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ON CLOSED RANGE OPERATORS AND THEIR CHARACTERIZATION VIA p–SCHATTEN IDEALS

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Abstract. Let S and T be two closed-range bounded linear operators defined on a complex Hilbert space $\mathscr{H}$. The main objective of this paper is to investigate the conditions under which the equality involving the p-Schatten norm, a specific case of symmetric or unitarily invariant norms, holds:

$$\|S^*XT^\dagger + S^\dagger XT^*\|_p = 2\|PXQ\|_p,$$

for any $X \in B(\mathscr{H})$, where $P = SS^\dagger$ and $Q = T^\dagger T$ with $1 < p < \infty$. Here, $S^\dagger$ and $T^\dagger$ denote the Moore-Penrose inverses of S and T, respectively.

1. Introduction and preliminaries

Closed range operators are an important class of operators in functional analysis, especially in the study of bounded linear operators on Hilbert spaces. These operators, characterized by their range being closed in the norm topology, have attracted significant attention due to their connection with compact operators and their role in operator-theoretic inequalities. The Moore-Penrose inverse [2, 15] is a key tool for studying closed range operators, extending the concept of an inverse for non-invertible operators. Applications of the Moore-Penrose inverse in Hilbert spaces can be found in [2].

An essential part of analyzing operator inequalities involves operator norms in ideal spaces, particularly the Schatten p-classes for $1 \leqslant p \leqslant \infty$. These norms generalize concepts like the trace norm and the Hilbert-Schmidt norm, and have been widely studied in matrix theory and compact operator analysis, see [7]. Schatten p-norms are specific cases of symmetric or unitarily invariant norms and are crucial for understanding the behavior of operators and establishing various inequalities.

Although this paper focuses on closed range operators and their characterization through p-Schatten ideals, partial isometries play a relevant role. Partial isometries, which act as isometries on the orthogonal complement of their null space, are fundamental in polar decomposition, where any bounded operator can be factored into a

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partial isometry and a positive operator. The significance of partial isometries has been extensively studied, as shown in [8, 9, 15].

In recent years, researchers have focused on using inequalities and elementary operators to characterize different families of linear operators, see [4] for recent developments. In this work, we aim to investigate conditions under which operator inequalities involving closed range operators and Schatten p-norms hold. These results provide further insights into the structure of bounded operators and their norm relations.

Before proceeding, we introduce some key concepts and notations. Let $\mathscr{H}$ be a complex Hilbert space, and let $B(\mathscr{H})$ denote the algebra of all bounded linear operators on $\mathscr{H}$, equipped with the operator norm $\|\cdot\|$, where

$$\|T\| = \sup\{\|Tz\| : z \in \mathscr{H}, \|z\| = 1\},$$

for any $T \in B(\mathscr{H})$.

The identity operator is denoted by I. As is customary, for any $X \in B(\mathscr{H})$, we write T^* for the adjoint of T, $\mathscr{R}(T)$ for its range, and $\mathscr{N}(T)$ for its kernel. Recall that an operator $T \in B(\mathscr{H})$ is said to be *normal* if $T^*T = TT^*$. Additionally, throughout this work, $Gl(\mathscr{H})$ refers to the group of invertible operators in $B(\mathscr{H})$, while $Gl_s(\mathscr{H})$ denotes the set of all invertible self-adjoint operators. The notation $CR(\mathscr{H})$ stands for the set of closed range operators, $B_0(\mathscr{H})$ represents the ideal of compact operators, and $U(\mathscr{H})$ refers to the set of unitary operators.

Let $T, S \in B(\mathscr{H})$, S is called a *generalized inverse* of T if

$$TST = T \qquad \text{and} \qquad STS = S.$$

Moreover, S is called the *Moore-Penrose inverse* of T if

$$TST = T, \quad STS = S, \quad (TS)^* = TS \qquad \text{and} \qquad (ST)^* = ST.$$

It is important to note that the generalized inverse is generally not unique. However, if the Moore-Penrose inverse of an operator T exists, it is unique and is denoted by $T^\dagger$. It is well known that T has a unique Moore-Penrose inverse if and only if $T \in CR(\mathscr{H})$ (see [9]) and if $T \in Gl(\mathscr{H})$, then $T^\dagger = T^{-1}$.

Recall that an operator $T \in B(\mathscr{H})$ is called a *partial isometry* if it satisfies $\|Tx\| = \|x\|$ for all $x \in \mathscr{N}(T)^\perp$, where $\mathscr{N}(T)^\perp$ denotes the orthogonal complement of the null space $\mathscr{N}(T)$. Moreover, T is referred to as an *isometry* if $\|Tx\| = \|x\|$ for every $x \in \mathscr{H}$, or equivalently, if $T^*T = I$.

For any operator $T \in B(\mathscr{H})$, the absolute value of T is defined as $|T| = (T^*T)^{1/2}$. It is always possible to represent T in the form $T = U|T|$, where $U \in B(\mathscr{H})$ is a partial isometry. This representation is known as the polar decomposition of T. Notably, if $T \in Gl(\mathscr{H})$, then $U \in U(\mathscr{H})$ and we have the relation $T = U|T| = |T^*|U$.

If $T \in B_0(\mathscr{H})$, then the singular values of T are defined as the eigenvalues of the positive operator $|T|$, typically listed as $s_1(T), s_2(T), \ldots$. For $1 \leqslant p < \infty$, the p-Schatten class is defined as

$$B_p(\mathscr{H}) = \{X \in B_0(\mathscr{H}) : \{s_j(X)\} \in l^p(\mathbb{N}^*)\}.$$

Here, $\mathbb{N}^*$ means the set of all positive integers and $l^p(\mathbb{N}^*)$ denotes the space of sequences $\{a_n\}_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty}|a_n|^p<\infty$. It is well-known that $B_p(\mathscr{H})$ forms a two-sided ideal in $B(\mathscr{H})$. Moreover, the p-Schatten norm, defined as $\|T\|_p=(\sum s_j(T)^p)^{1/p}$ for any $T\in B_p(\mathscr{H})$, turns $B_p(\mathscr{H})$ into a Banach space. The Schatten p-norm is a highly useful tool in the study of matrix theory and compact operators. Over the years, numerous researchers have investigated various results related to this norm. For further details, the reader is referred to [1, 3, 6, 17, 18] and the references therein.

It is well known that $B_2(\mathscr{H})$ is a Hilbert space, called the class of the Hilbert-Schmidt operator, with the inner product defined by $\langle T,S\rangle_2=tr(T^*S)$, where tr denotes the usual trace and $T,S\in B_2(\mathscr{H})$.

Note that for any $1\leqslant p<\infty$, we have $\|UXV\|_p=\|X\|_p$, where $U,V\in U(\mathscr{H})$ and $X\in B_p(\mathscr{H})$. This shows that the p-Schatten norms are special examples of unitarily invariant (or symmetric) norms. Recall that a norm N is called unitarily invariant if, for any operator $X\in\mathscr{B}(\mathscr{H})$, the following condition holds $N(UXV)=N(X)$ for any $U,V\in U(\mathscr{H})$.

Given a unitarily invariant norm N, we can define a symmetrically normed ideal $\mathscr{I}$ in the context of linear operators on Hilbert spaces. A symmetrically normed ideal is a set of operators that satisfies certain properties, such as being closed under linear combinations and containing limits of convergent sequences of operators. For further details on norm ideals or symmetrically normed ideals, we refer the reader to [7].

In recent years, there has been considerable interest among mathematicians in the characterization of different families or classes of linear and bounded operators through the use of inequalities and equalities involving norms and elementary operators. For further information, the reader may consult [5,10,14,20,22] and the references therein.

In [23], Weiss extended the classical Fuglede-Putnam theorem [19] by proving the following result: if $M,N,X\in B(\mathscr{H})$ are such that M and N are normal operators and $MX-XN\in B_2(\mathscr{H})$, then $M^*X-XN^*\in B_2(\mathscr{H})$ and

$$\|MX-XN\|_2=\|M^*X-XN^*\|_2.$$

This result relates to the classical Fuglede-Putnam theorem, which states that for normal operators M and N, if $MX-XN=0$ for some $X\in B(\mathscr{H})$, then $M^*X-XN^*=0$.

In [16], Nakamoto obtained a result similar to Weiss's but for a single normal operator. Specifically, he proved that an operator N is normal if and only if $\|NX-XN\|_2=\|N^*X-XN^*\|_2$ for every $X\in B_2(\mathscr{H})$.

In [21], Seddik showed that if $S\in Gl(\mathscr{H})$ satisfies

$$\|SXS^{-1}+S^{-1}XS\|\geqslant 2\|X\|, \tag{1}$$

for any $X\in\mathscr{B}(\mathscr{H})$ if and only if $S=\lambda M$, where $\lambda\in\mathbb{C}^*=\mathbb{C}\setminus\{0\}$ and $M\in Gl_s(\mathscr{H})$.

The primary aim of this paper is to study the p-Schatten norm equality

$$\|S^*XT^\dagger+S^\dagger XT^*\|_p=2\|PXQ\|_p, \tag{2}$$

for any $X\in B(\mathscr{H})$, where S and T are closed range operators on a Hilbert space $\mathscr{H}$, $P=SS^\dagger$, and $Q=T^\dagger T$. The focus of this work is to establish conditions under which this equality holds and to characterize the operators that satisfy such a relation.

The remainder of the paper is structured as follows: In Section 2, we present the main results concerning the p-Schatten norm equality (2), delving into its profound implications for operator theory. This section offers a comprehensive analysis of the conditions that ensure the validity of the equality and underscores its importance within the broader context of operator spaces and norm-related properties. Additionally, we will show that the closed range operators satisfying inequality (2) have adjoints that are scalar multiples of their Moore-Penrose inverse. This result provides a deeper understanding of the relationship between closed range operators and their pseudo-inverses, highlighting a fundamental structural property in the context of operator theory.

2. Main results

In this section, we present our key contributions. To set the stage for the main theorems, we first introduce two preliminary lemmas that lay the foundation for the results to follow. The first lemma, originally established by Khosravi in [11, Theorem 2.1], extends the inequality (1) to the setting of closed range operators. Specifically, we state:

LEMMA 1. *Let $S,T \in B(\mathscr{H})$ be bounded linear operators with closed ranges. Then, for any unitarily invariant norm $\|\cdot\|_{\mathscr{I}}$, the following inequality holds:*

$$2\|PXQ\|_{\mathscr{I}} \leqslant \|S^*XT^\dagger + S^\dagger XT^*\|_{\mathscr{I}}, \tag{3}$$

where $P = SS^\dagger$ and $Q = T^\dagger T$.

Our second lemma was established by Kittaneh in [12, Theorem 2]. Before presenting it, it is important to note that from this point onward, we will consider the symmetric ideal $\mathscr{I}$ to be the p-Schatten ideal.

Prior to exploring the main results of this manuscript, we briefly discuss the conditions under which the arithmetic-geometric mean (AGM) equality holds for the p-Schatten norm. In this context, Kittaneh derived the following norm equality within the framework of $B_p(\mathscr{H})$ ideals.

LEMMA 2. *Let $A,B,X \in B(\mathscr{H})$ and $1 < p < \infty$. Then*

$$\|AA^*X + XBB^*\|_p = 2\|A^*XB\|_p \tag{4}$$

if and only if $AA^*X = XBB^*$.

Recall that given $A,B,X \in B(\mathscr{H})$, we say that X intertwines A and B if and only if $AX - XB = 0$. We can then rephrase Lemma 2 as follows: the equality (4) holds if and only if X intertwines AA^* and BB^*.

We are now ready to present one of the main results of this manuscript. Lemma 2 offers a method to determine when inequality (3) becomes an equality for the p-norms.

THEOREM 1. *Let $S,T \in CR(\mathscr{H})$, $X \in B(\mathscr{H})$ and $1 < p < \infty$, then*

$$\|S^*XT^\dagger + S^\dagger XT^*\|_p = 2\|PXQ\|_p \tag{5}$$

if and only if $S^*XT^\dagger = S^\dagger XT^*$.

Proof. Since $P = SS^\dagger$ and $Q = T^\dagger T$ are orthogonal projections on $\mathscr{R}(S)$ and $\mathscr{R}(T^*)$, respectively, then $PS = S$ and $T^* = QT^*$. Thus

$$\begin{aligned} \|S^*S(S^\dagger XT^\dagger) + (S^\dagger XT^\dagger)TT^*\|_p &= \|S^*XT^\dagger + S^\dagger XT^*\|_p \\ &\geqslant 2\|PXQ\|_p \\ &= 2\|S(S^\dagger XT^\dagger)T\|_p. \end{aligned} \tag{6}$$

Then, by Lemma 2, we conclude that (5) holds if and only if

$$S^*S(S^\dagger XT^\dagger) = (S^\dagger XT^\dagger)TT^*,$$

or equivalently $S^*XT^\dagger = S^\dagger XT^*$. □

By imposing some additional conditions on the operators S and T, such as having a bounded left or right inverse, or being invertible, we obtain the following corollaries.

COROLLARY 1. *Let $S,T \in CR(\mathscr{H})$, $X \in B(\mathscr{H})$ such that $T^\dagger T = SS^\dagger = I$, $1 < p < \infty$ and $\|S^*XT^\dagger + S^\dagger XT^*\|_p = 2\|X\|_p$. Then $SS^*X = XT^*T$.*

Proof. By the hypothesis, $T^\dagger T = SS^\dagger = I$ implies that the equality

$$\|S^*XT^\dagger + S^\dagger XT^*\|_p = 2\|X\|_p$$

is equivalent to

$$\|S^*XT^\dagger + S^\dagger XT^*\|_p = 2\|PXQ\|_p.$$

Now, by Theorem 1, we have that $S^*XT^\dagger = S^\dagger XT^*$. Finally, by multiplying the last equation on the left by S and on the right by T, we obtain that $SS^*X = XT^*T$. □

The following corollary was previously stated in [12, Corollary 2]. However, we present an alternative proof here.

COROLLARY 2. *Let $S,T \in Gl(\mathscr{H})$, $X \in B(\mathscr{H})$ and $1 < p < \infty$. Then*

$$\|S^*XT^{-1} + S^{-1}XT^*\|_p = 2\|X\|_p \tag{7}$$

if and only if $SS^*X = XT^*T$.

Proof. Since S and T are invertible operators, then $S,T \in CR(\mathscr{H})$, $T^\dagger = T^{-1}$, $S^\dagger = S^{-1}$ and $P = SS^\dagger = T^\dagger T = Q = I$. Then, (7) holds if and only if (5) holds, which is equivalent to

$$S^*XT^{-1} = S^*XT^\dagger = S^\dagger XT^* = S^{-1}XT^*. \quad \square$$

We now aim to characterize the operators S and T that satisfy the identity in Theorem 1:

$$S^*XT^\dagger = S^\dagger XT^* \quad \text{for any } X \in \mathscr{B}(\mathscr{H}). \tag{8}$$

This characterization is essential for understanding the structure of these operators. To achieve this, we first recall some properties of elementary operators, which help us derive conditions under which the identity holds.

An operator $\Phi : \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{H})$ is defined by

$$\Phi(X) = A_1XB_1 + A_2XB_2 + \cdots + A_mXB_m,$$

where $A_i, B_i \in \mathscr{B}(\mathscr{H})$, with $1 \leqslant i \leqslant m$, is called an *elementary operator* on $\mathscr{B}(\mathscr{H})$, and we denote by $\Phi = \Phi_{\overline{A},\overline{B}}$, where $\overline{A} = (A_1, \ldots, A_m)$ and $\overline{B} = (B_1, \ldots, B_m)$. This class of operators includes many important operators of $\mathscr{B}(\mathscr{H})$ such as the inner derivation $\delta_A(X) = AX - XA$, the generalized derivation $\delta_{A,B}(X) = AX - XB$ (related to the equality (4)), the multiplication operator $\mathscr{M}_{A,B}(X) = AXB$, the symmetrized two-sided multiplication $\mathscr{U}_{A,B}(X) = AXB + BXA$, the operator $\mathscr{V}_{A,B}(X) = AXB - BXA$.

In the next result we focus on characterizing under what conditions on the m-operators $\overline{A}$ and $\overline{B}$ is the elementary operator equal to zero for any $X \in \mathscr{B}(\mathscr{H})$. The main tool used is the following result by Fong and Sourour [11, Theorem 1].

LEMMA 3. *Let*

$$\Phi(X) = A_1XB_1 + A_2XB_2 + \cdots + A_mXB_m,$$

with $X \in B(\mathscr{H})$. If $\{B_1, B_2, \cdots, B_n\}$ is linearly independent, where $n \leqslant m$ and c_{kj} denote constants for which $B_j = \sum_{k=1}^{n} c_{kj}B_k$ for any $n+1 \leqslant j \leqslant m$. Then $\Phi \equiv 0$ if and only if

$$A_k = -\sum_{j=n+1}^{m} c_{kj}A_j$$

for any $1 \leqslant k \leqslant n$. If $m = n$, then $A_1, A_2, \cdots, A_m$ are equal to zero.

We are now in a position to determine the pairs of operators S and T that satisfy the identity (8). By applying the previously discussed properties of elementary operators and leveraging the results from the lemmas, we can derive the precise conditions these operators must meet. This will allow us to fully characterize the pairs (S,T) that fulfill the required relationship.

THEOREM 2. *Let $S, T \in CR(\mathscr{H})$. Then the following conditions are equivalent:*

1. *$S^*XT^\dagger = S^\dagger XT^*$ for all $X \in B(\mathscr{H})$.*
2. *$S^* = \lambda S^\dagger$, $T^* = \lambda T^\dagger$ for some $\lambda \in \mathbb{R}$.*

Proof. Without loss of generality we can assume that $S \neq 0$ and $T \neq 0$. Following Lemma 3, we consider $\Phi_0 : B(\mathscr{H}) \to B(\mathscr{H})$, such that

$$\Phi_0(X) = S^*XT^\dagger - S^\dagger XT^*.$$

Note that (8) is equivalent to $\Phi_0 \equiv 0$.

If T^* is not a scalar multiple of $T^\dagger$, then $S^* = S^\dagger = 0$, a contradiction to the assumption that both S and T are non-zero operators. Hence T^* is a scalar multiple of $T^\dagger$, say $T^* = \lambda T^\dagger$. Then by Lemma 3, we have that $\Phi_0 \equiv 0$ if and only if $T^* = \lambda T^\dagger$ and $S^* = \lambda S^\dagger$.

Finally, if $\lambda \neq 0$, from the conditions that define the Moore-Penrose inverse, we have that

$$\frac{1}{\overline{\lambda}} SS^* = (SS^\dagger)^* = SS^\dagger = \frac{1}{\lambda} SS^*,$$

and this implies that $\lambda \in \mathbb{R}$. $\square$

We introduce the following subclass of operators of closed ranges,

$$\mathscr{K} = \{T \in CR(\mathscr{H}) : \exists \lambda \in \mathbb{R}, T^* = \lambda T^\dagger\}.$$

Notice that every partial isometry T satisfies $T^* = T^\dagger$, so taking $\lambda = 1$ shows $T \in \mathscr{K}$. Hence

$$\{T \in CR(\mathscr{H}) : T \text{ is an partial isometry}\} \subseteq \mathscr{K}.$$

We note that if $T \in \mathscr{K}$, then $|\lambda| = \|T\|\gamma(T) = \|T\|\|T^\dagger\|$, where

$$\gamma(T) = \inf\{\|Tx\| : \operatorname{dist}(x, \mathscr{N}(T)) = 1\},$$

or $\gamma(T) = +\infty$ if $T = 0$, is the reduced minimum modulus. Here

$$\operatorname{dist}(x, \mathscr{N}(T)) = \inf\{\|x - z\| : z \in \mathscr{N}(T)\},$$

represents the distance from x to the kernel of T. It is well-known that $\gamma(T) > 0$ if and only if $T \in CR(\mathscr{H})$, $\gamma(T) = \gamma(T^*)$, and $\gamma(T) = \frac{1}{\|T^\dagger\|}$, see [15].

We will demonstrate through examples that the aforementioned inclusion is strict.

EXAMPLE 1.

1. We begin with a simple finite dimensional example. Let $\mathscr{H} = \mathbb{C}^2$ and

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

We have that $T^* = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ and $T^\dagger = \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \end{pmatrix}$, i.e. $T^\dagger = \frac{1}{2}T^*$. Finally, as

$$TT^* = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix},$$

we conclude that T is not a partial isometry, since if T were a partial isometry, then TT^* would need to be a projection onto $\mathscr{R}(T)$.

2. An example of an infinite-dimensional operator T where the Moore-Penrose inverse $T^{\dagger}$ is a scalar multiple of the adjoint T^* but T is not a partial isometry, can be found among compact operators in infinite-dimensional spaces that are not isometries. Let $\mathscr{H} = l^2(\mathbb{N})$, the space of square-summable sequences and consider the operator $T : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ defined by:

$$T(x_1, x_2, x_3, \ldots) = (\lambda x_1, 0, 0, \ldots),$$

where λ is a positive scalar, with $\lambda \neq 1$. Thus, it is immediate that $T = T^*$, and

$$T^{\dagger}(x_1, x_2, x_3, \ldots) = \left(\frac{1}{\lambda} x_1, 0, 0, \ldots\right).$$

Hence, for this choice of operator, $T^{\dagger} = \frac{1}{\lambda^2} T^*$. Moreover, T is not a partial isometry because if we consider $x = (1, 0, 0, \ldots) \in \mathscr{N}(T)^{\perp}$, then

$$\|Tx\|_2^2 = \lambda^2 \neq 1 = \|x\|_2^2.$$

Combining Theorem 2 and basic properties of the p-Schatten norm, we can deduce the following statement.

COROLLARY 3. *Let $S, T \in CR(\mathscr{H})$ and $1 < p < \infty$. Then, the following conditions are equivalent:*

1. $\|S^* X T^{\dagger}\|_p + \|S^{\dagger} X T^*\|_p = 2\|PXQ\|_p$ *for all* $X \in B(\mathscr{H})$,
2. $\|S^* X T^{\dagger}\|_p + \|S^{\dagger} X T^*\|_p \leqslant 2\|PXQ\|_p$ *for all* $X \in B(\mathscr{H})$,
3. $\|S^* X T^{\dagger} + S^{\dagger} X T^*\|_p \leqslant 2\|PXQ\|_p$ *for all* $X \in B(\mathscr{H})$,
4. $\|S^* X T^{\dagger} + S^{\dagger} X T^*\|_p = 2\|PXQ\|_p$ *for all* $X \in B(\mathscr{H})$.
5. $S^* X T^{\dagger} = S^{\dagger} X T^*$ *for all* $X \in B(\mathscr{H})$.
6. $S^* = \lambda S^{\dagger}$, $T^* = \lambda T^{\dagger}$ *for some* $\lambda \in \mathbb{R}$.

In [5, Lemma 1], the second author proved that if $S \in Gl(\mathscr{H})$, then $\|S\|\|S^{-1}\| = 1$ if and only if $S = \|S\|V$ where $V \in U(\mathscr{H})$. In the following result, we extend this result assuming that the operator S has a closed range, and in particular, we obtain a characterizarion involving the set $\mathscr{K}$, previously defined.

PROPOSITION 1. *Let $S \in CR(\mathscr{H})$. Then the following conditions are equivalent:*

1. $\|S\|\|S^{\dagger}\| = 1$.
2. $S = \|S\|V$, *for some partial isometry* V.
3. $S \in \mathscr{K}$ *with* $|\lambda| = \|S\|^2$.

Proof. Without loss of generality, we assume that $S \neq 0$.

(1) $\Rightarrow$ (2) This is a consequence of [15, Corollary 3.2] and by considering $W := \frac{S}{\|S\|}$.

(2) $\Rightarrow$ (3) We have $S^* = \|S\|V^* = \|S\|V^\dagger$ and $S^\dagger = \|S\|^{-1}V^\dagger$, or equivalently,

$$S^\dagger \|S\| = \|S\|^{-1} S^*.$$

Thus, we conclude that $S^* = \|S\|^2 S^\dagger$.

(3) $\Rightarrow$ (1) By the hypothesis, we have that

$$\|S\| = \|S^*\| = \|S\|^2 \|S^\dagger\|.$$

Thus, $\|S\|\|S^\dagger\| = 1$. □

Next, we present a result similar to that obtained in Theorem 2 but with the assumption that S and T are bounded linear operators, and not necessarily with closed range. It is worth mentioning that the proof we present is slightly different from the previous one, which is why we decided to include it.

THEOREM 3. *Let $S, T \in B(\mathscr{H})$. Then the following conditions are equivalent:*

1. *X intertwines SS^* and T^*T for all $X \in B(\mathscr{H})$.*
2. *X intertwines $(SS^*)^r$ and $(T^*T)^r$ for all $X \in B(\mathscr{H})$ and for any $r > 0$.*
3. *$S^* = \lambda U$, $T = \lambda V$ where $U, V \in B(\mathscr{H})$ are isometries and $\lambda \in \mathbb{R}$ with $\lambda > 0$.*

   If, in addition, $S, T \in CR(\mathscr{H})$ with $T^\dagger T = SS^\dagger = I$, then each of the the previous conditions is equivalent to:

4. *$\|S^*\|\|(S^*)^\dagger\| = \|T\|\|T^\dagger\| = 1$ and $\|T\| = \|S^*\|$.*

Proof. (1) $\Leftrightarrow$ (2) It follows by Corollary 2 in [13].

(1) $\Rightarrow$ (3) If $XT^*T = SS^*X$ for all $X \in B(\mathscr{H})$, then

$$\delta_{SS^*, T^*T} \equiv 0.$$

Now, by Lemma 3, we conclude that there exists $\lambda_0 \in \mathbb{C}$ such that

$$SS^* = \lambda_0 I = T^*T$$

with $\lambda_0 = \|S^*\|^2 = \|T\|^2 > 0$. Therefore the result is obtained taking $U = \lambda_0^{-1/2} S^*$ and $V = \lambda_0^{-1/2} T$.

(3) $\Rightarrow$ (1) This is a consequence of the properties of isometries.

(3) $\Rightarrow$ (4) This follows by Proposition 1.

(4) $\Rightarrow$ (1) By Proposition 1, we have that

$$S^* = \|S^*\|U \text{ and } T = \|T\|V$$

with U,V partial isometries. Now, from the hypothesis $T^{\dagger}T = SS^{\dagger} = I$, we can conclude that T,S^* are injective, and therefore V,U are isometries. Therefore, for every $X \in B(\mathscr{H})$ it holds that

$$\begin{aligned} XT^*T &= \|T\|^2 XV^*V \\ &= \|T\|^2 X = \|S^*\|^2 X \\ &= \|S^*\|^2 U^*UX = SS^*X. \end{aligned}$$

This completes the proof as required. □

Combining Corollary 3 and Theorem 3, we obtain the following statement.

COROLLARY 4. *Let $S,T \in CR(\mathscr{H})$ such that $T^{\dagger}T = SS^{\dagger} = I$ and $1 < p < \infty$. Then, the following conditions are equivalent:*

1. $\|S^*XT^{\dagger}\|_p + \|S^{\dagger}XT^*\|_p = 2\|X\|_p$ *for all* $X \in B(\mathscr{H})$.
2. $\|S^*XT^{\dagger}\|_p + \|S^{\dagger}XT^*\|_p \leqslant 2\|X\|_p$ *for all* $X \in B(\mathscr{H})$.
3. $\|S^*XT^{\dagger} + S^{\dagger}XT^*\|_p \leqslant 2\|X\|_p$ *for all* $X \in B(\mathscr{H})$.
4. $\|S^*XT^{\dagger} + S^{\dagger}XT^*\|_p = 2\|X\|_p$ *for all* $X \in B(\mathscr{H})$.
5. *X intertwines SS^* and T^*T for all $X \in B(\mathscr{H})$.*
6. *X intertwines $(SS^*)^r$ and $(T^*T)^r$ for all $X \in B(\mathscr{H})$ and for any $r > 0$.*
7. *$S^* = \lambda U$, $T = \lambda V$ where $U,V \in B(\mathscr{H})$ isometries and $\lambda \in \mathbb{R}$ with $\lambda > 0$.*
8. $\|S^*\|\|(S^*)^{\dagger}\| = \|T\|\|T^{\dagger}\| = 1$ *and* $\|T\| = \|S^*\|$

Proof. The implications (1)⇒ (2), (2)⇒ (3), (3)⇒ (4) are trivial, and the equivalence between (5), (6), (7) and (8) is given by Theorem 3. As consequence of Corollary 1, we conclude that (4) implies (5).

Finally, if we assume that (7) holds. By Lemma 1, without loss of generality, we can assume that $X \in B_p(\mathscr{H})$; otherwise, the equality in (1) is trivially satisfied. Then, we have

$$\begin{aligned} 2\|X\|_p \leqslant \|S^*XT^{\dagger}\|_p + \|S^{\dagger}XT^*\|_p &\leqslant \|S^*\|\|X\|_p\|T^{\dagger}\| + \|S^{\dagger}\|\|X\|_p\|T^*\| \\ &= 2\|X\|_p. \end{aligned}$$

This achieves the proof. □

Finally, to conclude this paper, we will examine a more specific case by assuming that $S = T$ and both operators are invertible. To establish our characterization under these assumptions, we first present a corollary of Proposition 1 without proof. This corollary establishes important relationships between the properties of the invertible operator S and its norms.

COROLLARY 5. *Let $S \in Gl(\mathscr{H})$. Then the following conditions are equivalent:*

1. $\|S\|\|S^{-1}\| = 1$.
2. $S = \|S\|V$, *for some* $V \in U(\mathscr{H})$.
3. $S \in \mathscr{K}$ *with* $|\lambda| = \|S\|^2$.

It is worth mentioning that Corollary 5 is an extension of [5, Lemma 1], since it characterizes the invertible operators $S = \|S\|V$, with $V \in U(\mathscr{H})$, through the set $\mathscr{K}$.

As consequence of Corollary 4 and 5, we conclude the following statement which is a particular case of Theorem 6 in [5], when $\mathscr{I} = B_p(\mathscr{H})$ with $1 < p < \infty$, but with a different proof.

PROPOSITION 2. *Let $S \in Gl(\mathscr{H})$ and $1 < p < \infty$. Then the following conditions are equivalent:*

1. $\|S^*XS^{-1}\|_p + \|S^{-1}XS^*\|_p = 2\|X\|_p$ *for all* $X \in B(\mathscr{H})$.
2. $\|S^*XS^{-1}\|_p + \|S^{-1}XS^*\|_p \leqslant 2\|X\|_p$ *for all* $X \in B(\mathscr{H})$.
3. $\|S^*XS^{-1} + S^{-1}XS^*\|_p \leqslant 2\|X\|_p$ *for all* $X \in B(\mathscr{H})$.
4. $\|S^*XS^{-1} + S^{-1}XS^*\|_{\mathscr{I}} = 2\|X\|_{\mathscr{I}}$ *for every* $X \in \mathscr{I}$,
5. $S = \|S\|V$ *with* $V \in U(\mathscr{H})$.
6. $\|S\|\|S^{-1}\| = 1$.
7. $S \in \mathscr{K}$ *with* $|\lambda| = \|S\|^2$.

Conclusion

In this paper, we have studied the conditions under which the p-Schatten norm equality

$$\|S^*XT^\dagger + S^\dagger XT^*\|_p = 2\|PXQ\|_p,$$

holds for any bounded linear operator X in a Hilbert space. Our work provides a new perspective on closed range operators through the use of the Moore-Penrose inverse and p-Schatten norms. We have characterized the roles of the projections $P = SS^\dagger$ and $Q = T^\dagger T$ in establishing these norm equalities.

These findings contribute to a deeper understanding of the behavior of closed range operators and extend the scope of operator inequalities in the framework of norm ideals. This study may serve as a starting point for future work in exploring similar norm equalities for other classes of operators, as well as further applications of the Moore-Penrose pseudoinverse in operator theory. We hope that this work will inspire more research into the interplay between operator inequalities and norm ideals.

Declarations

Availability of data and materials. Not applicable.

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