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POSITIVE ANSWER TO THE HYPERINVARIANT SUBSPACE PROBLEM FOR M–HYPONORMAL OPERATORS

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To my daughter Dr. Rofaida

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Abstract. The question whether every operator on infinite dimensional Hilbert space H has a non-trivial invariant subspace or a non-trivial hyperinvariant subspace is one of the most difficult problem in operator theory. This problem is open for more than half a century. A subnormal operator has a non-trivial invariant subspace, but the existence of non-trivial invariant subspace for a hyponormal operator T still open. In this paper, we present a review of the history of the problem and we give an affirmative answer of the existence of a non-trivial hyperinvariant subspace for a m-hyponormal operator S such that $S = T + N$, where T is m-hyponormal and N is normal such that $TN = NT$.

1. Preliminaries

Let $B(H)$ be the algebra of all bounded linear operators acting on an infinite-dimensional separable complex Hilbert space H. Let T be an operator in $B(H)$. Hyponormal operators $(T^*T \geqslant TT^*)$, a natural extension of normal operators, have been studied by many mathematicians. Though there are many unsolved interesting problems related to hyponormal operators, e.g., the invariant subspace problem, recent trends in operator theory have focused on studying natural extensions of hyponormal operators; hence, we introduce non-hyponormal operators. The operator T is said to be M-hyponormal if there exists a real positive number M such that

$$M^2(T-\lambda)^*(T-\lambda) \geqslant (T-\lambda)(T-\lambda)^* \text{ for all } \lambda \in \mathbb{C}.$$

Consequently, if T is hyponormal, then T is M-hyponormal. The following facts follow from the above definition and other well-known facts regarding M-hyponormal operators.

(i) If T is M-hyponormal, then $T - \lambda I$ is M-hyponormal for each $\lambda \in \mathbb{C}$.

(ii) If T is M-hyponormal and $\mathscr{M} \subseteq H$ is a closed T-invariant subspace, then $T|_{\mathscr{M}}$ is M-hyponormal.

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(iii) If T is M-hyponormal, then $N(T-\lambda I) \subseteq N(T-\lambda I)^*$ for every $\lambda \in \mathbb{C}$.

(iv) Every quasinilpotent M-hyponormal operator is a zero operator.

(v) T is M-hyponormal if and only if there exists a positive real number M such that $M||(T-\lambda)x|| \geqslant ||(T-\lambda)^*x||$ for all $x \in H$ and for all $\lambda \in \mathbb{C}$.

Note that an operator T need not be hyponormal even though T and T^* are both M-hyponormal. To see this, consider the operator

$$T = \begin{pmatrix} U & K \\ 0 & U^* \end{pmatrix} : l_2 \oplus l_2 \to l_2 \oplus l_2,$$

where U is the unilateral shift on l_2 and $K : l_2 \to l_2$ is given by $g(x_l, x_2, x_3, \dots) = (2x_1, 0, 0, \dots)$. Direct calculations shows that

$$\frac{1}{2}||(T-z)x|| \leqslant ||(T-z)^*x|| \leqslant 2||(T-z)x||$$

for all $z \in \mathbb{C}$ and for all $x \in l_2 \oplus l_2$; that is, T and T^* are both dominant (M-hyponormal). However, since

$$\begin{pmatrix} I & 0 \\ 0 & I + \frac{3}{2}K \end{pmatrix} = T^*T \neq TT^* = \begin{pmatrix} I + \frac{3}{2}K & 0 \\ 0 & I \end{pmatrix},$$

T is not normal (hyponormal).

2. Main results

DEFINITION 2.1. An operator T is called a spectral operator with spectral measure $E(.)$, if for every Borel sets S,

(1) T commutes $E(S)$, and

(2) the spectrum of $T|_{\mathrm{ran}(E(S))}$ is contained in the closure of S.

The spectral theorem is the assertion that every normal operator T on a Hilbert space H can be expressed as an integral $T = \int_{\sigma(T)} z dE_z$, where E is a spectral measure defined on a Borel subsets of the complex plane $\mathbb{C}$. A well known theorem of Fuglede states that every operator that commutes with a normal operator T commutes with its adjoint T^*. This easily seen to be equivalent to saying that if E is the spectral measure associated with T, then for every Borel subsets of $\mathbb{C}$, the range of $E(S)$ is a hyperinvariant subspace for T.

The concept of a spectral operator is a generalization of a normal operator. Dunford [24] obtained a generalization of Fuglede's theorem to spectral operators.

THEOREM 2.1. (Fuglede Dunford theorem) *If T is a spectral operator with spectral measure $E(.)$, then for each Borel set S, the subspace $ran(S)$ is hyperinvariant for T.*

This means that every operator that commutes with T commutes with all projections $E(S)$.

Recall [34] that if T is M-hyponormal, then $T+N$ is M-hyponormal, where N is a normal operator which commutes with T.

A spectral operator S is M-hyponormal if and only if $S = T + N$, where T is M-hyponormal and N is normal [34].

THEOREM 2.2. *Assume that S is not multiple of the identity. If S is M-hyponormal such that $S = T + N$, where T is M - hyponormal and N is normal such that $TN = NT$, then S has a non-trivial hyperinvariant subspace.*

Proof. By [34], $S = T + N$ is a spectral operator. Let A be an operator which commutes with S, then A commutes with the resolution of the identity of S. Therefore, A commutes with the spectral projection of T, Hence if T is not a multiple of the identity, then T has spectral projections distinct from 0 and I and the ranges of the projections are hyperinvariant subspaces for S. If T is scalar a subspace M of H will be hyperinvariant just in case it is hyperinvariant for N, but S is non scalar, so N can not be the zero operator. on the other hand, N is normal, if E is the spectral measure associated with N, then for every Borel subsets of $\mathbb{C}$, the range of $E(N)$ is a hyperinvariant subspace for N and hence for S. Since $S = T + N$ where T is M-hyponormal and N is normal, Hence S is M-hyponormal by [34, Theorem 9]. Therefore an M-hyponormal operator has a hyperinvariant subspace. $\square$

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