Operators and Matrices
Volume 19, Number 2 (2025), 221–234
doi:10.7153/oam-2025-19-15

DAVIS–WIELANDT RADIUS INEQUALITIES FOR PRODUCTS OF HILBERT SPACE OPERATORS

XIAOMEI DONG*, YUZHEN GUO AND DEYU WU

(*Communicated by F. Kittaneh*)

Abstract. In this paper, we obtain some upper bounds for the Davis-Wielandt radius of the product of two bounded linear operators in a Hilbert space, from which we derive new upper bounds for the Davis-Wielandt radius of bounded linear operators. By providing some examples, we show that the upper bounds obtained here are better than the existing upper bounds in some situations.

1. Introduction

Let $\mathscr{B}(H)$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space H with inner product $\langle\cdot,\cdot\rangle$. For $T \in \mathscr{B}(H)$, T^* and $|T|$ denote the adjoint of T and absolute value of T (i.e., $|T| = (T^*T)^{\frac{1}{2}}$), respectively. The numerical range of $T \in \mathscr{B}(H)$ is defined as [20]

$$W(T) = \{\langle Tx, x\rangle : x \in H, \|x\| = 1\}.$$

The numerical radius and the operator norm of T, denoted as $w(T)$ and $\|T\|$, respectively, are defined as

$$w(T) = \sup\{|\langle Tx, x\rangle| : x \in H, \|x\| = 1\}$$

and

$$\|T\| = \sup\{\|Tx\| : x \in H, \|x\| = 1\}.$$

The numerical range and numerical radius play an important role in various fields of operator theory, and attracted many researchers over the years. Due to the importance of the numerical range and the numerical radius, there have many generalizations of the numerical range and the numerical radius, see, e.g. [1, 10, 12, 22, 25] and the references

Mathematics subject classification (2020): 47A12, 47A30, 47A50.
Keywords and phrases: Davis-Wielandt radius, numerical radius, inequality, bounded linear operator.
The research is supported by the Natural Science Foundation of Inner Mongolia (Grant Nos. 2023QN01012, 2023MS01011), Inner Mongolia Autonomous Region Introduced High-level Talent Scientific Research Support Project (Grant No. DC2300001432) and University Talent Research Start-up Fund of Inner Mongolia University of Technology (Grant No. DC2300001264).
* Corresponding author.

therein. One of these generalizations is the Davis-Wielandt shell and its radius of $T \in \mathscr{B}(H)$ which are defined as [13, 21]

$$DW(T) = \{(\langle Tx,x\rangle, \langle T^*Tx,x\rangle) : x \in H, \|x\| = 1\} \subseteq \mathbb{C} \times \mathbb{R}$$

and

$$dw(T) = \sup\{\sqrt{|\langle Tx,x\rangle|^2 + \|Tx\|^4} : x \in H, \|x\| = 1\},$$

respectively.

It is not difficult to verify that for $T \in \mathscr{B}(H)$, the Davis-Wielandt radius $dw(\cdot)$ satisfies the following inequality:

$$\max\{w(T), \|T\|^2\} \leqslant dw(T) \leqslant \sqrt{w^2(T) + \|T\|^4}. \tag{1.1}$$

Recently, a lot of mathematicians have studied the Davis-Wielandt radius inequalities. For instance, in [11, 14] the authors proved that if $T \in \mathscr{B}(H)$, then

$$dw^2(T) \leqslant \||T|^2 + |T|^4\|^{\frac{1}{2}} \||T^*|^2 + |T|^4\|^{\frac{1}{2}}. \tag{1.2}$$

Also in [8] proved that

$$dw^2(T) \leqslant \frac{1}{2}\||T|^2 + |T^*|^2 + 2|T|^4\|, \tag{1.3}$$

$$dw^2(T) \leqslant \||T|^2 + |T|^4\|, \tag{1.4}$$

$$dw^2(T) \leqslant \frac{1}{2}(w(T^2) + \|T\|^2) + \|T\|^4. \tag{1.5}$$

In [24] Zamani et. al. proved that

$$\begin{aligned} dw^2(T) \leqslant{}& \frac{1}{2}w(T^2) + \frac{1}{4}w(|T|^2 + |T^*|^2) \\ &+ 4w^2(T)(2w^2(T) - c^2(T) + 2w(T)\sqrt{w^2(T) - c^2(T)}), \end{aligned} \tag{1.6}$$

$$dw^2(T) \leqslant \max\{w(T), w(|T|^2)\}(w(|T|^4 + |T|^2) + 2w(|T|^2B))^{\frac{1}{2}}, \tag{1.7}$$

$$dw^2(T) \leqslant \|T\| \max\{w(T), w(|T|^2)\}(1 + \|T\|^2 + 2w(T))^{\frac{1}{2}}, \tag{1.8}$$

where $c(T) = \inf\{|\langle Tx,x\rangle| : x \in H, \|x\| = 1\}$. For more results on the Davis-Wielandt radius inequalities, we refer the readers to [4–7, 9, 15, 18, 23].

In this paper, we give some Davis-Wielandt radius inequalities for the product of two bounded linear operators. Furthermore, we obtain several new upper bounds for the Davis-Wielandt radius of bounded linear operators, and show that these upper bounds are better than the existing ones mentioned above by numerical examples.

2. Main results

To prove our main results, we need the following well-known lemmas.

LEMMA 2.1. ([19]) *Let $a,b \geqslant 0$, $0 \leqslant \alpha \leqslant 1$ and $p,q > 1$ such that $\frac{1}{p}+\frac{1}{q}=1$. Then*

(a) $a^{\alpha}b^{1-\alpha} \leqslant \alpha a+(1-\alpha)b \leqslant (\alpha a^r+(1-\alpha)b^r)^{\frac{1}{r}}$ *for* $r \geqslant 1$;

(b) $ab \leqslant \frac{a^p}{p}+\frac{b^q}{q} \leqslant (\frac{1}{p}a^{pr}+\frac{1}{q}b^{qr})^{\frac{1}{r}}$ *for* $r \geqslant 1$.

LEMMA 2.2. ([3]) *Let $a,b>0$ and $\alpha,\beta>1$ such that $\frac{1}{\alpha}+\frac{1}{\beta}=1$, then*

$$(a^{\frac{1}{\alpha}}b^{\frac{1}{\beta}})^m + r_0^m(a^{\frac{m}{2}}-b^{\frac{m}{2}})^2 \leqslant \left(\frac{a}{\alpha}+\frac{b}{\beta}\right)^m,$$

where $r_0=\min\{\frac{1}{\alpha},\frac{1}{\beta}\}$ and $m=1,2,\cdots$. In particular, if $\alpha=\beta=2$, then

$$(a^{\frac{1}{2}}b^{\frac{1}{2}})^m + \left(\frac{1}{2}\right)^m(a^{\frac{m}{2}}-b^{\frac{m}{2}})^2 \leqslant 2^{-m}(a+b)^m. \tag{2.1}$$

LEMMA 2.3. ([17]) *Let $T \in \mathscr{B}(H)$ be a nonnegative bounded linear operator and $x \in H$ be any unit vector. Then*

(a) $\langle Tx,x\rangle^p \leqslant \langle T^px,x\rangle$, $p \geqslant 1$;

(b) $\langle T^px,x\rangle \leqslant \langle Tx,x\rangle^p$, $0<p\leqslant 1$.

LEMMA 2.4. ([17]) *Let $T \in \mathscr{B}(H)$ and $x,y \in H$ be any vectors. If f,g are nonnegative continuous functions on $[0,\infty)$ which are satisfying the relation $f(t)g(t)=t$ $(t \geqslant 0)$, then*

$$|\langle Tx,y\rangle|^2 \leqslant \langle f^2(|T|)x,x\rangle\langle g^2(|T^*|)y,y\rangle.$$

In particular, if $f(t)=g(t)=t^{\frac{1}{2}}(t\geqslant 0)$, then

$$|\langle Tx,y\rangle|^2 \leqslant \langle |T|x,x\rangle\langle |T^*|y,y\rangle.$$

LEMMA 2.5. ([17]) *Let $A,B \in \mathscr{B}(H)$ such that $|A|B=B^*|A|$. If f,g are nonnegative continuous functions on $[0,\infty)$ which are satisfying the relation $f(t)g(t)=t$ $(t\in[0,\infty))$, then*

$$|\langle ABx,y\rangle| \leqslant r(B)\|f(|A|)x\|\|g(|A^*|)y\|$$

for every $x,y\in H$. Here $r(B)$ denotes the spectral radius of B.

Our first main result can be stated as follows.

THEOREM 2.1. *Let $A,B \in \mathscr{B}(H)$ such that $|A|B = B^*|A|$. If f and g are non-negative continuous functions on $[0,\infty)$ satisfying $f(t)g(t) = t$ $(t \geqslant 0)$, then*

$$dw^{2p}(AB) \leqslant 2^{p-1} \max\{r^{2p}(B), 1\} \min\{\beta, \gamma\}, \tag{2.2}$$

for all $p \geqslant 1$, where

$$\beta = \|f^{4p}(|A|) + (B^* f^{2p}(|A^*A|)B)^2\|^{\frac{1}{2}} \|g^{4p}(|A^*|) + (B^* g^{2p}(|A^*A|)B)^2\|^{\frac{1}{2}},$$
$$\gamma = \|f^{4p}(|A|) + (B^* g^{2p}(|A^*A|)B)^2\|^{\frac{1}{2}} \|g^{4p}(|A^*|) + (B^* f^{2p}(|A^*A|)B)^2\|^{\frac{1}{2}}.$$

Proof. Let $x \in H$ be a unit vector. Then

$$\begin{aligned}
&(|\langle ABx,x\rangle|^2 + \|ABx\|^4)^p \\
&\leqslant 2^{p-1} \left(|\langle ABx,x\rangle|^{2p} + \langle A^*ABx, Bx\rangle^{2p}\right) \\
&\leqslant 2^{p-1} \left(r^{2p}(B)\langle f^2(|A|)x,x\rangle^p \langle g^2(|A^*|)x,x\rangle^p \right. \\
&\quad \left. + \langle B^* f^2(|A^*A|)Bx,x\rangle^p \langle B^* g^2(|A^*A|)Bx,x\rangle^p\right) \qquad \text{(by Lemmas 2.4 and 2.5)} \\
&\leqslant 2^{p-1} \left(\max\{r^{2p}(B),1\}(\langle f^{2p}(|A|)x,x\rangle \langle g^{2p}(|A^*|)x,x\rangle \right. \\
&\quad \left. + \langle (B^* f^2(|A^*A|)B)^p x,x\rangle \langle (B^* g^2(|A^*A|)B)^p x,x\rangle)\right) \\
&\leqslant 2^{p-1} \max\{r^{2p}(B),1\}(\langle f^{2p}(|A|)x,x\rangle^2 + \langle (B^* f^2(|A^*A|)B)^p x,x\rangle^2)^{\frac{1}{2}} \\
&\quad \times (\langle g^{2p}(|A^*|)x,x\rangle^2 + \langle (B^* g^2(|A^*A|)B)^p x,x\rangle^2)^{\frac{1}{2}} \\
&\leqslant 2^{p-1} \max\{r^{2p}(B),1\} \langle (f^{4p}(|A|) + (B^* f^2(|A^*A|)B)^{2p})x,x\rangle^{\frac{1}{2}} \\
&\quad \times \langle (g^{4p}(|A^*|) + (B^* g^2(|A^*A|)B)^{2p})x,x\rangle^{\frac{1}{2}}. \qquad \text{(by Lemma 2.3(a))}
\end{aligned}$$

Now, taking supremum over all unit vectors in H, we get

$$\begin{aligned}
dw^{2p}(AB) \leqslant 2^{p-1} \max\{r^{2p}(B),1\} &\|f^{4p}(|A|) + (B^* f^2(|A^*A|)B)^{2p}\|^{\frac{1}{2}} \\
&\times \|g^{4p}(|A^*|) + (B^* g^2(|A^*A|)B)^{2p}\|^{\frac{1}{2}}.
\end{aligned}$$

Similarly, we can show that

$$\begin{aligned}
dw^{2p}(AB) \leqslant 2^{p-1} \max\{r^{2p}(B),1\} &\|f^{4p}(|A|) + (B^* g^2(|A^*A|)B)^{2p}\|^{\frac{1}{2}} \\
&\times \|g^{4p}(|A^*|) + (B^* f^2(|A^*A|)B)^{2p}\|^{\frac{1}{2}}.
\end{aligned}$$

This completes the proof of the inequality (2.2). □

REMARK 2.1. In particular, if we take $B = I$ in Theorem 2.1, then we get the inequality in [14, Theorem 3.2], i.e.,

$$\begin{aligned}
dw^{2p}(A) \leqslant 2^{p-1} \min\{&\|f^{4p}(|A|) + f^{4p}(|A^*A|)\|^{\frac{1}{2}} \|g^{4p}(|A^*|) + g^{4p}(|A^*A|)\|^{\frac{1}{2}}, \\
&\|f^{4p}(|A|) + g^{4p}(|A^*A|)\|^{\frac{1}{2}} \|g^{4p}(|A^*|) + f^{4p}(|A^*A|)\|^{\frac{1}{2}}\},
\end{aligned} \tag{2.3}$$

for all $p \geqslant 1$.

Considering $f(t) = g(t) = t^{\frac{1}{2}}$ in Theorem 2.1, we get the following corollary.

COROLLARY 2.1. *Let $A, B \in \mathscr{B}(H)$ such that $|A|B = B^*|A|$. Then*

$$dw^2(AB) \leqslant \max\{r^2(B), 1\}\||A|^2 + (|A|B)^4\|^{\frac{1}{2}}\||A^*|^2 + (|A|B)^4\|^{\frac{1}{2}}. \tag{2.4}$$

REMARK 2.2. In particular, if we take $B = I$, $A = T$ in the inequality (2.4), then we get the inequality (1.2). If we take $A = I$, $B = T$ in the inequality (2.4), then we have

$$dw^2(T) \leqslant \max\{r^2(T), 1\}\|I + T^4\|. \tag{2.5}$$

Now, we consider an example to show that the inequality (2.5) is better than the existing ones.

EXAMPLE 2.1. The inequalities in [11, Theorem 2.10], [24, Theorems 2.1, 2.2, 2.5, 2.11], [9, Theorem 2.4] and [4, Theorem 4] are shown, respectively, as follows

$$\begin{aligned}
dw^2(T) &\leqslant \frac{1}{4}w^2(|T| + i|T^*|) + \frac{1}{4}w(|T||T^*|) \\
&\quad + \frac{1}{8}\min\{\||T|^2 + |T^*|^2 + 8|T|^4\|, \||T|^2 + |T^*|^2 + 8|T^*|^4\|\}, \\
dw^2(T) &\leqslant w^2(|T|^2 - T) + 2\|T\|^2 w(T), \\
dw^2(T) &\leqslant \frac{1}{2}w(|T|^2 + 2|T|^4 + |T^*|^2) - \frac{1}{2}\inf_{\|x\|=1}(\|Tx\| - \|T^*x\|)^2, \\
dw^2(T) &\leqslant \frac{1}{4}\left(w((|T|^2 + T)^2) + w((|T|^2 - T)^2) + w(|T|^2 + 2|T|^4 + |T^*|^2)\right), \\
dw^2(T) &\leqslant (w(|T|^4 + |T|^8) + 2w^2(|T|^2T))^{\frac{1}{2}}, \\
dw^2(T) &\leqslant \frac{1}{2}\min\{\|(|T| + |T^*|)^2 + 2|T|^4\|, \|(|T| + |T^*|)^2 + 2|T^*|^4\|\} - c(|T|)c(|T^*|) \\
dw^2(T) &\leqslant \left\|\frac{1}{4}(|T| + |T^*|)^2 + |T|^4\right\|.
\end{aligned}$$

If we take $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, then from (2.5) we get $dw^2(T) \leqslant 1$, whereas the inequalities in (1.4), (1.5), (1.6), (1.7), (1.8), [11, Theorem 2.10], [24, Theorems 2.1, 2.2, 2.5, 2.11], [9, Theorem 2.4] and [4, Theorem 4] give $dw^2(T) \leqslant 2$, $dw^2(T) \leqslant 1.5$, $dw^2(T) \leqslant 1.25$, $dw^2(T) \leqslant 1.414$, $dw^2(T) \leqslant 1.732$, $dw^2(T) \leqslant 1.375$, $dw^2(T) \leqslant 2.457$, $dw^2(T) \leqslant 1.5$, $dw^2(T) \leqslant 1.354$, $dw^2(T) \leqslant 1.414$, $dw^2(T) \leqslant 1.5$ and $dw^2(T) \leqslant 1.25$, respectively. Thus, for this example, the upper bound of $dw^2(T)$ in the inequality (2.5) is better than the existing bounds mentioned above.

Next we obtain the following upper bound for the Davis-Wielandt radius of the product of two bounded linear operators.

THEOREM 2.2. *Let $A,B \in \mathscr{B}(H)$ such that $|A|B = B^*|A|$. If f and g are non-negative continuous functions on $[0,\infty)$ satisfying $f(t)g(t) = t$ $(t \geqslant 0)$, then for all $p \geqslant 1$, $\alpha,\beta > 1$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, we have*

$$dw^{2p}(AB) \leqslant 2^{p-1} \max\{r^{2p}(B), 1\} \left\| \frac{1}{\alpha} X + \frac{1}{\beta} Y \right\|, \tag{2.6}$$

where

$$X = f^{2\alpha p}(|A|) + (B^* f^2(|A^*A|)B)^{p\alpha},$$
$$Y = g^{2p\beta}(|A^*|) + (B^* g^2(|A^*A|)B)^{p\beta}.$$

Proof. Let $x \in H$ be a unit vector. Then

$$\begin{aligned}
&(|\langle ABx,x\rangle|^2 + \|ABx\|^4)^p \\
&\leqslant 2^{p-1} \left(|\langle ABx,x\rangle|^{2p} + \langle A^*ABx, Bx\rangle^{2p} \right) \\
&\leqslant 2^{p-1} \left(r^{2p}(B) \langle f^2(|A|)x,x\rangle^p \langle g^2(|A^*|)x,x\rangle^p \right. \\
&\quad \left. + \langle B^* f^2(|A^*A|)Bx,x\rangle^p \langle B^* g^2(|A^*A|)Bx,x\rangle^p \right) \\
&\leqslant 2^{p-1} \left(r^{2p}(B) \left(\frac{1}{\alpha} \langle f^2(|A|)x,x\rangle^{p\alpha} + \frac{1}{\beta} \langle g^2(|A^*|)x,x\rangle^{p\beta} \right) \right. \\
&\quad \left. + \frac{1}{\alpha} \langle B^* f^2(|A^*A|)Bx,x\rangle^{p\alpha} + \frac{1}{\beta} \langle B^* g^2(|A^*A|)Bx,x\rangle^{p\beta} \right) \quad \text{(by Lemma 2.1(b))} \\
&\leqslant 2^{p-1} \left(r^{2p}(B) \left(\frac{1}{\alpha} \langle f^{2p\alpha}(|A|)x,x\rangle + \frac{1}{\beta} \langle g^{2p\beta}(|A^*|)x,x\rangle \right) \right. \\
&\quad \left. + \frac{1}{\alpha} \langle (B^* f^2(|A^*A|)B)^{p\alpha} x,x\rangle + \frac{1}{\beta} \langle (B^* g^2(|A^*A|)B)^{p\beta} x,x\rangle \right) \quad \text{(by Lemma 2.3(a))} \\
&\leqslant 2^{p-1} \max\{r^{2p}(B), 1\} \left\langle \left(\frac{1}{\alpha} (f^{2p\alpha}(|A|) + (B^* f^2(|A^*A|)B)^{p\alpha}) \right. \right. \\
&\quad \left. \left. + \frac{1}{\beta} (g^{2p\beta}(|A^*|) + (B^* g^2(|A^*A|)B)^{p\beta}) \right) x,x \right\rangle.
\end{aligned}$$

Taking the supremum over all unit vectors $x \in H$, we have

$$\begin{aligned}
dw^{2p}(AB) \leqslant 2^{p-1} \max\{r^{2p}(B), 1\} \left\| \frac{1}{\alpha} (f^{2p\alpha}(|A|) + (B^* f^2(|A^*A|)B)^{p\alpha}) \right. \\
\left. + \frac{1}{\beta} (g^{2p\beta}(|A^*|) + (B^* g^2(|A^*A|)B)^{p\beta}) \right\|.
\end{aligned}$$

This completes the proof. □

Considering $f(t) = g(t) = t^{\frac{1}{2}}$, $\alpha = \beta = 2$ and $p = 1$ in Theorem 2.2, we get the following corollary.

COROLLARY 2.2. *Let $A,B \in \mathscr{B}(H)$ such that $|A|B = B^*|A|$. Then*

$$dw^2(AB) \leqslant \frac{1}{2}\max\{r^2(B),1\}\||A|^2+|A^*|^2+2(|A|B)^4\|. \tag{2.7}$$

REMARK 2.3. In particular, if we take $B = I$, $A = T$ in the inequality (2.7), then we get the inequality (1.3). If we take $A = I$, $B = T$ in the inequality (2.7), then we get the inequality (2.5) again.

In the next theorem, we obtain another upper bound for the Davis-Wielandt radius of the product of two bounded linear operators.

THEOREM 2.3. *Let $A,B \in \mathscr{B}(H)$ such that $|A|B = B^*|A|$. If f and g are non-negative continuous functions on $[0,\infty)$ satisfying $f(t)g(t) = t$ $(t \geqslant 0)$, then for all $p \geqslant 1$, we have*

$$\begin{aligned} dw^{2p}(AB) \leqslant 2^{p-2}\max\{r^{2p}(B),1\}(\|f^{4p}(|A|)+g^{4p}(|A^*|)+(B^*f^2(|A^*A|)B)^{2p} \\ +(B^*g^2(|A^*A|)B)^{2p}\| - \inf_{\|x\|=1}\xi_1(x) - \inf_{\|x\|=1}\xi_2(x)), \end{aligned}$$

where

$$\begin{aligned} \xi_1(x) &= (\langle f^{4p}(|A|)x,x\rangle^{\frac{1}{2}} - \langle g^{4p}(|A^*|)x,x\rangle^{\frac{1}{2}})^2, \\ \xi_2(x) &= (\langle (B^*f^2(|A^*A|)B)^{2p}x,x\rangle^{\frac{1}{2}} - \langle (B^*g^2(|A^*A|)B)^{2p}x,x\rangle^{\frac{1}{2}})^2. \end{aligned}$$

Proof. Let $x \in H$ be a unit vector. Then

$$\begin{aligned} &(|\langle ABx,x\rangle|^2+\|ABx\|^4)^p \\ &\leqslant 2^{p-1}\left(|\langle ABx,x\rangle|^{2p}+\langle A^*ABx,Bx\rangle^{2p}\right) \\ &\leqslant 2^{p-1}\left(r^{2p}(B)\langle f^2(|A|)x,x\rangle^p\langle g^2(|A^*|)x,x\rangle^p\right. \\ &\quad \left.+\langle B^*f^2(|A^*A|)Bx,x\rangle^p\langle B^*g^2(|A^*A|)Bx,x\rangle^p\right) \\ &\leqslant 2^{p-1}\left(r^{2p}(B)\langle f^{2p}(|A|)x,x\rangle\langle g^{2p}(|A^*|)x,x\rangle\right. \\ &\quad \left.+\langle (B^*f^2(|A^*A|)B)^px,x\rangle\langle (B^*g^2(|A^*A|)B)^px,x\rangle\right) \\ &\leqslant 2^{p-2}\max\{r^{2p}(B),1\}\left(\langle f^{4p}(|A|)x,x\rangle+\langle g^{4p}(|A^*|)x,x\rangle\right. \\ &\quad -(\langle f^{4p}(|A|)x,x\rangle^{\frac{1}{2}} - \langle g^{4p}(|A^*|)x,x\rangle^{\frac{1}{2}})^2 \\ &\quad +\langle (B^*f^2(|A^*A|)B)^{2p}x,x\rangle+\langle (B^*g^2(|A^*A|)B)^{2p}x,x\rangle \\ &\quad \left.-(\langle (B^*f^2(|A^*A|)B)^{2p}x,x\rangle^{\frac{1}{2}} - \langle (B^*g^2(|A^*A|)B)^{2p}x,x\rangle^{\frac{1}{2}})^2\right). \quad \text{(by Lemma 2.2)} \end{aligned}$$

Now taking the supremum over all unit vectors $x \in H$, we deduce the required inequality. □

Considering $f(t) = g(t) = t^{\frac{1}{2}}$ and $p = 1$ in Theorem 2.3, we get the following corollary.

COROLLARY 2.3. *Let $A,B\in\mathscr{B}(H)$ such that $|A|B=B^*|A|$. Then*

$$dw^2(AB)\leqslant\frac{1}{2}\max\{r^2(B),1\}\||A|^2+|A^*|^2+2(B^*|A|^2B)^2\| \tag{2.8}$$
$$-\frac{1}{2}\inf_{\|x\|=1}(\langle|A|^2x,x\rangle^{\frac{1}{2}}-\langle|A^*|^2x,x\rangle^{\frac{1}{2}})^2,$$

which is an improvement of the inequality (2.7) in Corollary 2.2.

REMARK 2.4. In particular, if we take $B=I$, $A=T$ in the inequality (2.8), then we have

$$dw^2(T)\leqslant\frac{1}{2}\||T|^2+|T^*|^2+2|T|^4\|-\inf_{\|x\|=1}(\langle|T|^2x,x\rangle^{\frac{1}{2}}-\langle|T^*|^2x,x\rangle^{\frac{1}{2}})^2,$$

which is an improvement of the inequality (1.3). If we take $A=I$, $B=T$ in the inequality (2.8), then we get the inequality (2.5) again.

In order to prove our next result, we need the following lemma.

LEMMA 2.6. ([16]) *Let $x,y,e\in H$ with $\|e\|=1$ and $\alpha\in\mathbb{C}\backslash\{0\}$. Then*

$$|\langle x,e\rangle\langle e,y\rangle|\leqslant\frac{1}{|\alpha|}(\max\{1,|\alpha-1|\}\|x\|\|y\|+|\langle x,y\rangle|). \tag{2.9}$$

THEOREM 2.4. *Let $A,B\in\mathscr{B}(H)$. Then for $\alpha\in\mathbb{C}\backslash\{0\}$ and $p\geqslant1$, we have*

$$dw^{2p}(AB)\leqslant2^{p-1}\left(\frac{\max\{1,|1-\alpha|\}}{2|\alpha|}\||B|^{4p}+|A^*|^{4p}\|+\frac{1}{|\alpha|}w(|A^*|^{2p}|B|^{2p})+\|AB\|^{4p}\right).$$

Proof. Let $x\in H$ be a unit vector. Then

$$\begin{aligned}
&|\langle ABx,x\rangle|^{2p}\\
&=|\langle Bx,A^*x\rangle|^{2p}\\
&\leqslant\|Bx\|^{2p}\|A^*x\|^{2p}\\
&=\langle|B|^2x,x\rangle^p\langle|A^*|^2x,x\rangle^p\\
&\leqslant|\langle|B|^{2p}x,x\rangle\langle x,|A^*|^{2p}x\rangle|\\
&\leqslant\frac{1}{|\alpha|}\left(\max\{1,|1-\alpha|\}\||B|^{2p}\|\||A^*|^{2p}\|+|\langle|B|^{2p}x,|A^*|^{2p}x\rangle|\right)\\
&\leqslant\frac{1}{|\alpha|}\left(\frac{\max\{1,|1-\alpha|\}}{2}\||B|^{2p}\|^2+\||A^*|^{2p}\|^2+|\langle|A^*|^{2p}|B|^{2p}x,x\rangle|\right)\\
&=\frac{\max\{1,|1-\alpha|\}}{2|\alpha|}\langle(|B|^{4p}+|A^*|^{4p})x,x\rangle+\frac{1}{|\alpha|}|\langle|A^*|^{2p}|B|^{2p}x,x\rangle|\\
&\leqslant\frac{\max\{1,|1-\alpha|\}}{2|\alpha|}\||B|^{4p}+|A^*|^{4p}\|+\frac{1}{|\alpha|}w(|A^*|^{2p}|B|^{2p}).
\end{aligned}$$

Therefore

$$\begin{aligned}
&(|\langle ABx,x\rangle|^2+\|ABx\|^4)^p\\
&\leqslant 2^{p-1}(|\langle ABx,x\rangle|^{2p}+\|ABx\|^{4p})\\
&\leqslant 2^{p-1}\left(\frac{\max\{1,|1-\alpha|\}}{2|\alpha|}\||B|^{4p}+|A^*|^{4p}\|+\frac{1}{|\alpha|}w(|A^*|^{2p}|B|^{2p})+\|AB\|^{4p}\right).
\end{aligned}$$

Taking the supremum over all unit vectors $x\in H$, we have

$$dw^{2p}(AB)\leqslant 2^{p-1}\left(\frac{\max\{1,|1-\alpha|\}}{2|\alpha|}\||B|^{4p}+|A^*|^{4p}\|+\frac{1}{|\alpha|}w(|A^*|^{2p}|B|^{2p})+\|AB\|^{4p}\right).$$

This completes the proof. □

Considering $\alpha=2,\ p=1$ in Theorem 2.5, we get the following corollary.

COROLLARY 2.4. *Let $A,B\in\mathscr{B}(H)$. Then*

$$dw^2(AB)\leqslant\frac{1}{4}\||B|^4+|A^*|^4\|+\frac{1}{2}w(|A^*|^2|B|^2)+\|AB\|^4. \tag{2.10}$$

REMARK 2.5. By letting $A=B=I$ in Corollary 2.4, we obtain

$$dw^2(I)=2=\frac{1}{4}\||B|^4+|A^*|^4\|+\frac{1}{2}w(|A^*|^2|B|^2)+\|AB\|^4.$$

Therefore, the inequality of Corollary 2.4 is sharp.

REMARK 2.6. In particular, if we take $B=I,\ A=T$ in Corollary 2.4, then we have

$$dw^2(T)\leqslant\frac{1}{4}\|I+|T^*|^4\|+\frac{1}{2}w(|T^*|^2)+\|T\|^4. \tag{2.11}$$

If we take $A=I,\ B=T$ in Corollary 2.4, then we have

$$dw^2(T)\leqslant\frac{1}{4}\||T|^4+I\|+\frac{1}{2}w(|T|^2)+\|T\|^4. \tag{2.12}$$

EXAMPLE 2.2. The inequalities in [24, Theorems 2.13, 2.14] and [9, Theorem 2.4] are shown, respectively, as follows

$$\begin{aligned}
&dw^2(T)\leqslant\max\{\|T\|^2,\|T\|^4\}+\sqrt{2}w(|T|^2T),\\
&dw^2(T)\leqslant\frac{1}{2}(w(|T|^4+|T|^2)+w(|T|^4-|T|^2))+\sqrt{2}w(|T|^2T),\\
&dw^2(T)\leqslant\frac{1}{2}\min\{\|(|T|+|T^*|)^2+2|T|^4\|,\|(|T|+|T^*|)^2+2|T^*|^4\|\}-c(|T|)c(|T^*|).
\end{aligned}$$

If we consider $T = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$, then from (2.12) we get $dw^2(T) \leqslant 22.25$, whereas the inequalities in (1.6), (1.7), (1.8), [24, Theorems2.13, 2.14] and [9, Theorem 2.4] give $dw^2(T) \leqslant 260$, $dw^2(T) \leqslant 24$, $dw^2(T) \leqslant 24$, $dw^2(T) \leqslant 27.314$, $dw^2(T) \leqslant 27.314$ and $dw^2(T) \leqslant 24$, respectively. Thus, for this example, the upper bounds of $dw^2(T)$ in (2.12) is better than the existing bounds mentioned above.

Next we need the following lemma.

LEMMA 2.7. ([2]) *Let $x, y, e \in H$ with $\|e\| = 1$. Then*

$$|\langle x, e\rangle\langle e, y\rangle|^p \leqslant \frac{1+\alpha}{2}\|x\|^p\|y\|^p + \frac{1-\alpha}{2}|\langle x, y\rangle|^p.$$

for every $0 \leqslant \alpha \leqslant 1$ and $p \geqslant 1$.

Now we prove the following theorem by applying the lemma 2.7.

THEOREM 2.5. *Let $A, B \in \mathscr{B}(H)$. Then for $0 \leqslant \alpha \leqslant 1$ and $p \geqslant 1$, we have*

$$dw^{2p}(AB) \leqslant 2^{p-1}\left(\frac{1+\alpha}{4}\||B|^{4p} + |A^*|^{4p}\| + \frac{1-\alpha}{2}w(|A^*|^{2p}|B|^{2p}) + \|AB\|^{4p}\right).$$

Proof. Let $x \in H$ be a unit vector. Then

$$\begin{aligned}
&|\langle ABx, x\rangle|^{2p} \\
&= |\langle Bx, A^*x\rangle|^{2p} \\
&\leqslant \|Bx\|^{2p}\|A^*x\|^{2p} \\
&= |\langle |B|^2x, x\rangle\langle x, |A^*|^2x\rangle|^p \\
&\leqslant \frac{1+\alpha}{2}\||B|^2x\|^p\||A^*|^2x\|^p + \frac{1-\alpha}{2}|\langle |B|^{2p}x, |A^*|^{2p}x\rangle| \\
&\leqslant \frac{1+\alpha}{4}(\langle |B|^4x, x\rangle^p + \langle |A^*|^4x, x\rangle^p) + \frac{1-\alpha}{2}|\langle |B|^{2p}x, |A^*|^{2p}x\rangle| \\
&\leqslant \frac{1+\alpha}{4}\langle (|B|^{4p} + |A^*|^{4p})x, x\rangle + \frac{1-\alpha}{2}|\langle |A^*|^{2p}|B|^{2p}x, x\rangle| \\
&\leqslant \frac{1+\alpha}{4}\||B|^{4p} + |A^*|^{4p}\| + \frac{1-\alpha}{2}w(A^*|^{2p}|B|^{2p}).
\end{aligned}$$

Therefore

$$\begin{aligned}
&(|\langle ABx, x\rangle|^2 + \|ABx\|^4)^p \\
&\leqslant 2^{p-1}(|\langle ABx, x\rangle|^{2p} + \|ABx\|^{4p}) \\
&\leqslant 2^{p-1}\left(\frac{1+\alpha}{4}\||B|^{4p} + |A^*|^{4p}\| + \frac{1-\alpha}{2}w(|A^*|^{2p}|B|^{2p}) + \|AB\|^{4p}\right).
\end{aligned}$$

Taking the supremum over all unit vectors $x \in H$, we have

$$dw^{2p}(AB) \leqslant 2^{p-1}\left(\frac{1+\alpha}{4}\||B|^{4p}+|A^*|^{4p}\|+\frac{1-\alpha}{2}w(|A^*|^{2p}|B|^{2p})+\|AB\|^{4p}\right).$$

This completes the proof. □

REMARK 2.7. If we take $B=I$, $A=T$ and $p=1$ in Theorem 2.5, then we have

$$dw^2(T) \leqslant \frac{1+\alpha}{4}\|I+|T^*|^4\|+\frac{1-\alpha}{2}w(|T^*|^2)+\|T\|^4. \tag{2.13}$$

If we take $A=I$, $B=T$ and $p=1$ in Theorem 2.5, then we have

$$dw^2(T) \leqslant \frac{1+\alpha}{4}\||T|^4+I\|+\frac{1-\alpha}{2}w(|T|^2)+\|T\|^4. \tag{2.14}$$

In particular, if we take $\alpha=0$ in inequalities (2.13) and (2.14), then we get the inequalities (2.11) and (2.12), respectively.

Our final result for the Davis-Wielandt radius of the product of two bounded linear operators can be stated as follows.

THEOREM 2.6. *Let $A,B \in \mathscr{B}(H)$. Then for all $p \geqslant 1$, $\alpha,\beta>1$ with $\frac{1}{\alpha}+\frac{1}{\beta}=1$, we have*

$$dw^{2p}(AB) \leqslant 2^{p-1}\left(\left\|\frac{1}{\alpha}|B|^{2p\alpha}+\frac{1}{\beta}|A^*|^{2p\beta}+(B^*|A|^2B)^{2p}\right\|-\inf_{\|x\|=1}\delta(x)\right),$$

where

$$\delta(x)=\min\left\{\frac{1}{\alpha},\frac{1}{\beta}\right\}\left(\langle|B|^{2p\alpha}x,x\rangle^{\frac{1}{2}}-\langle|A^*|^{2p\beta}x,x\rangle^{\frac{1}{2}}\right)^2.$$

Proof. Let $x \in H$ be a unit vector. Then

$$\begin{aligned}
&|\langle ABx,x\rangle|^{2p}\\
&=|\langle Bx,A^*x\rangle|^{2p}\\
&\leqslant \|Bx\|^{2p}\|A^*x\|^{2p}\\
&=\langle|B|^2x,x\rangle^p\langle|A^*|^2x,x\rangle^p\\
&\leqslant \langle|B|^{2p}x,x\rangle\langle|A^*|^{2p}x,x\rangle\\
&\leqslant \langle|B|^{2p\alpha}x,x\rangle^{\frac{1}{\alpha}}\langle|A^*|^{2p\beta}x,x\rangle^{\frac{1}{\beta}}\\
&\leqslant \frac{1}{\alpha}\langle|B|^{2p\alpha}x,x\rangle+\frac{1}{\beta}\langle|A^*|^{2p\beta}x,x\rangle-\min\left\{\frac{1}{\alpha},\frac{1}{\beta}\right\}\left(\langle|B|^{2p\alpha}x,x\rangle^{\frac{1}{2}}-\langle|A^*|^{2p\beta}x,x\rangle^{\frac{1}{2}}\right)^2\\
&=\left\langle\left(\frac{1}{\alpha}|B|^{2p\alpha}+\frac{1}{\beta}|A^*|^{2p\beta}\right)x,x\right\rangle-\min\left\{\frac{1}{\alpha},\frac{1}{\beta}\right\}\left(\langle|B|^{2p\alpha}x,x\rangle^{\frac{1}{2}}-\langle|A^*|^{2p\beta}x,x\rangle^{\frac{1}{2}}\right)^2.
\end{aligned}$$

Therefore

$$\begin{aligned}
&(|\langle ABx,x\rangle|^2+\|ABx\|^4)^p\\
&\leqslant 2^{p-1}(|\langle ABx,x\rangle|^{2p}+\langle B^*|A|^2Bx,x\rangle^{2p})\\
&\leqslant 2^{p-1}\left(\left\langle\left(\frac{1}{\alpha}|B|^{2p\alpha}+\frac{1}{\beta}|A^*|^{2p\beta}\right)x,x\right\rangle\right.\\
&\quad\left.-\min\left\{\frac{1}{\alpha},\frac{1}{\beta}\right\}\left(\langle|B|^{2p\alpha}x,x\rangle^{\frac{1}{2}}-\langle|A^*|^{2p\beta}x,x\rangle^{\frac{1}{2}}\right)^2+\langle(B^*|A|^2B)^{2p}x,x\rangle\right)\\
&\leqslant 2^{p-1}\left(\left\|\frac{1}{\alpha}|B|^{2p\alpha}+\frac{1}{\beta}|A^*|^{2p\beta}+(B^*|A|^2B)^{2p}\right\|-\inf_{\|x\|=1}\delta(x)\right).
\end{aligned}$$

Now taking the supremum over all unit vectors $x\in H$, we deduce the required inequality. □

Considering $\alpha=\beta=2$ and $p=1$ in Theorem 2.6, we get the following corollary.

COROLLARY 2.5. *Let $A,B\in\mathscr{B}(H)$, then*

$$\begin{aligned}
dw^2(AB)\leqslant&\frac{1}{2}\||B|^4+|A^*|^4+2(B^*|A|^2B)^2\|\\
&-\frac{1}{2}\inf_{\|x\|=1}(\langle|B|^4x,x\rangle^{\frac{1}{2}}-\langle|A^*|^4x,x\rangle^{\frac{1}{2}})^2.
\end{aligned}\tag{2.15}$$

REMARK 2.8. In particular, if we take $B=I$, $A=T$ in the inequality (2.15), then we have

$$dw^2(T)\leqslant\frac{1}{2}\|I+|T^*|^4+2|T|^4\|-\frac{1}{2}\inf_{\|x\|=1}(1-\langle|T^*|^4x,x\rangle^{\frac{1}{2}})^2.\tag{2.16}$$

If we take $A=I$, $B=T$ in the inequality (2.15), then we have

$$dw^2(T)\leqslant\frac{1}{2}\|I+3|T|^4\|-\frac{1}{2}\inf_{\|x\|=1}(\langle|T|^4x,x\rangle^{\frac{1}{2}}-1)^2.\tag{2.17}$$

EXAMPLE 2.3. The inequalities in [8, Theorems 2.2] and [11, Theorem 2.10, 2.13] are shown, respectively, as follows

$$\begin{aligned}
dw^2(T)&\leqslant\frac{1}{2}\{w^2(T+T^*T)+w^2(T-T^*T)\},\\
dw^2(T)&\leqslant\frac{1}{4}w^2(|T|+i|T^*|)+\frac{1}{4}w(|T||T^*|)\\
&\quad+\frac{1}{8}\min\{\||T|^2+|T^*|^2+8|T|^4\|,\||T|^2+|T^*|^2+8|T^*|^4\|\},\\
dw^2(T)&\leqslant\min\{w^2(Re(T)+i|T|^2)+\|Im(T)\|^2,w^2(Im(T)+i|T|^2)+\|Re(T)\|^2\}.
\end{aligned}$$

If we consider $T = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$, then from (2.16) and (2.17) we get $dw^2(T) \leqslant 1.5$ and $dw^2(T) \leqslant 2$, whereas the inequalities in (1.2), (1.4), (1.5), [8, Theorems 2.2] and [11, Theorem 2.10, 2.13] give $dw^2(T) \leqslant 18.438$, $dw^2(T) \leqslant 19.999$, $dw^2(T) \leqslant 18.992$, $dw^2(T) \leqslant 21.354$, $dw^2(T) \leqslant 18.370$ and $dw^2(T) \leqslant 18.499$, respectively. Thus, for this example, the upper bounds of $dw^2(T)$ in (2.16) and (2.17) are better than the existing bounds mentioned above.

REFERENCES

[1] A. ABU-OMAR, F. KITTANEH, *A generalization of the numerical radius*, Linear Algebra Appl. **569**, 323–334 (2019).

[2] M. ALDOLAT, F. KITTANEH, *Upper bounds for the numerical radii of powers of Hilbert space operators*, Quaest. Math. **47**, 341–352 (2024).

[3] Y. AL-MANASRAH, F. KITTANEH, *A generalization of two refined Young inequalities*, Positivity. **19**, 757–768 (2015).

[4] M. W. ALOMARI, *On the Davis-Wielandt radius inequalities of Hilbert space operators*, Linear Multilinear A. **71**, 1804–1828 (2023).

[5] M. W. ALOMARI, M. BAKHERAD AND M. HAJMOHAMADI, *A generalization of the Davis-Wielandt radius for operators*, Bol. Soc. Mat. Mex. **30**, 1–18 (2024).

[6] N. ALTWAIJRY, S. S. DRAGOMIR, K. FEKI, S. FURUICHI, *New bounds for the Euclidean numerical radius of two Operators in Hilbert spaces*, Symmetry. **17**, 7 (2025).

[7] N. ALTWAIJRY, S. S. DRAGOMIR, K. FEKI, *Improved bounds for the Euclidean numerical radius of operator pairs in Hilbert spaces*, Mathematics. **12**, 2838 (2024).

[8] P. BHUNIA, A. BHANJA, S. BAG, K. PAUL, *Bounds for the Davis-Wielandt radius of bounded linear operators*, Ann. Funct. Anal. **12**, 1–23 (2021).

[9] P. BHUNIA, A. BHANJA, K. PAUL, *New inequalities for Davis-Wielandt radius of Hilbert space operators*, Bull. Malays. Math. Sci. Soc. **2**, 1–17 (2021).

[10] P. BHUNIA, K. PAUL, *Some improvements of numerical radius inequalities of operators and operator matrices*, Linear Multilinear A. **70**, 1995–2013 (2022).

[11] P. BHUNIA, K. PAUL, S. BARIK, *Further refinements of Davis-Wielandt radius inequalities*, Oper. Matrices. **17**, 767–778 (2023).

[12] N. CHANDRA ROUT, S. SAHOO, D. MISHRA, *Some A-numerical radius inequalities for semi-Hilbertian space operators*, Linear Multilinear A. **69**, 980–996 (2020).

[13] C. DAVIS, *The shell of a Hilbert space operator*, Acta Sci. Math. (Szeged) **29**, 69–86 (1968).

[14] X. M. DONG, D. Y. WU, *Davis-Wielandt radius inequalities for off-diagonal operator matrices*, Linear Multilinear A. **71**, 1158–1181 (2023).

[15] M. GUESBA, S. BARIK, P. BHUNIA, K. PUAL, *A-Davis-Wielandt radius bounds of semi-Hilbertian space operators*, Bull. Iran. Math. Soc. **50**, 1–21 (2024).

[16] M. KHOSRAVI, R. DRNOVSEK, M. S. MOSLEHIAN, *A commutator approach to Buzano's inequality*, Filomat. **26**, 827–832 (2012).

[17] F. KITTANEH, *Notes on some inequalities for Hilbert space operators*, Publ Res. Inst. for Math. Sci. **24**, 283–293 (1988).

[18] S. F. MOGHADDAM, A. K. MIRMOSTAFAEE AND M. JANFADA, *Some sharp estimations for Davis-Wielandt radius in $B(H)$*, Mediterr. J. Math. **19**, 283 (2022).

[19] M. SATTARI, M. S. MOSLEHIAN, T. YAMAZAKI, *Some generalized numerical radius inequalities for Hilbert space operators*, Linear Algebra Appl. **470**, 216–227 (2015).

[20] O. TOEPLITZ, *Das algebraische Analogon zu einem Satze von Fejér*, Math. Z. **2**, 187–197 (1918).

[21] H. WIELANDT, *On eigenvalues of sums of normal matrices*, Pacific J. Math. **5**, 633–638 (1955).

[22] A. ZAMANI, *Characterization of numerical radius parallelism in C^*-algebras*, Positivity. **23**, 397–411 (2019).

[23] A. ZAMANI, M. S. MOSLEHIAN, M.-T. CHIEN, H. NAKAZATO, *Norm-parallelism and the Davis-Wielandt radius of Hilbert space operators*, Linear Multilinear A. **67**, 2147–2158 (2019).

[24] A. ZAMANI, K. SHEBRAWI, *Some upper bounds for the Davis-Wielandt radius of Hilbert spaces operators*, Mediterr. J. Math. **17**, 1–13 (2019).
[25] A. ZAMANI, P. WÓJCIK, *Another generalization of the numerical radius for Hilbert space operators*, Linear Algebra Appl. **609**, 114–128 (2021).

(Received January 9, 2025)

Xiaomei Dong
College of Sciences
Inner Mongolia University of Technology
Hohhot, 010051, China
e-mail: xiaomd1205@126.com

Yuzhen Guo
College of Sciences
Inner Mongolia University of Technology
Hohhot, 010051, China
e-mail: gyz0504@126.com

Deyu Wu
School of Mathematical Sciences
Inner Mongolia University
Hohhot, 010021, China
e-mail: wudeyu2585@163.com