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AN EXTENSION OF A HLAWKA TYPE INEQUALITY

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Abstract. In this paper, we present an extension of a multiple Hlawka type inequality, which solves a conjecture of Wang affirmatively [6].

1. Introduction

We use $|A|$ to denote the determinant of a $n\times n$ matrix A.

Let A, B be positive definite $n\times n$ matrices. It is well known that (e.g. ([4], p. 511))

$$|A+B|\geqslant |A|+|B|. \tag{1}$$

In 1973, D. J. Hartfiel [2] generalized (1) to

$$|A+B|\geqslant |A|+|B|+(2^n-2)\sqrt{|AB|}. \tag{2}$$

When it comes to three positive defintie $n\times n$ matrices, we have ([7], p. 215)

$$|A+B+C|+|C|\geqslant |A+C|+|B+C|. \tag{3}$$

In 2014, M. Lin [5] generalized (3) to

$$|A+B+C|+|C|-|A+C|-|B+C|\geqslant |A+B|-|A|-|B|. \tag{4}$$

In 2022, Y. Hong and F. Qi [3] generalized (4) to

$$|A+B+C|-|A+B|-|A+C|-|B+C|+|A|+|B|+|C|\geqslant (3^n-3\cdot 2^n+3)\sqrt[3]{|ABC|}. \tag{5}$$

By using tensor product, W. Berndt and S. Sra [1] extended (1) and (4) to multiple version, which is called the Hlawka type inequality.

THEOREM 1. ([1], Conjecture 3.1 and Corollary 3.4) *Let $A_1,A_2,\cdots,A_m$ be positive definite $n\times n$ matrices. For each $k=1,2,\cdots,m$, define*

$$s_k(A_1,A_2,\cdots,A_m)=\sum_{1\leqslant i_1<i_2<\cdots<i_k\leqslant m}|A_{i_1}+A_{i_2}+\cdots+A_{i_k}|.$$

Then

$$\sum_{k=1}^{m}(-1)^{m-k}s_k(A_1,A_2,\cdots,A_m)\geqslant 0. \tag{6}$$

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In 2024, F. Wang [6] provided an elementary proof of Theorem 1 and proposed the following conjecture.

THEOREM 2. ([6], Conjecture 1.8) *Let $A_1,A_2,\cdots,A_m$ be positive definite $n\times n$ matrices. For each $k=1,2,\cdots,m$, define*

$$s_k(A_1,A_2,\cdots,A_m)=\sum_{1\leqslant i_1<i_2<\cdots<i_k\leqslant m}|A_{i_1}+A_{i_2}+\cdots+A_{i_k}|.$$

Then

$$\sum_{k=1}^{m}(-1)^{m-k}s_k(A_1,A_2,\cdots,A_m)\geqslant\Big(\sum_{k=1}^{m}(-1)^{m-k}\binom{m}{k}k^n\Big)\sqrt[m]{|A_1A_2\cdots A_m|},\qquad(7)$$

where $\binom{m}{k}=\frac{m!}{(m-k)!k!}$.

REMARK 1. It's easy to see that (7) reduces to (2) when $m=2$ and to (5) when $m=3$.

The main goal of this paper is to prove Theorem 2, which is an extension of the Hlawka type inequality.

2. Proof of Theorem 2

Note that all the inequalities above are correct not only for positive definite matrices but also for positive semidefintie matrices, as a positive semidefinite matrix can be seen as a limit of positive definite matrices.

For convenience, we write

$$\sum_{k=1}^{m}(-1)^{m-k}s_k(A_1,A_2,\cdots,A_m)=S(A_1,\cdots,A_m),$$

$$\Big(\sum_{k=1}^{m}(-1)^{m-k}\binom{m}{k}k^n\Big)=f_m(n).$$

We need a lemma.

LEMMA 1.

$$(m+1)f_{m+1}(n-1)+(m+1)f_m(n-1)=f_{m+1}(n).$$

Proof.

$$\begin{aligned}
&(m+1)\sum_{k=1}^{m+1}(-1)^{m+1-k}\binom{m+1}{k}k^{n-1}+(m+1)\sum_{k=1}^{m}(-1)^{m-k}\binom{m}{k}k^{n-1}\\
=&(m+1)^n+\sum_{k=1}^{m}(-1)^{m+1-k}k^{n-1}\left(\binom{m+1}{k}-\binom{m}{k}\right)(m+1)\\
=&(m+1)^n+\sum_{k=1}^{m}(-1)^{m+1-k}k^{n-1}\binom{m+1}{k}\cdot k\\
=&\sum_{k=1}^{m+1}(-1)^{m+1-k}\binom{m+1}{k}k^n. \quad \square
\end{aligned}$$

Now we are in a position to prove Theorem 2.

Proof. The proof of Theorem 2 is by induction on m and n.

For $m=2$ and any n, it follows from the inequality (2).

Suppose the inequality holds for $m-1$ and any n.

We use $A_{k,n-1},\ k=1,\cdots,m,$ to denote the principal submatrices of order $n-1$ in the upper left corner of the matrices A_k respectively.

Write

$$A_k=\begin{pmatrix} A_{k,n-1} & \alpha_k \\ \alpha_k^* & a_k \end{pmatrix}.$$

Then

$$A_1+A_2+\cdots+A_m=\begin{pmatrix} A_{1,n-1}+A_{2,n-1}+\cdots+A_{m,n-1} & \alpha_1+\alpha_2+\cdots+\alpha_m \\ (\alpha_1+\alpha_2+\cdots+\alpha_m)^* & a_1+a_2+\cdots a_m \end{pmatrix}.$$

By the definition of Schur complement,

$$\begin{aligned}
&\frac{|A_1+A_2+\cdots+A_m|}{|A_{1,n-1}+A_{2,n-1}+\cdots+A_{m,n-1}|}=a_1+a_2+\cdots+a_m\\
&-(\alpha_1+\alpha_2+\cdots+\alpha_m)^*(A_{1,n-1}+A_{2,n-1}+\cdots+A_{m,n-1})^{-1}(\alpha_1+\alpha_2+\cdots+\alpha_m),
\end{aligned}$$

$$\frac{|A_k|}{|A_{k,n-1}|}=a_k-\alpha_k^*A_{k,n-1}^{-1}\alpha_k.$$

Define

$$A_k'=\begin{pmatrix} A_{k,n-1} & \alpha_k \\ \alpha_k^* & \alpha_k^*A_{k,n-1}^{-1}\alpha_k \end{pmatrix}.$$

Then

$$A_1'+A_2'+\cdots+A_m'=\begin{pmatrix} \sum_{k=1}^m A_{k,n-1} & \sum_{k=1}^m \alpha_k \\ (\sum_{k=1}^m \alpha_k)^* & \sum_{k=1}^m \alpha_k^*A_{k,n-1}^{-1}\alpha_k \end{pmatrix}.$$

Since $A_{k,n-1}$ and its Schur complement $(=0)$ are positive semidefinite, A_k' are all positive semidefinite matrices.

Notice that

$$
\begin{aligned}
&|A_{1,n-1}+A_{2,n-1}+\cdots+A_{m,n-1}|\Big(\frac{|A_1+A_2+\cdots+A_m|}{|A_{1,n-1}+A_{2,n-1}+\cdots+A_{m,n-1}|}-\sum_{k=1}^{m}\frac{|A_k|}{|A_{k,n-1}|}\Big)\\
=&|A_{1,n-1}+A_{2,n-1}+\cdots+A_{m,n-1}|\Big(\sum_{k=1}^{m}\alpha_k^*A_{k,n-1}^{-1}\alpha_k-(\sum_{k=1}^{m}\alpha_k)^*(\sum_{k=1}^{m}A_{k,n-1})^{-1}(\sum_{k=1}^{m}\alpha_k)\Big)\\
=&|A_{1,n-1}+A_{2,n-1}+\cdots+A_{m,n-1}|\frac{|A'_1+A'_2+\cdots+A'_m|}{|A_{1,n-1}+A_{2,n-1}+\cdots+A_{m,n-1}|}\\
=&|A'_1+A'_2+\cdots+A'_m|.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&|A_{i_1,n-1}+A_{i_2,n-1}+\cdots+A_{i_k,n-1}|\Big(\frac{|A_{i_1}+A_{i_2}+\cdots+A_{i_k}|}{|A_{i_1,n-1}+A_{i_2,n-1}+\cdots+A_{i_k,n-1}|}-\sum_{j=1}^{k}\frac{|A_{i_j}|}{|A_{i_j,n-1}|}\Big)\\
=&|A'_{i_1}+A'_{i_2}+\cdots+A'_{i_k}|.\ (1\leqslant i_1<i_2<\cdots<i_k\leqslant m)
\end{aligned}
$$

By Theorem 1, we know

$$S(A'_1,A'_2,\cdots,A'_m)\geqslant 0.$$

It's equivalent to

$$
\begin{aligned}
&S(A_1,A_2,\cdots,A_m)\\
\geqslant&\sum_{k=1}^{m}(-1)^{m-k}\sum_{1\leqslant i_1<i_2<\cdots<i_k\leqslant m}|A_{i_1,n-1}+A_{i_2,n-1}+\cdots+A_{i_k,n-1}|\Big(\sum_{j=1}^{k}\frac{|A_{i_j}|}{|A_{i_j,n-1}|}\Big).
\end{aligned}\tag{8}
$$

The inequality (8) is the most important inequality in our proof.

When $n=1$, the inequality (7) reduces to Theorem 1 as $f_m(1)=0$.

Suppose Theorem 2 is true for m and $n-1$. We consider m and n.

We have

$$
\begin{aligned}
&\frac{|A_{i_1}|}{|A_{i_1,n-1}|}+\cdots+\frac{|A_{i_k}|}{|A_{i_k,n-1}|}\\
=&\frac{|A_1|}{|A_{1,n-1}|}+\cdots+\frac{|A_m|}{|A_{m,n-1}|}-\sum_{j\in J}\frac{|A_j|}{|A_{j,n-1}|},
\end{aligned}
$$

where $J=\{1,2,\ldots,m\}-\{i_1,i_2,\cdots,i_k\}$.

Denote the right part of inequality (8) by I.

By calculation,

$$
\begin{aligned}
I=&\sum_{k=1}^{m}(-1)^{m-k}\sum_{1\leqslant i_1<i_2<\cdots<i_k\leqslant m}|A_{i_1,n-1}+A_{i_2,n-1}+\cdots+A_{i_k,n-1}|\\
&\Big(\frac{|A_1|}{|A_{1,n-1}|}+\cdots+\frac{|A_m|}{|A_{m,n-1}|}-\sum_{j\in J}\frac{|A_j|}{|A_{j,n-1}|}\Big)\\
=&S(A_{1,n-1},A_{2,n-1},\cdots,A_{m,n-1})\Big(\frac{|A_1|}{|A_{1,n-1}|}+\cdots+\frac{|A_m|}{|A_{m,n-1}|}\Big)\\
&+\sum_{k=1}^{m-1}(-1)^{m-1-k}\sum_{1\leqslant i_1<i_2<\cdots<i_k\leqslant m}|A_{i_1,n-1}+A_{i_2,n-1}+\cdots+A_{i_k,n-1}|\Big(\sum_{j\in J}\frac{|A_j|}{|A_{j,n-1}|}\Big)\\
=&S(A_{1,n-1},A_{2,n-1},\cdots,A_{m,n-1})\Big(\frac{|A_1|}{|A_{1,n-1}|}+\cdots+\frac{|A_m|}{|A_{m,n-1}|}\Big)\\
&+S(A_{2,n-1},A_{3,n-1},\cdots,A_{m,n-1})\frac{|A_1|}{|A_{1,n-1}|}\\
&+S(A_{1,n-1},A_{3,n-1},\cdots,A_{m,n-1})\frac{|A_2|}{|A_{2,n-1}|}\\
&+\cdots+S(A_{1,n-1},A_{2,n-1},\cdots,A_{m-1,n-1})\frac{|A_m|}{|A_{m,n-1}|}.
\end{aligned}
$$

And by assumption,

$$
\begin{aligned}
I\geqslant&f_m(n-1)\sqrt[m]{|A_{1,n-1}A_{2,n-1}\cdots A_{m,n-1}|}\Big(\frac{|A_1|}{|A_{1,n-1}|}+\cdots+\frac{|A_m|}{|A_{m,n-1}|}\Big)\\
&+f_{m-1}(n-1)\sqrt[m-1]{|A_{2,n-1}A_{3,n-1}\cdots A_{m,n-1}|}\frac{|A_1|}{|A_{1,n-1}|}\\
&+f_{m-1}(n-1)\sqrt[m-1]{|A_{1,n-1}A_{3,n-1}\cdots A_{m,n-1}|}\frac{|A_2|}{|A_{2,n-1}|}\\
&+\cdots+f_{m-1}(n-1)\sqrt[m-1]{|A_{1,n-1}A_{2,n-1}\cdots A_{m-1,n-1}|}\frac{|A_m|}{|A_{m,n-1}|}.
\end{aligned}
$$

And by basic inequality,

$$
\begin{aligned}
I&\geqslant mf_m(n-1)\sqrt[m]{|A_1\cdots A_m|}+mf_{m-1}(n-1)\sqrt[m]{|A_1\cdots A_m|}\\
&=f_m(n)\sqrt[m]{|A_1\cdots A_m|}.
\end{aligned}
$$

We are done. □

We have an interesting corollary.

COROLLARY 1. *Let $A_1, A_2, \cdots, A_m$ be positive definite $n \times n$ matrices. Then*

$$\begin{aligned}|\sum_{i=1}^{m} A_i| \geqslant & f_1(n) \sum_i |A_i| + f_2(n) \sum_{i<j} \sqrt{|A_i A_j|} \\ & + f_3(n) \sum_{i<j<t} \sqrt[3]{|A_i A_j A_t|} + \cdots + f_m(n) \sqrt[m]{|A_1 A_2 \cdots A_m|}.\end{aligned}$$

Proof. Notice that

$$\begin{aligned}& |A_1 + A_2 + \cdots + A_m| \\ = & S(A_1, A_2, \cdots, A_m) + \sum_{1 \leqslant i_1 < i_2 < \cdots < i_{m-1} \leqslant m} S(A_{i_1}, \cdots, A_{i_{m-1}}) \\ & + \sum_{1 \leqslant i_1 < i_2 < \ldots < i_{m-2} \leqslant m} S(A_{i_1}, \cdots, A_{i_{m-2}}) + \cdots + \sum_{i=1}^{m} S(A_i).\end{aligned}$$

Then use Theorem 2. □

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