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THE PROPERTIES OF THE PRODUCT OF TWO ORTHOGONAL PROJECTIONS

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Abstract. In this paper, we use the block operator technique and Halmos' CS decomposition to study some necessary and sufficient conditions for an operator on a complex separable Hilbert space to be written as a product of two orthogonal projections.

1. Introduction

Throughout this paper, a complex separable Hilbert space is denoted by $\mathscr{H}$. The set of all linear operators from $\mathscr{H}_1$ to $\mathscr{H}_2$ is denoted by $\mathscr{L}(\mathscr{H}_1,\mathscr{H}_2)$. In particular, the subclass of closed operators in $\mathscr{L}(\mathscr{H}_1,\mathscr{H}_2)$ is denoted by $\mathscr{C}(\mathscr{H}_1,\mathscr{H}_2)$, and the subclass of bounded operators is denoted by $\mathscr{B}(\mathscr{H}_1,\mathscr{H}_2)$. To illustrate the main results, we need some notation and terminology. Given $B\in\mathscr{L}(\mathscr{H})$, $\mathscr{D}(B)$, $\mathscr{R}(B)$, $\mathscr{N}(B)$, $\sigma(B)$, and B^* denote the domain, the range, the null space, the spectrum, and the adjoint of B, respectively. Let B_1, $B_2\in\mathscr{L}(\mathscr{H}_1,\mathscr{H}_2)$ with $\mathscr{D}(B_1)\subset\mathscr{D}(B_2)$. If $B_1x=B_2x$ for all $x\in\mathscr{D}(B_1)$, then B_2 is called an extension of B_1 and B_1 is called a restriction of B_2. These relations are denoted by $B_1=B_2|_{\mathscr{D}(B_1)}$. Recall that if B is a closed linear operator and A is a bounded linear operator such that $\mathscr{R}(A)\subset\mathscr{D}(B)$, then the well-defined linear operator $B\cdot A$ is bounded (see [10, Chapter X]). Now a densely defined closed linear operator E is said to be idempotent if $\mathscr{R}(E)\subset\mathscr{D}(E)$ and $E\cdot E=E$. Unbounded idempotent operators were first considered by Ota [21]. If $P\in\mathscr{B}(\mathscr{H})$ satisfies $P^2=P$ and $\mathscr{N}(P)=\mathscr{R}(P)^\perp$, then P is called an orthogonal projection. Given any $B\in\mathscr{B}(\mathscr{H})$, $|B|:=(B^*B)^{\frac{1}{2}}$ is the positive part of B and $B=U|B|=|B^*|U$ is the polar decomposition of B, with U the partial isometry such that $\mathscr{N}(U)=\mathscr{N}(B)$. If $\mathscr{M}$ is a closed subspace of $\mathscr{H}$, then its orthogonal complement is denoted by $\mathscr{M}^\perp$. We use $P_{\mathscr{M}}$ to denote the orthogonal projection of $\mathscr{H}$ onto $\mathscr{M}$. The main goal of this paper is the research of the set

$$\mathfrak{X}=\{PQ:P,Q\in\mathscr{B}(\mathscr{H}),P=P^2=P^*,Q=Q^2=Q^*\}.$$

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We give some properties of this set in terms of CS decomposition, Tseng inverses, group inverses and core inverses. If $\mathscr{M}$ and $\mathscr{N}$ are subspaces of $\mathscr{H}$ and $\mathscr{M}\cap\mathscr{N}=\{0\}$, then the direct sum of $\mathscr{M}$ and $\mathscr{N}$ is denoted by $\mathscr{M}\dotplus\mathscr{N}$. In particular, if $\mathscr{M}\subset\mathscr{N}^{\perp}$, we denote the orthogonal sum by $\mathscr{M}\oplus\mathscr{N}$.

Let $B\in\mathscr{B}(\mathscr{H})$. If there exists $B^D\in\mathscr{B}(\mathscr{H})$ such that

$$BB^D=B^DB,\ \ B^DBB^D=B^D,\ \ B^{k+1}B^D=B^k$$

for some non-negative integer k, then B^D is called the Drazin inverse of B, and the smallest non-negative integer k satisfying the above equations is called the Drazin index of B, abbreviated as $\mathrm{ind}(B)$ (see [4]). $\mathscr{B}^1(\mathscr{H})$ stands for the set of bounded linear operators on $\mathscr{H}$ with indices less or equal to one,

$$\mathscr{B}^1(\mathscr{H})=\{B\in\mathscr{B}(\mathscr{H}):\mathrm{ind}(B)\leqslant 1\}.$$

If there exists $S\in\mathscr{B}(\mathscr{H})$ satisfying

$$BS=SB,\ \ SBS=S,\ \ BSB=B,$$

then S is called the group inverse of B. The group inverse of B exists if and only if $\mathrm{ind}(B)\leqslant 1$. We denote the group inverse of B by $B^{\sharp}$. If there exists $B^{\circledast}\in\mathscr{B}(\mathscr{H})$ satisfying the following three operator equations:

$$BB^{\circledast}B=B,\ \ \mathscr{R}(B^{\circledast})=\mathscr{R}(B),\ \ \mathscr{N}(B^{\circledast})=\mathscr{N}(B^*),$$

then $B^{\circledast}$ is called the core inverse of B (see [23]).

Let $B\in\mathscr{B}(\mathscr{H}_1,\mathscr{H}_2)$. Recall [4] that an operator $B^g\in\mathscr{L}(\mathscr{H}_2,\mathscr{H}_1)$ is called a Tseng inverse of B if $\mathscr{R}(B)\subset\mathscr{D}(B^g)$ and

$$B^gBx=P_{\overline{\mathscr{R}(B^g)}}x,\quad x\in\mathscr{H}_1,$$
$$BB^gy=P_{\overline{\mathscr{R}(B)}}y,\quad y\in\mathscr{D}(B^g).$$

We denote the maximal Tseng inverse of B by $B^{\dagger}$, which satisfies

$$\mathscr{D}(B^{\dagger})=\mathscr{R}(B)\oplus\mathscr{R}(B)^{\perp},$$

$$\mathscr{N}(B^{\dagger})=\mathscr{R}(B)^{\perp}.$$

For bounded operators with closed range, it is easy to see that the maximal Tseng inverse coincides with the Moore-Penrose inverse, we thus denote the Moore-Penrose inverse of B by $B^{\dagger}$ as well. In such case, $B^{\dagger}$ is bounded and

$$BB^{\dagger}B=B,\ B^{\dagger}BB^{\dagger}=B^{\dagger},\ (BB^{\dagger})^*=BB^{\dagger},\ (B^{\dagger}B)^*=B^{\dagger}B.$$

There is an extensive bibliography on pairs of projections (see [5–9,15,24]). Crimmins revealed that if B is the product of two orthogonal projections, then B has the canonical factorization $B=P_{\overline{\mathscr{R}(B)}}P_{\mathscr{N}(B)^{\perp}}$. Radjavi and Willians [22] gave a proof of

this theorem by Crimmins which implies that $B \in \mathfrak{X}$ if and only if $B^2 = BB^*B$. Corach and Maestripieri concluded in [11] that this is also equivalent to $\|Bx\|^2 = \langle Bx, x\rangle$, for all $x \in \mathscr{N}(B)^\perp$. Böttcher and Spitkovsky (see [6,7]) characterized the Drazin invertibility, Moore-Penrose invertibility, and core invertibility of operators in the von Neumann algebra generated by two orthogonal projections and provided explicit representations for the corresponding inverses. Besides, Corach et al. (see [1,2]) generalized the products of two orthogonal projections to the forms of the products of projections and positive operators as well as the products of idempotent operators. In [2], the equivalent conditions for bounded linear operator B belonging to the set $\mathfrak{Y} = \{EF : E^2 = E, F^2 = F\}$ were given, and found that $B \in \mathfrak{Y}$ if and only if there exists an idempotent operator C such that $\mathscr{R}(C) = \overline{\mathscr{R}(B)}$ and $\mathscr{R}(B - B^2) \subset \mathscr{R}(B(I - C))$. The results proved in [26] show that the set of all operators which are less than the identity operator under the diamond order $\leqslant^\diamond$ is precisely the set of all products of two orthogonal projections, where $A \leqslant^\diamond B$ iff $\overline{\mathscr{R}(A)} \subseteq \overline{\mathscr{R}(B)}$, $\overline{\mathscr{R}(A^*)} \subseteq \overline{\mathscr{R}(B^*)}$ and $AA^*A = AB^*A$ for $A, B \in \mathscr{B}(\mathscr{H})$.

Inspired by the result $B \in \mathfrak{X}$ if and only if $B^2 = BB^*B$, we are interested in other equivalent conditions under which B can be expressible as the product of two orthogonal projections. In this paper, the general explicit descriptions for all operators in $\mathfrak{X}$ are established. For an operator $B \in \mathfrak{X}$, based on the polar decomposition of B, the operator-matrix representation of the maximal Tseng inverse of B is obtained. Furthermore, some important equivalent conditions for $B \in \mathfrak{X}$ ($\mathscr{R}(B)$ is not necessarily closed) are presented. It is shown that $B \in \mathfrak{X}$ if and only if $B^* \cdot B^\dagger = B^*|_{\mathscr{D}(B^\dagger)}$ if and only if $|B^*|^\dagger B^* = |B^*|$, also, for all $x \in \mathscr{R}(B)$, this is the case if and only if $\langle B^\dagger x, x\rangle = \|x\|^2$. Finally, we characterize the block operator-matrix representations of the Drazin inverse, group inverse and core inverse of B respectively, and some other equivalent conditions for $B \in \mathfrak{X}$ are given.

2. Preliminaries

In this section, we begin with some lemmas which play important roles in the subsequent main results.

LEMMA 2.1. (see [4,20]) *Let $B \in \mathscr{C}(\mathscr{H}_1, \mathscr{H}_2)$ be dense. Then:*

(1) $B^\dagger \in \mathscr{C}(\mathscr{H}_2, \mathscr{H}_1)$;

(2) $\mathscr{N}(B^\dagger) = \mathscr{N}(B^*)$;

(3) $\mathscr{R}(B^\dagger) = \mathscr{D}(B) \cap \mathscr{N}(B)^\perp$;

(4) $\mathscr{N}(B^{*\dagger}) = \mathscr{N}(B)$, $B^{*\dagger} = B^{\dagger *}$;

(5) $(B^*B)^\dagger = B^\dagger B^{*\dagger}$, $(BB^*)^\dagger = B^{*\dagger} B^\dagger$.

LEMMA 2.2. (see [5,19]) *Let P and Q be two orthogonal projections in $\mathscr{B}(\mathscr{H})$. Denote $\mathscr{H}_1 = \mathscr{R}(P) \cap \mathscr{R}(Q)$, $\mathscr{H}_2 = \mathscr{R}(P) \cap \mathscr{R}(Q)^\perp$, $\mathscr{H}_3 = \mathscr{R}(P)^\perp \cap \mathscr{R}(Q)$, $\mathscr{H}_4 = \mathscr{R}(P)^\perp \cap \mathscr{R}(Q)^\perp$, $\mathscr{H}_5 = \mathscr{R}(P) \ominus (\mathscr{H}_1 \oplus \mathscr{H}_2)$, $\mathscr{H}_6 = \mathscr{R}(P)^\perp \ominus (\mathscr{H}_3 \oplus \mathscr{H}_4)$. Then*

$$\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2 \oplus \mathscr{H}_3 \oplus \mathscr{H}_4 \oplus \mathscr{H}_5 \oplus \mathscr{H}_6, \tag{2.1}$$

and with respect to this space decomposition, P and Q have the operator matrices

$$\begin{aligned} P &= I_1 \oplus I_2 \oplus 0 \oplus 0 \oplus \begin{pmatrix} I_5 & 0 \\ 0 & 0 \end{pmatrix}, \\ Q &= I_1 \oplus 0 \oplus I_3 \oplus 0 \oplus \begin{pmatrix} C^2 & CSR \\ R^*CS & R^*S^2R \end{pmatrix}, \end{aligned} \tag{2.2}$$

where I_i is the identity operator on the corresponding subspace $\mathscr{H}_i$, $i = 1,2,3,5$. R is an isometric isomorphism from $\mathscr{H}_6$ onto $\mathscr{H}_5$. S and C are two self-adjoint operators on $\mathscr{H}_5$ such that $0 \leqslant S \leqslant I_5$, $0 \leqslant C \leqslant I_5$, $S^2 + C^2 = I_5$ and $\mathscr{N}(S) = \mathscr{N}(C) = \{0\}$.

LEMMA 2.3. (see [17,18]) *Let $B \in \mathscr{B}(\mathscr{H},\mathscr{K})$ and $S \in \mathscr{B}(\mathscr{K},\mathscr{H})$ be two closed range operators. If $\mathscr{R}(S) = \mathscr{R}(B^*)$, then BS is Moore-Penrose invertible and $(BS)^\dagger = S^\dagger B^\dagger$.*

LEMMA 2.4. (see [4,18]) *Let $B \in \mathscr{B}(\mathscr{H})$ be Drazin invertible, $k = ind(B)$. Then $\mathscr{R}(BB^D) = \mathscr{R}(B^D) = \mathscr{R}(B^k)$, $\mathscr{N}(BB^D) = \mathscr{N}(B^D) = \mathscr{N}(B^k)$. If $ind(B) \leqslant 1$, then B is group invertible with $\mathscr{R}(B^\sharp) = \mathscr{R}(B)$ and $\mathscr{N}(B^\sharp) = \mathscr{N}(B)$.*

LEMMA 2.5. (see [23]) *Let $B \in \mathscr{B}^1(\mathscr{H})$. Then:*

(i) $B^\circledast = B^\sharp BB^\dagger$;

(ii) $B^\circledast B = B^\sharp B$;

(iii) $BB^\circledast = P_{\mathscr{R}(B)}$;

(iv) $(B^\circledast)^\dagger = (B^\circledast)^\sharp = (B^\circledast)^\circledast = BP_{\mathscr{R}(B)}$.

3. Main result

Recall that [11] if $B \in \mathfrak{X}$, then $B^\dagger$ is a densely defined close projection with $\mathscr{D}(B^\dagger) = \mathscr{R}(B) \oplus \mathscr{R}(B)^\perp$, and $B^\dagger$ is bounded if and only if $\mathscr{R}(B)$ is closed. In what follows, we first present the block operator-matrix representation of $B^\dagger$, which is of remarkable importance in characterizing $\mathfrak{X}$. Here the CS decomposition (Halmos' two projections theory) which is given by Lemma 2.2 is the cornerstone of our research. In this paper, we assume one of $\mathscr{H}_5$ and $\mathscr{H}_6$ is nontrivial.

PROPOSITION 3.1. *Let $B \in \mathfrak{X}$. Then with respect to the space decomposition* (2.1)*,*

$$B = I_1 \oplus 0 \oplus 0 \oplus 0 \oplus \begin{pmatrix} C^2 & CSR \\ 0 & 0 \end{pmatrix} \tag{3.1}$$

and

$$B^\dagger = I_1 \oplus 0 \oplus 0 \oplus 0 \oplus H, \tag{3.2}$$

where

$$H = \begin{pmatrix} I_5 & 0 \\ R^*SC^\dagger & 0 \end{pmatrix} \text{ on } \mathscr{R}(C) \oplus \mathscr{H}_6.$$

Proof. Let $B = PQ \in \mathfrak{X}$, P and Q are given by (2.2). Then (see [4]) $B^\dagger$ is an unbounded pseudoinverse of B with $\mathscr{D}(B^\dagger) = \mathscr{R}(B) \oplus \mathscr{R}(B)^\perp$ and $\mathscr{R}(B^\dagger) = \overline{\mathscr{R}(B^*)}$. Next, we characterize the operator-matrix representation of $B^\dagger$ by the polar decomposition of B. According to Lemma 2.2, a direct calculation shows that

$$B = I_1 \oplus 0 \oplus 0 \oplus 0 \oplus \begin{pmatrix} C^2 & CSR \\ 0 & 0 \end{pmatrix}$$

with respect to the space decomposition (2.1), where R is an isometric isomorphism from $\mathscr{H}_6$ onto $\mathscr{H}_5$, S and C are two self-adjoint operators on $\mathscr{H}_5$ such that $0 \leqslant S \leqslant I_5$, $0 \leqslant C \leqslant I_5$, $S^2 + C^2 = I_5$ and $\mathscr{N}(S) = \mathscr{N}(C) = \{0\}$. It is easy to check that C commutes with S. In this case,

$$B^* = I_1 \oplus 0 \oplus 0 \oplus 0 \oplus \begin{pmatrix} C^2 & 0 \\ R^*CS & 0 \end{pmatrix} \tag{3.3}$$

and

$$BB^* = I_1 \oplus 0 \oplus 0 \oplus 0 \oplus \begin{pmatrix} C^2 & 0 \\ 0 & 0 \end{pmatrix} \tag{3.4}$$

with respect to the space decomposition (2.1). By (3.4) it yields

$$|B^*| = (BB^*)^{\frac{1}{2}} = I_1 \oplus 0 \oplus 0 \oplus 0 \oplus \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.5}$$

Let

$$U = I_1 \oplus 0 \oplus 0 \oplus 0 \oplus \begin{pmatrix} C & SR \\ 0 & 0 \end{pmatrix} \tag{3.6}$$

with respect to the space decomposition (2.1). Combining with (3.1) and (3.3), it is easy to obtain that $\mathscr{N}(U) = \mathscr{N}(B)$ and $\mathscr{N}(U^*) = \mathscr{N}(B^*)$. The second equation follows $\mathscr{R}(U) = \overline{\mathscr{R}(B)}$. Hence, $B = |B^*|U$ is the left polar decomposition of B and is unique. It was proved in [11] that $B^\dagger = U^*|B^*|^\dagger$ is the right polar decomposition of $B^\dagger$ on $\mathscr{D}(B^\dagger)$. By (3.5) we have

$$|B^*|^\dagger = I_1 \oplus 0 \oplus 0 \oplus 0 \oplus \begin{pmatrix} C^\dagger & 0 \\ 0 & 0 \end{pmatrix} \tag{3.7}$$

with respect to the space decomposition (2.1). It should be noticed that $\mathscr{R}(B)$ is closed if and only if $\mathscr{R}(BB^*)$ is closed if and only if $\mathscr{R}(C)$ is closed by (3.4). Hence, $C^\dagger$ is a densely defined close operator if $\mathscr{R}(B)$ is not closed, and $\mathscr{D}(C^\dagger) = \mathscr{R}(C)$. Consider $\mathscr{N}(C) = \{0\}$, it follows that $\overline{\mathscr{R}(C)} = \mathscr{H}_5$ and $CC^\dagger x = P_{\overline{\mathscr{R}(C)}} x = x$, for all $x \in \mathscr{R}(C) \subset \mathscr{H}_5$. Thus

$$B^\dagger = U^*|B^*|^\dagger = I_1 \oplus 0 \oplus 0 \oplus 0 \oplus H,$$

where

$$H = P_{\mathscr{H}_5 \oplus \mathscr{H}_6}(B^\dagger)|_{\mathscr{H}_5 \oplus \mathscr{H}_6} = \begin{pmatrix} I_5 & 0 \\ R^*SC^\dagger & 0 \end{pmatrix} \text{ on } \mathscr{R}(C) \oplus \mathscr{H}_6. \quad \square$$

REMARK 3.2. In general, if $\mathscr{R}(B)$ is closed, then $B^\dagger = B^*(BB^*)^\dagger$ (see [15]). However, it should be noted that this equality is not valid when $\mathscr{R}(B)$ is not closed. Indeed,

$$\mathscr{D}((BB^*)^\dagger) = \mathscr{D}(B^{*\dagger}B^\dagger) \subsetneqq \mathscr{D}(B^\dagger),$$

see [25] for more details.

Based on the above results, we have the block operator-matrix representations of B and $B^\dagger$, which is helpful for analyzing the properties of operators in $\mathfrak{X}$ more intuitively. Let $\mathscr{J}_{\mathfrak{X}}$ denote all isometric parts of elements in $\mathfrak{X}$. It was shown in [11] that $\mathfrak{X} = \{V^2 : V \in \mathscr{J}_{\mathfrak{X}}\}$ and the map $B \to V$ is a bijection between $\mathfrak{X}$ and $\mathscr{J}_{\mathfrak{X}}$. Next we provide an alternative proof.

COROLLARY 3.3. *Let $B = PQ$, and let $B = |B^*|U$ be the polar decomposition of B. Then the following statements hold:*

(1) $B = U^2$;

(2) $P = (UU^*)|_{\mathscr{H}_2^\perp} \oplus I_2$;

(3) $Q = (U^*U)|_{\mathscr{H}_3^\perp} \oplus I_3$.

Proof. Since $B \in \mathfrak{X}$, by (3.6) in Proposition 3.1 it follows that

$$U^2 = I_1 \oplus 0 \oplus 0 \oplus 0 \oplus \begin{pmatrix} C^2 & CSR \\ 0 & 0 \end{pmatrix} = B,$$

$$UU^* = I_1 \oplus 0 \oplus 0 \oplus 0 \oplus \begin{pmatrix} I_5 & 0 \\ 0 & 0 \end{pmatrix} = 0 \oplus (UU^*)|_{\mathscr{H}_2^\perp},$$

where $(UU^*)|_{\mathscr{H}_2^\perp}$ is the restriction of UU^* to $\mathscr{H}_2^\perp$, and

$$U^*U = I_1 \oplus 0 \oplus 0 \oplus 0 \oplus \begin{pmatrix} C^2 & CSR \\ R^*CS & R^*S^2R \end{pmatrix} = 0 \oplus (U^*U)|_{\mathscr{H}_3^\perp}$$

with respect to the space decomposition (2.1). Using (2.2), we obtain $(UU^*)|_{\mathscr{H}_2^\perp} \oplus I_2 = P$ and $(U^*U)|_{\mathscr{H}_3^\perp} \oplus I_3 = Q$. This completes the proof. □

COROLLARY 3.4. *Let $B \in \mathfrak{X}$. Then B^* commutes with $B^\dagger$ on $\mathscr{D}(B^\dagger)$.*

Proof. Let $B = PQ \in \mathfrak{X}$. According to Proposition 3.1, since C^2 in (3.3) satisfying $\mathscr{R}(C^2) \subset \mathscr{R}(C) = \mathscr{D}(C^\dagger)$, $B^\dagger B^*$ is well defined on $\mathscr{H}$. A direct calculation follows that

$$B^*B^\dagger = B^\dagger B^* \text{ on } \mathscr{D}(B^\dagger). \quad \square$$

The following theorem is the main result of this paper, which presents the sufficient and necessary conditions for $B \in \mathfrak{X}$.

THEOREM 3.5. *Let $B \in \mathscr{B}(\mathscr{H})$. Then the following statements are equivalent:*

(1) $B \in \mathfrak{X}$;

(2) $B^\dagger \cdot B^\dagger = B^\dagger$;

(3) $B^* \cdot B^\dagger = B^*|_{\mathscr{D}(B^\dagger)}$;

(4) $B^2 B^\dagger = BB^*|_{\mathscr{D}(B^\dagger)}$;

(5) $B^* B = B^\dagger B^2$;

(6) $|B^*|^\dagger B^* = |B^*|$.

Proof. Let $B = PQ \in \mathfrak{X}$. By (3.2) in Proposition 3.1, it is easy to check that $\mathscr{D}(B^\dagger) = (\mathscr{H} \ominus \mathscr{H}_5) \oplus \mathscr{R}(C)$ and $B^\dagger \cdot B^\dagger$ is well defined on $\mathscr{D}(B^\dagger)$. In this case, a straightforward calculation shows that

$$B^\dagger \cdot B^\dagger = B^\dagger \ \text{ on } \ \mathscr{D}(B^\dagger).$$

Combining with (3.3), it yields

$$B^* B^\dagger = B^* \ \text{ on } \ \mathscr{D}(B^\dagger).$$

Besides, by (3.1) we can obtain that

$$B^2 = I_1 \oplus 0 \oplus 0 \oplus 0 \oplus \begin{pmatrix} C^4 & C^3SR \\ 0 & 0 \end{pmatrix}$$

with respect to the space decomposition (2.1). It follows that

$$B^2 B^\dagger = BB^* \ \text{ on } \ \mathscr{D}(B^\dagger),$$

$$B^\dagger B^2 = B^* B \ \text{ on } \ \mathscr{H}.$$

Finally, by (3.3) and (3.7) we conclude that $|B^*|^\dagger B^*$ is well defined on $\mathscr{H}$, because $\mathscr{R}(C^2) \subset \mathscr{R}(C) \subset \mathscr{D}(C^\dagger)$. A direct calculation yields $|B^*|^\dagger B^* = |B^*|$. $(1) \Rightarrow (2) - (6)$ have been proved, we next present (2)–(6)$\Rightarrow$(1).

If (2) holds: $B^\dagger \cdot B^\dagger = B^\dagger$ on $\mathscr{D}(B^\dagger) = \mathscr{R}(B) \oplus \mathscr{R}(B)^\perp$, obviously, $B^\dagger$ is idempotent, and by Lemma 2.1 and [9, Theorem 3.1], we have

$$\mathscr{D}(B^\dagger) = \mathscr{R}(B^\dagger) \dot{+} \mathscr{N}(B^\dagger) = \mathscr{N}(B)^\perp \dot{+} \mathscr{R}(B)^\perp.$$

Consider $\mathscr{R}(B^\dagger B) \subset \mathscr{R}(B^\dagger) \subset \mathscr{D}(B^\dagger)$, it follows that

$$BB^\dagger \cdot B^\dagger B = BB^\dagger B,$$

and then

$$P_{\overline{\mathscr{R}(B)}} P_{\mathscr{N}(B)^\perp} = BP_{\mathscr{N}(B)^\perp} = B.$$

Thus $B \in \mathfrak{X}$.

If (3) holds: $B^*|_{\mathscr{D}(B^\dagger)} = B^*B^\dagger$, in this way

$$B^{*\dagger}B^*B = B^{*\dagger}B^*B^\dagger B,$$

that is

$$P_{\mathscr{N}(B^*)^\perp}B = P_{\overline{\mathscr{R}(B)}}B = B = P_{\mathscr{N}(B^*)^\perp}P_{\mathscr{N}(B)^\perp}.$$

Therefore, $B \in \mathfrak{X}$.

If (4) holds: $B^2B^\dagger = BB^*|_{\mathscr{D}(B^\dagger)}$, multiplying by $B^\dagger$ on both sides of the equation, we get

$$P_{\mathscr{N}(B)^\perp}BB^\dagger = P_{\overline{\mathscr{R}(B^*)}}B^*|_{\mathscr{D}(B^\dagger)} = B^*|_{\mathscr{D}(B^\dagger)}.$$

Since $\mathscr{R}(B) \subset \mathscr{D}(B^\dagger)$, multiplying by B, it follows

$$P_{\mathscr{N}(B)^\perp}BB^\dagger B = P_{\mathscr{N}(B)^\perp}B = B^*B,$$

and taking adjoints $B^*P_{\mathscr{N}(B)^\perp} = B^*B$. Multiplying by $B^{*\dagger}$, we have $P_{\mathscr{N}(B^*)^\perp}P_{\mathscr{N}(B)^\perp} = B$.

If (5) holds: $B^\dagger B^2 = B^*B$, that is $P_{\mathscr{N}(B)^\perp}B = B^*B$. Based on the proof of (4), we obtain $B = P_{\overline{\mathscr{R}(B)}}P_{\mathscr{N}(B)^\perp} \in \mathfrak{X}$ as wanted.

If (6) holds: $|B^*|^\dagger B^* = |B^*|$, then

$$P_{\overline{\mathscr{R}(|B^*|)}}B^* = |B^*||B^*|^\dagger B^* = |B^*||B^*| = BB^*.$$

Since $\mathscr{R}(|B^*|) = \mathscr{R}(B)$ (see [14, 16]), it follows that $P_{\overline{\mathscr{R}(B)}}B^* = BB^*$, and taking adjoints $BP_{\overline{\mathscr{R}(B)}} = BB^*$. Multiplying by $B^\dagger$ on both sides of the equation, one has

$$P_{\mathscr{N}(B)^\perp}P_{\overline{\mathscr{R}(B)}} = P_{\overline{\mathscr{R}(B^*)}}B^* = B^*.$$

Hence, $B \in \mathfrak{X}$. □

Based on Theorem 3.5, it is easy to verify that $B \in \mathfrak{X}$ if and only if $B^\dagger B^* = B^*$. Therefore, we obtain that if $B \in \mathfrak{X}$, then $B^\dagger B^* = B^*B^\dagger$ on $\mathscr{D}(B^\dagger)$, which has also been shown in Corollary 3.4. But the converse does not necessarily hold, see the following example.

EXAMPLE 3.6. Given a self-adjoint operator

$$B = \begin{pmatrix} B_{11} & 0 \\ 0 & 0 \end{pmatrix}$$

with respect to the space decomposition $\mathscr{H} = \mathscr{N}(B)^\perp \oplus \mathscr{N}(B)$. If $\mathscr{R}(B)$ is closed, then $B_{11} : \mathscr{N}(B)^\perp \to \mathscr{N}(B)^\perp$ is invertible. In this case,

$$B^\dagger = \begin{pmatrix} B_{11}^{-1} & 0 \\ 0 & 0 \end{pmatrix}$$

with respect to $\mathscr{H} = \mathscr{N}(B)^{\perp} \oplus \mathscr{N}(B)$. Hence, we obtain

$$B^*B^{\dagger} = B^{\dagger}B^* \neq B^*.$$

By Theorem 3.5, we know that B is not in $\mathfrak{X}$.

According to Theorem 3.5, we can obtain the following essential corollaries.

COROLLARY 3.7. *Let $B \in \mathscr{B}(\mathscr{H})$. Then the following statements are equivalent:*

(1) $B \in \mathfrak{X}$;

(2) $B(B^{\dagger})^2 = BB^{\dagger}$;

(3) $B^{*\dagger}(B^{\dagger})^2 = B^{*\dagger}B^{\dagger}$;

(4) $(B^{\dagger})^2B = B^{\dagger}B$;

(5) $(B^{\dagger})^2B^{*\dagger} = B^{\dagger}B^{*\dagger}$;

(6) $BB^*B^{\dagger} = BB^*|_{\mathscr{D}(B^{\dagger})}$;

(7) $B^*B^{\dagger}B = B^*B$;

(8) $B^*B^{\dagger}B^{*\dagger} = B^*B^{*\dagger}$;

(9) $B^{*\dagger}B^*B^{\dagger} = B^{*\dagger}B^*|_{\mathscr{D}(B^{\dagger})}$;

(10) $BB^{\dagger}B^* = BB^*$;

(11) $B^{*\dagger}B^{\dagger}B^{*} = B^{*\dagger}B^*$;

(12) $B^{\dagger}B^*B^{*\dagger} = B^*B^{*\dagger}$;

(13) $B^{\dagger}B^*B = B^*B$;

(14) $B^*B^2B^{\dagger} = B^*BB^*|_{\mathscr{D}(B^{\dagger})}$;

(15) $B^{\dagger}B^2B^{\dagger} = B^{*}|_{\mathscr{D}(B^{\dagger})}$.

Proof. First prove that (2), (3), (4) and (5) hold if and only if $B^{\dagger} \cdot B^{\dagger} = B^{\dagger}$. The "if" part is trivial. Next, we prove the "only if" part.

(2): If $B(B^{\dagger})^2 = BB^{\dagger}$, multiplying by $B^{\dagger}$ on both sides of this equation, we get

$$P_{\mathscr{N}(B)^{\perp}}(B^{\dagger})^2 = P_{\mathscr{N}(B)^{\perp}}B^{\dagger}.$$

Since $\mathscr{R}(B^{\dagger}) = \mathscr{N}(B)^{\perp}$, it follows $(B^{\dagger})^2 = B^{\dagger}$.

(3): If $B^{*\dagger}(B^{\dagger})^2 = B^{*\dagger}B^{\dagger}$, multiplying by B^* on both sides of this equation, one has $(B^{\dagger})^2 = B^{\dagger}$.

(4): If $(B^\dagger)^2B = B^\dagger B$, then for all $x \in \mathscr{D}(B^\dagger)$, we have $(B^\dagger)^2BB^\dagger x = B^\dagger BB^\dagger x$, that is

$$B^\dagger(B^\dagger BB^\dagger)x = B^\dagger B^\dagger x = B^\dagger x.$$

Hence, $B^\dagger \cdot B^\dagger = B^\dagger$ on $\mathscr{D}(B^\dagger)$.

(5): If $(B^\dagger)^2B^{*\dagger} = B^\dagger B^{*\dagger}$, multiplying by B^*B on both sides of this equation, we have

$$(B^\dagger)^2 P_{\mathscr{N}(B^*)^\perp} B = (B^\dagger)^2 B = B^\dagger B.$$

According to (4), it follows that $B^\dagger \cdot B^\dagger = B^\dagger$ on $\mathscr{D}(B^\dagger)$.

By item (2) of Theorem 3.5, (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5).

Analogously, we prove that (6), (7), (8) and (9) hold if and only if $B^*B^\dagger = B^*|_{\mathscr{D}(B^\dagger)}$. Just need to verify the "only if" part.

(6): If $BB^*B^\dagger = BB^*|_{\mathscr{D}(B^\dagger)}$, multiplying by $B^\dagger$ on both sides of this equation, one has $P_{\overline{\mathscr{R}(B^*)}} B^*B^\dagger = P_{\overline{\mathscr{R}(B^*)}} B^*|_{\mathscr{D}(B^\dagger)}$, that is $B^*B^\dagger = B^*|_{\mathscr{D}(B^\dagger)}$.

(7): If $B^*B^\dagger B = B^*B$, then for all $x \in \mathscr{D}(B^\dagger)$, we obtain

$$B^*B^\dagger BB^\dagger x = B^*BB^\dagger x,$$

that is $B^*B^\dagger x = B^*x$. Therefore, $B^*B^\dagger = B^*$ on $\mathscr{D}(B^\dagger)$.

(8): If $B^*B^\dagger B^{*\dagger} = B^*B^{*\dagger}$, multiplying by B^*B on both sides of this equation, one has $B^*B^\dagger B = B^*B$. Applying (7), it gives $B^*B^\dagger = B^*$ on $\mathscr{D}(B^\dagger)$.

(9): If $B^{*\dagger}B^*B^\dagger = B^{*\dagger}B^*|_{\mathscr{D}(B^\dagger)}$, multiplying by B^* on both sides of the equation, then $B^*B^\dagger = B^*|_{\mathscr{D}(B^\dagger)}$.

Based on item (3) of Theorem 3.5, (1) $\Leftrightarrow$ (6) $\Leftrightarrow$ (7) $\Leftrightarrow$ (8) $\Leftrightarrow$ (9).

Similarly, we can conclude that (10), (11) and (12) hold if and only if $B^\dagger B^* = B^*$ if and only if $B \in \mathfrak{X}$.

(1) $\Leftrightarrow$ (13): If $B \in \mathfrak{X}$, then $B^\dagger B^* = B^*$, it further yields $B^\dagger B^*B = B^*B$. Conversely, if $B^\dagger B^*B = B^*B$, multiplying by B on both sides of this equation, one has $P_{\overline{\mathscr{R}(B)}} B^*B = BB^*B$, and taking adjoints $B^*BP_{\overline{\mathscr{R}(B)}} = B^*BB^*$. multiplying by $B^{*\dagger}$, it follows $B^{*\dagger}B^*BP_{\overline{\mathscr{R}(B)}} = B^{*\dagger}B^*BB^*$, that is $BP_{\overline{\mathscr{R}(B)}} = BB^*$, multiplying by $B^\dagger$,

$$P_{\mathscr{N}(B)^\perp} P_{\overline{\mathscr{R}(B)}} = P_{\overline{\mathscr{R}(B^*)}} B^* = B^*.$$

Hence, $B \in \mathfrak{X}$.

(1) $\Leftrightarrow$ (14): It is easy to check that (14) holds if and only if $B^2B^\dagger = BB^*|_{\mathscr{D}(B^\dagger)}$, by item (4) of Theorem 3.5, this is the case if and only if $B \in \mathfrak{X}$.

(1) $\Leftrightarrow$ (15): If $B \in \mathfrak{X}$, by item (5) of Theorem 3.5, we have $B^\dagger B^2 = B^*B$. For all $x \in \mathscr{D}(B^\dagger) = \mathscr{R}(B) \oplus \mathscr{R}(B)^\perp = \mathscr{R}(B) \oplus \mathscr{N}(B^*)$, it follows $B^\dagger B^2B^\dagger x = B^*BB^\dagger x = B^*x$, that is $B^\dagger B^2B^\dagger = B^*|_{\mathscr{D}(B^\dagger)}$. Conversely, if $B^\dagger B^2B^\dagger = B^*|_{\mathscr{D}(B^\dagger)}$, multiplying by B, one has $B^\dagger B^2 = B^*B$ and $B \in \mathfrak{X}$. $\square$

COROLLARY 3.8. (see [11,22]) *$B \in \mathfrak{X}$ if and only if $BB^*B = B^2$.*

Proof. It is easy to check that $BB^*B = B^2$ if and only if $B^*B = B^\dagger B^2$. By item (5) of Theorem 3.5, we obtain that $B \in \mathfrak{X}$ if and only if $BB^*B = B^2$. □

COROLLARY 3.9. *Let $B \in \mathscr{B}(\mathscr{H})$. Then the following statements are equivalent:*

(1) $B \in \mathfrak{X}$;

(2) $B^*B^2 = (B^*B)^2$;

(3) $B^2B^* = (BB^*)^2$.

Proof. (1) $\Leftrightarrow$ (2) If $B \in \mathfrak{X}$, by Corollary 3.8 we have $BB^*B = B^2$, which further implies that $B^*B^2 = (B^*B)^2$. Conversely, if $B^*B^2 = (B^*B)^2$, multiplying by $B^{*\dagger}$ on both sides of this equation, one has $B^2 = BB^*B$. Hence, $B \in \mathfrak{X}$.

(1) $\Leftrightarrow$ (3) Analogously, if $B \in \mathfrak{X}$, then $B^2B^* = (BB^*)^2$. Conversely, if $B^2B^* = (BB^*)^2$, it gives $BB^{*2} = (BB^*)^2$, multiplying by $B^\dagger$ we obtain $B^{*2} = B^*BB^*$. Applying Corollary 3.8, it follows $B \in \mathfrak{X}$. □

In [11], it was proved that $B \in \mathfrak{X}$ if and only if $\|Bx\|^2 = \langle Bx, x\rangle$ for any $x \in \mathscr{N}(B)^\perp$. Motivated by this, we present the following results.

THEOREM 3.10. *Let $B \in \mathscr{B}(\mathscr{H})$. Then the following statements are equivalent:*

(1) $B \in \mathfrak{X}$;

(2) $\langle B^\dagger x, x\rangle = \|x\|^2$, *for all* $x \in \mathscr{R}(B)$;

(3) $\langle B^\dagger x, BB^*x\rangle = \|B^*x\|^2$, *for all* $x \in \mathscr{R}(B)$.

Proof. (1) $\Rightarrow$ (2) $-$ (3): Let $B = PQ$, P and Q as in (2.2) with respect to the space decomposition (2.1). By Proposition 3.1, it follows that $\mathscr{R}(B) \subset \mathscr{H}_1 \oplus \mathscr{H}_5$, then the following conclusions can be obtained by direct calculations:

$$\langle B^\dagger x, x\rangle = \|x\|^2$$

and

$$\langle B^\dagger x, BB^*x\rangle = \|B^*x\|^2,$$

for all $x \in \mathscr{R}(B)$.

(2) $\Rightarrow$ (1): If $\langle B^\dagger x, x\rangle = \|x\|^2$, $\forall x \in \mathscr{R}(B)$, then

$$\langle B^*B^\dagger By, y\rangle = \langle B^\dagger By, By\rangle = \|By\|^2 = \langle B^*By, y\rangle,$$

for all $y \in \mathscr{H}$. Hence, $B^*B^\dagger B = B^*B$. By item (7) of Corollary 3.7, it follows that $B \in \mathfrak{X}$.

(3)⇒(1): On the one hand,

$$\langle B^{\dagger}By, BB^{*}By\rangle = \langle B^{*}BB^{*}B^{\dagger}By, y\rangle$$

for all $y \in \mathscr{H}$. On the other hand,

$$\|B^{*}By\|^{2} = \langle (B^{*}B)^{2}y, y\rangle$$

for all $y \in \mathscr{H}$. If $\langle B^{\dagger}x, BB^{*}x\rangle = \|B^{*}x\|^{2}$, $\forall x \in \mathscr{R}(B)$, it yields $B^{*}BB^{*}B^{\dagger}By = (B^{*}B)^{2}y$, or

$$B^{*}B(B^{*}B^{\dagger}By - B^{*}By) = 0.$$

Since

$$(B^{*}B^{\dagger}B - B^{*}B)y \in \mathscr{R}(B^{*}) \subset \mathscr{N}(B)^{\perp} = \mathscr{N}(B^{*}B)^{\perp},$$

we obtain $(B^{*}B^{\dagger}B - B^{*}B)y = 0$ for all $y \in \mathscr{H}$. Therefore, $B^{*}B^{\dagger}B = B^{*}B$, it follows $B \in \mathfrak{X}$.

COROLLARY 3.11. *If $B \in \mathfrak{X}$, then $\langle B^{\dagger}x, Bx\rangle = \|B^{*}x\|^{2}$, for all $x \in \mathscr{R}(B)$.*

Proof. Let $B = PQ$, P and Q as in (2.2) with respect to the space decomposition (2.1). By a direct calculation, it follows that

$$\langle B^{\dagger}x, BB^{*}x\rangle = \langle B^{\dagger}x, Bx\rangle,$$

for all $x \in \mathscr{R}(B)$. Applying item (3) of Theorem 3.10, we have $\langle B^{\dagger}x, Bx\rangle = \|B^{*}x\|^{2}$, for all $x \in \mathscr{R}(B)$. □

The Drazin invertibility and core invertibility of idempotents were studied by many authors (see [3, 6, 12, 13]). In [7], Böttcher and Spitkovsky gave the operator-matrix representation of the Drazin inverse for all operators from the algebra generated by P and Q, where P and Q are two orthogonal projections. Next, we present an alternative proof for the operator PQ. □

PROPOSITION 3.12. *Let $B \in \mathfrak{X}$. If B is a closed range operator, then*

$$B^{D} = B^{\#} = I \oplus 0 \oplus 0 \oplus 0 \oplus \begin{pmatrix} C^{-2} & C^{-3}SR \\ 0 & 0 \end{pmatrix} \tag{3.8}$$

and

$$B^{\circledast} = I \oplus 0 \oplus 0 \oplus 0 \oplus \begin{pmatrix} C^{-2} & 0 \\ 0 & 0 \end{pmatrix} \tag{3.9}$$

with respect to the space decomposition (2.1).

Proof. Let $B \in \mathfrak{X}$, then B has the matrix form (3.1). Observe that $\mathscr{R}(B)$ is closed if and only if $\mathscr{R}(BB^{*})$ is closed. Combining with (3.4) in Proposition 3.1, it follows that

$\mathscr{R}(B)$ is closed if and only if $\mathscr{R}(C)$ is closed. Since C is self-adjoint and $\mathscr{N}(C)=0$, this is the case if and only if $0\notin\sigma(C)$. Let

$$L=I\oplus 0\oplus 0\oplus 0\oplus\begin{pmatrix}C^{-2} & C^{-3}SR\\ 0 & 0\end{pmatrix} \tag{3.10}$$

with respect to the space decomposition (2.1). By (3.10), a direct calculation shows that

$$\begin{cases}BL=LB;\\ LBL=L;\\ B^2L=B.\end{cases}$$

Hence, we obtain ind(B)=1, and

$$B^D=B^\sharp=I\oplus 0\oplus 0\oplus 0\oplus\begin{pmatrix}C^{-2} & C^{-3}SR\\ 0 & 0\end{pmatrix}$$

with respect to the space decomposition (2.1). Applying (3.2) in Proposition 3.1 and item (i) of Lemma 2.5, it follows that

$$B^{\circledast}=B^\sharp BB^\dagger=I\oplus 0\oplus 0\oplus 0\oplus\begin{pmatrix}C^{-2} & 0\\ 0 & 0\end{pmatrix}$$

with respect to the space decomposition (2.1). □

REMARK 3.13. Let $B\in\mathfrak{X}$ and $\mathscr{R}(B)$ be closed. Based on the matrix form of $B^\sharp$, we can also conclude

$$B^\sharp=B^{\dagger *}B^\sharp=B^{\dagger *}B^\dagger B^{\dagger *}.$$

In the following theorem, we present the equivalent conditions for $B\in\mathfrak{X}$ involving the Moore-Penrose inverse, group inverse and core inverse of B.

THEOREM 3.14. *Let $B\in\mathscr{B}^1(\mathscr{H})$. Then the following statements are equivalent:*

(1) $B\in\mathfrak{X}$;

(2) $BB^\sharp=B^{\dagger *}$;

(3) $B^{\circledast}B=B^{\dagger *}$;

(4) $B^{\circledast}B^\dagger=B^{\circledast}$;

(5) $B^{\dagger *}B^\dagger=B^{\circledast}$.

Proof. Since $B\in\mathscr{B}^1(\mathscr{H})$, $\mathscr{R}(B)$ is closed (see [4]).

$(1)\Leftrightarrow(2)$: If $B\in\mathfrak{X}$, then B has the matrix form (3.1), by (3.8) we obtain

$$BB^\sharp=I\oplus 0\oplus 0\oplus 0\oplus\begin{pmatrix}I & C^{-1}SR\\ 0 & 0\end{pmatrix}$$

with respect to the space decomposition (2.1). Notice that $\mathscr{R}(B)$ is closed, $B^\dagger \in \mathscr{B}(\mathscr{H})$ satisfies

$$B^\dagger = B^*(BB^*)^\dagger = I \oplus 0 \oplus 0 \oplus 0 \oplus \begin{pmatrix} I & 0 \\ R^*C^{-1}S & 0 \end{pmatrix}$$

with respect to the space decomposition (2.1). Thus $BB^\sharp = B^{\dagger *}$.

Conversely, if $BB^\sharp = B^{\dagger *}$, then

$$B = BB^\sharp B = B^{\dagger *}B,$$

multiplying both sides of the equation by B^* and $B^\dagger$ respectively, it follows that

$$B^* = B^*BB^\dagger = B^*B^{\dagger *}BB^\dagger = P_{\mathscr{N}(B)^\perp}P_{\mathscr{R}(B)}.$$

Hence, $B \in \mathfrak{X}$.

(1) $\Leftrightarrow$ (3): By item (ii) of Lemma 2.5, the proof is straightforward.

(1) $\Leftrightarrow$ (4): If $B \in \mathfrak{X}$, then $B^\circledast$ has the matrix form (3.9). In this case, with respect to the space decomposition (2.1),

$$B^\circledast B^\dagger = I \oplus 0 \oplus 0 \oplus 0 \oplus \begin{pmatrix} C^{-2} & 0 \\ 0 & 0 \end{pmatrix} = B^\circledast.$$

Conversely, if $B^\circledast B^\dagger = B^\circledast$, then

$$BB^\circledast B^\dagger B = BB^\circledast B = B.$$

Follows from (iii) of Lemma 2.5, knowing the fact that $P_{\mathscr{R}(B)}P_{\mathscr{N}(B)^\perp} = B \in \mathfrak{X}$.

(1) $\Leftrightarrow$ (5): If $B \in \mathfrak{X}$, by (3.9) a direct calculation yields $B^\circledast = B^{\dagger *}B^\dagger$. Conversely, if $B^{\dagger *}B^\dagger = B^\circledast$, then

$$B^\circledast B = B^{\dagger *}B^\dagger B = B^{\dagger *}P_{\mathscr{N}(B)^\perp}.$$

According to Lemma 2.1, it gives $\mathscr{N}(B)^\perp = \mathscr{N}(B^{\dagger *})^\perp$. Therefore, $B^\circledast B = B^{\dagger *}$. By (3) we obtain $B \in \mathfrak{X}$. □

COROLLARY 3.15. *Let $B \in \mathscr{B}^1(\mathscr{H})$. Then the following statements are equivalent:*

(1) $B \in \mathfrak{X}$;

(2) $B^{*\dagger}B^\sharp = (B^*BB^*)^\dagger$;

(3) $BB^\sharp(B^*B)^\dagger = (B^*BB^*)^\dagger$;

(4) $BB^*BB^\sharp = B$;

(5) $BB^\sharp B^*B = B$;

(6) $B^*BB^\sharp = B^*B^{\dagger *}$;

(7) $B^{\dagger}BB^{\sharp} = B^{\dagger}B^{\dagger *}$;

(8) $BB^{\sharp}B^{*} = B^{\dagger *}B^{*}$;

(9) $BB^{\sharp}B^{\dagger} = B^{\dagger *}B^{\dagger}$;

(10) $B^{\sharp\dagger}BB^{\sharp} = (B^{*}B^{\sharp})^{\dagger}$;

(11) $BB^{\sharp}B^{\sharp\dagger} = (B^{\sharp}B^{*})^{\dagger}$;

(12) $B^{\dagger}B^{\circledast}B^{\dagger} = B^{\dagger}B^{\circledast}$;

(13) $B^{*}B^{\circledast}B^{\dagger} = B^{*}B^{\circledast}$;

(14) $B^{\circledast}B^{\dagger}B = B^{\circledast}B$;

(15) $B^{\circledast}B^{\dagger}B^{\dagger *} = B^{\circledast}B^{\dagger *}$;

(16) $(B^{\circledast})^{\dagger}B^{\sharp} = (B^{\circledast})^{\sharp}B^{\sharp} = (B^{\circledast})^{\circledast}B^{\sharp} = B^{\dagger *}$.

Proof. Since $\mathscr{R}(B)$ is closed, $\mathscr{R}(BB^{*}) = \mathscr{R}(B)$, it follows from Lemma 2.3 that $(B^{*}BB^{*})^{\dagger} = (BB^{*})^{\dagger}B^{*\dagger}$. Consider $(BB^{*})^{\dagger} = B^{*\dagger}B^{\dagger}$, and hence $(B^{*}BB^{*})^{\dagger} = B^{*\dagger}B^{\dagger}B^{*\dagger}$.

(1) $\Rightarrow$ (2): According to Remark 3.13, (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (1): If $B^{*\dagger}B^{\sharp} = (BB^{*})^{\dagger}B^{*\dagger}$ holds, multiplying by BB^{*} on both sides of the equation, we obtain $BB^{*}B^{*\dagger}B^{\sharp} = BB^{*}(BB^{*})^{\dagger}B^{*\dagger}$, or, equivalently, $BP_{\mathscr{R}(B^{*})}B^{\sharp} = P_{\mathscr{R}(BB^{*})}B^{*\dagger}$. On the one hand,

$$BP_{\mathscr{R}(B^{*})}B^{\sharp} = (BP_{\mathscr{N}(B)^{\perp}})B^{\sharp} = BB^{\sharp}.$$

On the other hand, combining with Lemma 2.1, it follows that $\mathscr{R}(B^{*\dagger}) = \overline{\mathscr{R}(B)} = \mathscr{R}(B)$ and

$$P_{\mathscr{R}(BB^{*})}B^{*\dagger} = P_{\mathscr{R}(B)}B^{*\dagger} = B^{*\dagger}.$$

Hence, $BB^{\sharp} = B^{\dagger *}$. According to item (2) of Theorem 3.14, we can get $B \in \mathfrak{X}$.

(1) $\Rightarrow$ (3): If $B \in \mathfrak{X}$, Theorem 3.14 guarantees $BB^{\sharp} = B^{\dagger *}$. Thus,

$$BB^{\sharp}(B^{*}B)^{\dagger} = B^{\dagger *}(B^{*}B)^{\dagger} = B^{\dagger *}B^{\dagger}B^{\dagger *} = (B^{*}BB^{*})^{\dagger}.$$

(3) $\Rightarrow$ (1): If $BB^{\sharp}(B^{*}B)^{\dagger} = B^{\dagger *}(B^{*}B)^{\dagger}$, then $BB^{\sharp}(B^{*}B)^{\dagger}(B^{*}B) = B^{\dagger *}(B^{*}B)^{\dagger}(B^{*}B)$, or, equivalently, $BB^{\sharp}P_{\mathscr{N}(B^{*}B)^{\perp}} = B^{\dagger *}P_{\mathscr{N}(B^{*}B)^{\perp}}$. On the one hand,

$$\mathscr{N}(B^{*}B)^{\perp} = \mathscr{N}(B)^{\perp} = \mathscr{N}(B^{\dagger *})^{\perp}.$$

On the other hand, combining with Lemma 2.4, it follows that $\mathscr{N}(B^{*}B)^{\perp} = \mathscr{N}(B^{\sharp})^{\perp}$. Hence, $BB^{\sharp} = B^{\dagger *}$ and $B \in \mathfrak{X}$ is satisfied.

(1) $\Rightarrow$ (4): If $B \in \mathfrak{X}$, then $BB^{\sharp} = B^{\dagger *}$. Furthermore,

$$BB^{*}BB^{\sharp} = BB^{*}B^{\dagger *} = BP_{\mathscr{R}(B^{*})} = BP_{\mathscr{N}(B)^{\perp}} = B.$$

$(4) \Rightarrow (1)$: If $BB^*BB^\sharp = B = BB^*B^{\dagger *}$, multiplying by $(BB^*)^\dagger$, we have $(BB^*)^\dagger BB^* BB^\sharp = (BB^*)^\dagger BB^* B^{\dagger *}$, or, equivalently, $P_{\mathscr{N}(BB^*)^\perp} BB^\sharp = P_{\mathscr{N}(BB^*)^\perp} B^{\dagger *}$. Since $\mathscr{R}(B)$ is closed, the inclusion

$$BB^\sharp = P_{\mathscr{N}(BB^*)^\perp} BB^\sharp = P_{\mathscr{N}(BB^*)^\perp} B^{\dagger *} = B^{\dagger *}$$

results from

$$\mathscr{N}(BB^*)^\perp = \mathscr{R}(BB^*) = \mathscr{R}(B) = \mathscr{R}(B^{\dagger *}).$$

$(1) \Rightarrow (5)$: Similar to the method of $(1) \Rightarrow (4)$, it can be proved.

$(5) \Rightarrow (1)$: If $BB^\sharp B^* B = B = B^{*\dagger} B^* B$, multiplying by $(B^*B)^\dagger$, we obtain

$$BB^\sharp B^* B(B^*B)^\dagger = B^{*\dagger} B^* B(B^*B)^\dagger,$$

it yields $BB^\sharp P_{\mathscr{R}(B^*B)} = B^{*\dagger} P_{\mathscr{R}(B^*B)}$. Consider

$$\mathscr{R}(B^*B) = \mathscr{N}(B^*B)^\perp = \mathscr{N}(B)^\perp = \mathscr{N}(B^{*\dagger})^\perp = \mathscr{N}(B^\sharp)^\perp,$$

we get $BB^\sharp = B^{*\dagger}$ and $B \in \mathscr{X}$.

$(1) \Rightarrow (6) - (9)$: If $B \in \mathscr{X}$, by item (2) of Theorem 3.14, (6)–(9) hold obviously.

$(6) \Rightarrow (1)$: If $B^*BB^\sharp = B^*B^{*\dagger}$, then

$$B^{*\dagger} B^* BB^\sharp = B^{*\dagger} B^* B^{*\dagger},$$

or $P_{\mathscr{N}(B^*)^\perp} BB^\sharp = P_{\mathscr{N}(B^*)^\perp} B^{*\dagger}$. Follows from $\mathscr{N}(B^*)^\perp = \mathscr{R}(B) = \mathscr{R}(B^{*\dagger})$, knowing the fact that $BB^\sharp = B^{*\dagger}$ and $B \in \mathscr{X}$.

$(7) \Rightarrow (1)$: If $B^\dagger BB^\sharp = B^\dagger B^{\dagger *}$, multiplying by B on both sides of the equation, it can be proved.

$(8) \Rightarrow (1)$: If $BB^\sharp B^* = B^{*\dagger} B^*$, multiplying by $(B^{*\dagger})$, we can obtain $BB^\sharp = B^{*\dagger}$ and $B \in \mathscr{X}$.

$(9) \Rightarrow (1)$: If $BB^\sharp B^\dagger = B^{\dagger *} B^\dagger$; multiplying by B, it follows $BB^\sharp = B^{*\dagger}$.

$(1) \Rightarrow (10)$: If $B \in \mathscr{X}$, then $BB^\sharp = B^{*\dagger}$. Consider $\mathscr{R}(B) = \mathscr{R}(B^\sharp)$, it follows that $(B^*B^\sharp)^\dagger = B^{\sharp\dagger} B^{*\dagger}$ and

$$B^{\sharp\dagger} BB^\sharp = B^{\sharp\dagger} B^{*\dagger} = (B^*B^\sharp)^\dagger.$$

$(10) \Rightarrow (1)$: If $B^{\sharp\dagger} BB^\sharp = (B^*B^\sharp)^\dagger = B^{\sharp\dagger} B^{*\dagger}$, multiplying by $B^\sharp$, we have

$$B^\sharp B^{\sharp\dagger} BB^\sharp = B^\sharp B^{\sharp\dagger} B^{*\dagger},$$

then $P_{\mathscr{R}(B^\sharp)} BB^\sharp = P_{\mathscr{R}(B^\sharp)} B^{*\dagger}$. By Lemma 2.4, it yields $BB^\sharp = B^{*\dagger}$ and $B \in \mathscr{X}$.

$(1) \Leftrightarrow (11)$: Similar to the method of $(1) \Leftrightarrow (10)$.

$(1) \Leftrightarrow (12) - (15)$: Based on Theorem 3.14, (1) holds if and only if $B^{\circledast} B^\dagger = B^{\circledast}$. So we prove that (12), (13), (14), (15) hold if and only if $B^{\circledast} B^\dagger = B^{\circledast}$. The "if" part is trivial. Next, we prove the "only if" part.

(12): If $B^\dagger B^{\circledast} B^\dagger = B^\dagger B^{\circledast}$, multiplying by B on both sides of the equation, we have $P_{\mathscr{R}(B)} B^{\circledast} B^\dagger = P_{\mathscr{R}(B)} B^{\circledast}$. Consider $\mathscr{R}(B^{\circledast}) = \mathscr{R}(B)$, thus $B^{\circledast} B^\dagger = B^{\circledast}$.

(13): If $B^* B^{\circledast} B^\dagger = B^* B^{\circledast}$, multiplying by $(B^{*\dagger})$, we can obtain $B^{\circledast} B^\dagger = B^{\circledast}$.

(14): If $B^{\circledast} B^\dagger B = B^{\circledast} B$, multiplying by $B^\dagger$, it follows $B^{\circledast} B^\dagger B B^\dagger = B^{\circledast} P_{\mathscr{N}(B^*)^\perp}$. Notice that $\mathscr{N}(B^{\circledast}) = \mathscr{N}(B^*)$, so it yields $B^{\circledast} B^\dagger = B^{\circledast}$.

(15): If $B^{\circledast} B^\dagger B^{\dagger *} = B^{\circledast} B^{\dagger *}$, multiplying by B^* on both sides of the equation, it can be proved.

(1) $\Leftrightarrow$ (16) Since $\mathscr{R}(B^\sharp) = \mathscr{R}(B)$, we conclude that $BB^\sharp = BP_{\mathscr{R}(B)} B^\sharp$, so the asserted equivalence follows from (2) of Theorem 3.14 and (iv) of Lemma 2.5. □

REMARK 3.16. If $B \in \mathfrak{X}$ and $\mathscr{R}(B)$ is closed, then $BB^\sharp (BB^\sharp)^\dagger = (BB^\sharp B^*)^\dagger$. Indeed, by Theorem 3.14 we have $BB^\sharp = B^{*\dagger}$ and $\mathscr{R}(BB^\sharp) = \mathscr{R}(B^{*\dagger}) = \mathscr{R}(B)$, combining with Lemma 2.3, it follows $(BB^\sharp B^*)^\dagger = B^{*\dagger}(BB^\sharp)^\dagger$. Hence,

$$BB^\sharp (BB^\sharp)^\dagger = B^{*\dagger}(BB^\sharp)^\dagger = (BB^\sharp B^*)^\dagger.$$

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