(u_{j+1})}{f(u_j)}$ exists. Set $\sigma = \lim\limits_{n\to+\infty} \frac{f(u_{j+1})}{f(u_j)}$. Taking limits on n, we have that

$$\begin{cases} \lambda = (1+1/\sigma)^{p-1}+(1+\sigma)^{p-1}, \\ \lambda = \tau + (1+1/\sigma)^{p-1}, \end{cases}$$

where $\tau = 2^{p-1}(n-k-1)$. Hence, $(1+\sigma)^{p-1} = \tau$. It leads to $\sigma = \tau^{q-1}-1$ as $(p-1)(q-1) = 1$. Therefore, $\lambda = (1+1/\sigma)^{p-1}+(1+\sigma)^{p-1} = \tau + \frac{\tau}{(\tau^{q-1}-1)^{p-1}}$ and our proof is complete. □

Combining Theorems 3.1, 3.2 and 3.3, we have the following result.

THEOREM 3.4. *For any graph $G \in \mathscr{G}_{n,k}$ such that $k \leqslant n-3$ and $\tau = 2^{p-1}(n-k-1)$, we have*

$$\lambda_p(G) < \tau + \frac{\tau}{(\tau^{q-1}-1)^{p-1}}.$$

4. Graphs with cut edges

In this part, we determine the graph which attains the maximal signless p-Laplacian spectral radius among all graphs with fixed number of cut edges. Let $\mathscr{G}^k_n$ be the set of all connected graphs with order n and k cut edges. By G^k_n, we denote the graph obtained from K_{n-k} by adding k independent vertices to one vertex of K_{n-k}. Let $K_{1,k}$ be a star with vertex set $V(K_{1,k}) = \{v_0, v_1, \cdots, v_k\}$, where v_0 is the center of the star. Let $K(a_0, \{a_1, \cdots, a_k\})$ be a graph obtained from $K_{1,k}$ by replacing v_i with clique $K_{a_i} (a_i \geqslant 1, i = 0, 1, \cdots, k)$ and $E[K(a_0, \{a_1, \cdots, a_k\})] = \left(\cup_{i=0}^k E(K_{a_i})\right) \cup \{v_0 v_i \mid i = 1, 2, \cdots, k\}$. Denote

$$\mathscr{K}^k_n = \left\{ K(a_0, \{a_1, \cdots, a_k\}) : a_i \geqslant 1 (0 \leqslant i \leqslant k), \sum_{i=0}^{k} a_i = n \right\}.$$

Obviously, $G^k_n = K(n-k, \{\underbrace{1, \cdots, 1}_{k}\})$.

THEOREM 4.1. *Among all connected graphs with n vertices and k cut edges, G^k_n is the unique graph that attains the maximal signless p-Laplacian spectral radius.*

Proof. Pick $G \in \mathscr{G}^k_n$ such that $\lambda_p(G)$ is as large as possible. Let $E_1 = \{e_1, e_2, \cdots, e_k\}$ be the set of the cut edges of G. By Lemma 2.2, each component of $G - E_1$ is a clique.

If $k = 0$, then $G = K_n$ and the theorem holds naturally. Hence, we may assume that $k \geqslant 1$. Denote the components of $G - E_1$ by $K_{a_0}, K_{a_1}, \cdots, K_{a_k}$, where $a_0, a_1, \cdots, a_k$ are the numbers of the vertices of these components, respectively. Then $a_0 + a_1 + \cdots + a_k = n$.

Let $V_{a_i} = \{v \in V(K_{a_i}) : v \text{ is an end vertex of the cut edges of } G\}$.

In the sequel, we will prove some claims.

CLAIM 1. $|V_{a_i}| = 1$ *for* $0 \leqslant i \leqslant k$.

Proof. By way of contradiction, suppose that $|V_{a_i}| > 1$ for some $i, 0 \leqslant i \leqslant k$. Let $u, u' \in V_{a_i}$ and assume that $f(u) \geqslant f(u')$. Denote $N(u') \setminus N(u) = \{w_1, w_2, \cdots, w_s\}$. Then $s \geqslant 1$ since $u' \in V_{a_i}$. Let $G' = G - \{u'w_1, \cdots, u'w_s\} + \{uw_1, \cdots, uw_s\}$. And $G' \in \mathscr{G}_n^k$. By Lemma 2.4, $\lambda_p(G') > \lambda_p(G)$, a contradiction. Therefore, $|V_{a_i}| = 1$. □

CLAIM 2. $G \in \mathscr{K}_n^k$.

Proof. Suppose that $G \notin \mathscr{K}_n^k$. Then there exist $v \in V(K_{a_i})$ and $v' \in V(K_{a_j}), 0 \leqslant i, j \leqslant k, i \neq j$ such that $|N(v) \setminus V(K_{a_i})| \geqslant 2$ and $|N(v') \setminus V(K_{a_j})| \geqslant 2$. According to Claim 1, we have $\{v\} = V_{a_i}$ and $\{v'\} = V_{a_j}$. Without loss of generality, we may assume that $f(v') \leqslant f(v)$. Denote $N(v') \setminus (V(K_{a_j}) \bigcup \{v\}) = \{z_1, \cdots, z_t\}$. Thus $t \geqslant 1$ by $|N(v') \setminus V(K_{a_j})| \geqslant 2$. Let $G' = G - \{v'z_1, \cdots, v'z_t\} + \{vz_1, \cdots, vz_t\}$. Clearly, $G' \in \mathscr{G}_n^k$. We have $\lambda_p(G') > \lambda_p(G)$ by Lemma 2.4, a contradiction. □

By Claim 1, suppose that $V_{a_i} = \{v_i\}, 0 \leqslant i \leqslant k$. By Claim 2, we may assume that $v_0v_j \in E(G), 1 \leqslant j \leqslant k$. Without loss of generality, we may assume that $a_k \geqslant a_{k-1} \geqslant \cdots \geqslant a_1 \geqslant 1$.

CLAIM 3. $G \cong K\big(a_0, \{\underbrace{1, \cdots, 1}_{k-1}, n - a_0 - k + 1\}\big)$.

Proof. Assume that $a_i > 1$ for some i, $1 \leqslant i \leqslant k-1$. Thus $a_k > 1$. Without loss of generality, assume that $f(v_k) \geqslant f(v_i)$. Denote $N(v_i) \setminus \{v_0\} = \{w_1, \cdots, w_{a_i - 1}\}$. Let $G' = G - \{v_iw_1, \cdots, v_iw_{a_i-1}\} + \{v_kw_1, \cdots, v_kw_{a_i-1}\}$. Clearly, $G' \in \mathscr{G}_n^k$. By Lemma 2.4, we have $\lambda_p(G') > \lambda_p(G)$, a contradiction. □

CLAIM 4. $a_0 = n - k$.

Proof. Obviously, $a_0 = n - (a_1 + a_2 + \cdots + a_k) \leqslant n - k$. Suppose that $a_0 < n - k$. Thus $a_k > 1$ by Claim 3. Denote $N(v_k) \setminus \{v_0\} = \{w_1, \cdots, w_{a_k - 1}\}$ and $N(v_0) \setminus \{v_1, \cdots, v_k\} = \{z_1, \cdots, z_{a_0 - 1}\}$. If $f(v_k) \geqslant f(v_0)$, let

$$G' = G - \{v_0v_1, \cdots, v_0v_{k-1}, v_0z_1, \cdots, v_0z_{a_0-1}\} + \{v_kv_1, \cdots, v_kv_{k-1}, v_kz_1, \cdots, v_kz_{a_0-1}\}.$$

If $f(v_k) \leqslant f(v_0)$, let

$$G' = G - \{v_kw_1, \cdots, v_kw_{a_k-1}\} + \{v_0w_1, \cdots, v_0w_{a_k-1}\}.$$

Then in either case, $G' \in \mathscr{G}_n^k$. By Lemma 2.4, we have $\lambda_p(G') > \lambda_p(G)$, a contradiction. □

Combining Claims 3 and 4, we deduce our result. □

Now we would like to state the following proposition, which will be used in the proof of Theorem 4.2.

PROPOSITION 4.1. *Let G be a star with $n+1$ vertices. Then the only real eigenvalues of G are* $0, 1$ *and* $(1 + n^{q-1})^{p-1}$.

Proof. Denote by $V(G)=\{v,v_1,v_2,\cdots,v_n\}$, where $d_G(v)=n$. Denote by f an eigenvector of G corresponding to the signless p-Laplacian eigenvalue λ. Thus we have

$$\lambda f(v)^{p-1}=\sum_{i=1}^{n}(f(v)+f(v_i))^{p-1}$$

and

$$\lambda f(v_i)^{p-1}=(f(v_i)+f(v))^{p-1}\quad \text{for}\quad i=1,\cdots,n.$$

Since $(p-1)(q-1)=1$, we have

$$(\lambda^{q-1}-1)f(v_i)=f(v)\quad \text{for}\quad i=1,\cdots,n$$

Thus if $\lambda=1$, we have $f(v)=0$. And if f is a vector satisfying that $\sum_{i=1}^{n}f(v_i)^{p-1}=0$ and $f(v)=0$, then f is an eigenvector of G corresponding to the eigenvalue 1.

Now assume $\lambda\neq 0,1$, thus $f(v_i)=(\lambda^{q-1}-1)^{-1}f(v)$, which implies that

$$(f(v)+f(v_i))^{p-1}=\lambda f(v_i)^{p-1}=\lambda(\lambda^{q-1}-1)^{-(p-1)}f(v)^{p-1}.$$

Then we have

$$\lambda f(v)^{p-1}=n\lambda(\lambda^{q-1}-1)^{-(p-1)}f(v)^{p-1},$$

which implies that

$$\lambda^{q-1}=1+n^{q-1}.$$

Besides, for a real number $c\neq 0$, we have f is an eigenvector corresponding to the eigenvalue 0 if f satisfies that $f(v)=c$ and $f(v_i)=-c$ for $i=1,2,\cdots,n$. □

Now we prepare to give the upper bound of $\lambda_p(G_n^k)$.

THEOREM 4.2. *Let G_n^k be defined as above, then we have*

$$\lambda_p(G_n^k)<\begin{cases}\tilde{h}(\tau)=\tau+\frac{k\tau}{(\tau^{q-1}-1)^{p-1}}, & \text{if } 1\leqslant k\leqslant n-1-\left(\frac{1+(n-1)^{q-1}}{2}\right)^{p-1};\\ \tilde{h}(\tau'), & \text{if } n-1-\left(\frac{1+(n-1)^{q-1}}{2}\right)^{p-1}<k\leqslant n-3.\end{cases}$$

where $\tilde{h}(t)=(n-k-1)\frac{t-2^{p-1}(n-k-2)}{\left[(t-2^{p-1}(n-k-2))^{q-1}-1\right]^{p-1}}+\frac{kt}{(t^{q-1}-1)^{p-1}}$, $\tau=2^{p-1}(n-k-1)$ *and* $\tau'=(1+(n-1)^{q-1})^{p-1}$.

Proof. For short, let $\lambda=\lambda_p(G_n^k)$. Let f be the eigenvector corresponding to λ whose entries are labelled as x,y,z at vertices in G_n^k of degree $n-k-1$, $n-1$, 1, respectively. Since G_n^k contains a complete subgraph of order $n-k$ and a star $K_{1,n-1}$, by Lemma 2.2, we have $\lambda>\max\{\tau,\tau'\}$. Here $\tau=\lambda_p(K_{n-k})=2^{p-1}(n-k-1)$, $\tau'=\lambda_p(K_{1,n-1})=(1+(n-1)^{q-1})^{p-1}$ follows from Lemma 2.3 and Proposition 4.1, respectively.

Now we can check that if $1\leqslant k\leqslant n-1-\left(\frac{1+(n-1)^{q-1}}{2}\right)^{p-1}$, then $\tau\geqslant\tau'$; if $n-1-\left(\frac{1+(n-1)^{q-1}}{2}\right)^{p-1}<k\leqslant n-3$, then $\tau<\tau'$.

Moreover, from eigenequations, we have that

$$\begin{cases} \lambda = 2^{p-1}(n-k-2) + \left(1+\frac{y}{x}\right)^{p-1}, \\ \lambda = (n-k-1)\left(1+\frac{x}{y}\right)^{p-1} + k\left(1+\frac{z}{y}\right)^{p-1}, \\ \lambda = \left(1+\frac{y}{z}\right)^{p-1}. \end{cases}$$

Solving these equations, we have that

$$\lambda = (n-k-1)\frac{\lambda - 2^{p-1}(n-k-2)}{[(\lambda - 2^{p-1}(n-k-2))^{q-1}-1]^{p-1}} + \frac{k\lambda}{(\lambda^{q-1}-1)^{p-1}}.$$

It is easy to check that

$$\tilde{h}(t) = (n-k-1)\frac{t - 2^{p-1}(n-k-2)}{[(t - 2^{p-1}(n-k-2))^{q-1}-1]^{p-1}} + \frac{kt}{(t^{q-1}-1)^{p-1}}$$

is an monotone decreasing continuous function on $(2^{p-1}(n-k-2)+1, +\infty)$.

Notice that $\tau > 2^{p-1}(n-k-2)+1$. Hence, $\lambda > \max\{\tau, \tau'\} > 2^{p-1}(n-k-2)+1$. Thus we have

$$\lambda = \tilde{h}(\lambda) < \begin{cases} \tilde{h}(\tau), & \text{if } \tau \geqslant \tau'; \\ \tilde{h}(\tau'), & \text{if } \tau < \tau'. \end{cases} \qquad \square$$

Let $q(G) = \lambda_2(G)$ be the signless Laplacian spectral radius of any graph G. We have the following corollary.

COROLLARY 4.1. *For any connected graph G with order n and k cut edges, we have*

$$q(G) < \begin{cases} 2n-k-2+\frac{k}{2n-2k-3}, & \text{if } 1 \leqslant k \leqslant \frac{n}{2}-1; \\ n-1+\frac{n-k-1}{2k-n+3}+\frac{k}{n-1}, & \text{if } \frac{n}{2}-1 < k \leqslant n-3. \end{cases}$$

5. Graphs with diameter d

In this part, we determine the graph which attains the maximal signless p-Laplacian spectral radius among all graphs with fixed diameter. Let $\mathscr{G}_n^d$ be the family of all connected graphs of order n and diameter d. Given a positive integer a, let $P_s = x_1x_2\cdots x_{s-1}x_s$ and $P_t = y_1y_2\cdots y_{t-1}y_t$ be two paths with $a = s+t$. Let $K_{s,t}^{n-a}$ be a graph obtained from a complete graph K_{n-a} by joining all vertices of K_{n-a} to an end vertex x_s of P_s and an end vertex y_t of P_t. Denote by $\mathscr{K}^* = \{K_{s,t}^{n-d} | s+t = d, s,t \geqslant 1\}$. It is easy to see that $K_{s,t}^{n-d} \in \mathscr{K}^* \subseteq \mathscr{G}_n^d$.

FACT 5.1. Let $G = K_{s,t}^{n-d} \in \mathscr{K}^*$ be the maximal graph. Then $f(y_{i+1}) > f(x_i)$ and $f(x_{i+1}) > f(y_i)$ for $1 \leqslant i \leqslant \min\{s,t\}-1$.

Proof. If $f(x_1) \geqslant f(y_2)$, then $G' = G - \{y_1y_2\} + \{x_1y_1\}$ is a graph in $\mathscr{K}^*$ that has larger signless p-Laplacian spectral radius than G by Lemma 2.4. Hence, $f(y_2) > f(x_1)$. Similarly, $f(x_2) > f(y_1)$.

If $f(x_2) \geqslant f(y_3)$, then $G' = G - \{x_1x_2, y_2y_3\} + \{x_1y_3, x_2y_2\}$ is a graph in $\mathscr{K}^*$ that has larger signless p-Laplacian spectral radius than G by Lemma 2.5. Hence, $f(y_3) > f(x_2)$. Similarly, $f(x_3) > f(y_2)$. The same discussion can be repeatedly applied to show that Fact 5.1 holds for $3 \leqslant i \leqslant \min\{s,t\} - 1$. □

FACT 5.2. Let $G = K_{s,t}^{n-d} \in \mathscr{K}^*$ be the maximal graph. If $f(x_s) > f(y_t)$, then $f(x_{s-1}) > f(y_{t-1})$; if $f(y_t) > f(x_s)$, then $f(y_{t-1}) > f(x_{s-1})$.

Proof. Let $G' = G - \{x_sx_{s-1}, y_ty_{t-1}\} + \{x_sy_{t-1}, y_tx_{s-1}\}$. If $f(x_s) > f(y_t)$ and $f(x_{s-1}) \leqslant f(y_{t-1})$, then by Lemma 2.5, $\lambda_p(G') > \lambda_p(G)$, which is a contradiction. Similarly, if $f(y_t) > f(x_s)$, then $f(y_{t-1}) > f(x_{s-1})$. □

FACT 5.3. Let $G = K_{s,t}^{n-d} \in \mathscr{K}^*$ be the maximal graph. If $f(y_t) > f(x_s)$, then $t \geqslant s$.

Proof. Using a similar method as Claim 3 in [4], we know that $\frac{f(x_{i+1})}{f(x_i)} = \frac{f(y_{i+1})}{f(y_i)}$ for $1 \leqslant i \leqslant \min\{s,t\} - 1$. If $s > t$, then $\frac{f(x_t)}{f(x_{t-1})} = \frac{f(y_t)}{f(y_{t-1})}$. Combining with Lemma 2.6, we have $\frac{f(x_s)}{f(x_i)} > \frac{f(x_t)}{f(x_i)} = \frac{f(y_t)}{f(y_i)}$ for $1 \leqslant i \leqslant \min\{s,t\} - 1 = t - 1$. Therefore, $f(y_i) > f(x_i)$ for $1 \leqslant i \leqslant t - 1$. Together with Fact 5.1, we obtain $f(x_s) > \cdots > f(y_t) > f(x_t) > f(y_{t-1}) > f(x_{t-1}) > f(y_{t-2}) > f(x_{t-2}) > \cdots > f(y_2) > f(x_2) > f(y_1) > f(x_1)$, which contradicts the fact that $f(y_t) > f(x_s)$. □

Now we can propose the following proposition.

PROPOSITION 5.1. *Let $G = K_{s,t}^{n-d} \in \mathscr{K}^*$ be the maximal graph. Then $|s-t| \leqslant 1$.*

Proof. Without loss of generality, assume that $t \geqslant s$. By way of contradiction, suppose that $t \geqslant s+2$. Then by Fact 5.3, we have $f(y_t) \geqslant f(x_s)$.

If $f(y_t) = f(x_s)$, then by $\frac{f(x_s)}{f(x_i)} = \frac{f(y_s)}{f(y_i)} < \frac{f(y_t)}{f(y_i)}$, we have $f(x_i) > f(y_i)$ for $1 \leqslant i \leqslant s$. Together with Fact 5.1, we obtain

$$f(x_s) > f(y_s) > f(x_{s-1}) > f(y_{s-1}) > \cdots > f(x_1) > f(y_1).$$

Moreover, $f(y_{s+1}) \leqslant f(y_{t-1}) < f(y_t) = f(x_s)$. Let

$$G' = G - \{x_sx_{s-1}, y_sy_{s+1}\} + \{x_sy_s, x_{s-1}y_{s+1}\}.$$

By Lemma 2.5, we have $\lambda_p(G') > \lambda_p(G^*)$, which yields a contradiction.

If $f(y_t) > f(x_s)$, then by repeating the same discussion as in the proof of Fact 5.2, we have $f(y_{t-i}) > f(x_{s-i})$ for $1 \leqslant i \leqslant s-1$. Now, if $f(y_{t-i-1}) < f(x_{s-i})$ for any $0 \leqslant i \leqslant s-1$, then

$$f(y_t) > f(x_s) > f(y_{t-1}) > f(x_{s-1}) > \cdots > f(y_{t-s+1}) > f(x_1) > f(y_{t-s}) \geqslant f(y_2),$$

the last inequality holds as $t \geqslant s+2$. Hence $f(x_1) > f(y_2)$, which contradicts to Fact 5.1. Hence, there exists some i such that $f(y_{t-i-1}) \geqslant f(x_{s-i})$. Especially, taking i_0 to be the minimum index such that $f(y_{t-i-1}) \geqslant f(x_{s-i})$. Notice that $i_0 \neq 0$. Otherwise, if $i_0 = 0$, then we have $f(y_{t-1}) \geqslant f(x_s)$. Denote by $V(K_{n-d}) = \{w_1, w_2, \cdots, w_{n-d}\}$, where $n-d \geqslant 2$ since $s,t \geqslant 1$. Let $G' = G - \{x_s w_i \mid 2 \leqslant i \leqslant n-d\} + \{y_{t-1} w_i \mid 2 \leqslant i \leqslant n-d\}$. Clearly, $G' \in \mathscr{K}^*$. According to Lemma 2.4, we have $\lambda_p(G') > \lambda_p(G)$, which derives a contradiction. Thus it holds that $1 \leqslant i_0 \leqslant s-1$. Based on this, we have

$$f(y_{t-i_0+1}) > f(x_{s-i_0+1}) > f(y_{t-i_0}) > f(y_{t-i_0-1}) \geqslant f(x_{s-i_0}).$$

Let $G' = G - \{x_{s-i_0+1}x_{s-i_0}, y_{t-i_0}y_{t-i_0-1}\} + \{x_{s-i_0+1}y_{t-i_0-1}, x_{s-i_0}y_{t-i_0}\}$. By Lemma 2.5, we have $\lambda_p(G') > \lambda_p(G)$, which is a contradiction. □

PROPOSITION 5.2. *Let $2 \leqslant d \leqslant n-2$. If G is the maximal graph in $\mathscr{G}_n^d$, then $G \in \mathscr{K}^*$.*

Proof. Let $P_{d+1} = u_1 u_2 \cdots u_d u_{d+1}$ be the diametrical path of G that connecting u_1 and u_{d+1}. Denote $S = V(G) \setminus V(P_{d+1})$. Obviously, $S \neq \emptyset$ by $n > d+1$. Since G is connected, $E(S, V(P_{d+1})) \neq \emptyset$. For $s \in S$, let $N_{P_{d+1}}(s) = \{u_i \in V(P_{d+1}) \mid su_i \in E(G)\}$. By Lemma 2.2, the set S induces a clique K_{n-d-1} in G.

CLAIM 1. *For any vertex $s \in S$, $N_{P_{d+1}}(s) = \{u_i, u_{i+1}, u_{i+2}\}$ for some $1 \leqslant i \leqslant d-1$.*

Proof. It is easy to see that $|N_{P_{d+1}}(s)| \leqslant 3$, otherwise, P is not the diametrical path. If $|N_{P_{d+1}}(s)| \leqslant 2$, then we may add an edge to G which connecting s and a vertex in P such that the diameter keeps unchanged. However, adding new edges to G increases the signless p-Laplacian spectral radius of G by Lemma 2.2, a contradiction. Thus we have $|N_{P_{d+1}}(s)| = 3$. Let $N_{P_{d+1}}(s) = \{u_i, u_j, u_k\}$. Without loss of generality, assume that $i < j < k$. If $k > i+2$, then P is not the diametrical path. Hence, we have $j = i+1$ and $k = i+2$. So Claim 1 holds. □

CLAIM 2. *For any $s,t \in S$ with $s \neq t$, $N_{P_{d+1}}(s) = N_{P_{d+1}}(t)$.*

Proof. By Claim 1, we set $N_{P_{d+1}}(s) = \{u_i, u_{i+1}, u_{i+2}\}$ and $N_{P_{d+1}}(t) = \{u_j, u_{j+1}, u_{j+2}\}$. By way of contradiction, suppose that $N_{P_{d+1}}(s) \neq N_{P_{d+1}}(t)$, which means that

$$|N_{P_{d+1}}(s) \bigcap N_{P_{d+1}}(t)| \leqslant 2.$$

Notice that $st \in E(G)$ since S induces a clique, and so we have $|N_{P_{d+1}}(s) \bigcap N_{P_{d+1}}(t)| = 2$ (otherwise, it will contradict the fact that the diameter of G is d). Without loss of generality, assume that $N_{P_{d+1}}(t) = \{u_{i+1}, u_{i+2}, u_{i+3}\}$ and $f(u_{i+3}) \geqslant f(u_i)$. Let $G' = G - \{su_i\} + \{su_{i+3}\}$, and $G' \in \mathscr{G}_n^d$. We have $\lambda_p(G') > \lambda_p(G)$ by Lemma 2.4, a contradiction. □

Combining Claims 1 and 2, we deduce the result. □

THEOREM 5.1. *Among all connected graphs of order n and diameter d, the maximal graph is obtained by the following graphs:*

$$\begin{cases} \text{the complete graph } K_n, \text{ if } d = 1, \\ \text{the graph } K^{n-d}_{\lfloor \frac{d}{2} \rfloor, \lceil \frac{d}{2} \rceil}, \qquad \text{if } d \geqslant 2. \end{cases}$$

Proof. When $d = 1$, the complete graph K_n is the unique graph with order n and diameter 1, then the conclusion holds. When $d = n-1$, $P_n \cong K^1_{\lfloor \frac{d}{2} \rfloor, \lceil \frac{d}{2} \rceil}$ is the unique graph with order n and diameter $n-1$, the conclusion holds. When $2 \leqslant d \leqslant n-2$, we have $G \in \mathscr{K}^*$ by Proposition 5.2. According to Proposition 5.1, we deduce that $G \cong K^{n-d}_{\lfloor \frac{d}{2} \rfloor, \lceil \frac{d}{2} \rceil}$. Thus, we obtain our result. □

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