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NUCLEAR VOLTERRA COMPOSITION OPERATORS BETWEEN BANACH TYPE SPACES OF CONTROLLED GROWTH

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Abstract. In this paper, we provide a complete characterization of nuclear Volterra composition operators $T_g^\phi : \mathscr{H}_\nu^\infty \to \mathscr{H}_\mu^\infty$ and $S_g^\phi : \mathscr{H}_\nu^\infty \to \mathscr{H}_\mu^\infty$ acting between weighted type spaces in terms of the symbols g and ϕ, as well as the weights ν and μ, where ν and μ are normal weights in the sense of Shields and Williams. Furthermore, we also characterize nuclear Volterra composition operators acting between little weighted type spaces and Bloch spaces of order β.

Notation

This article involves extensive notation, which is generally explained throughout the text. However, due to the complexity of the topic, we provide a summary of the notation used in the article, which will greatly assist readers in following the material.

1. $\mathbb{D}$ is the open unit disk in the complex plane $\mathbb{C}$.
2. $\mathscr{H}(\mathbb{D})$ is the class of all analytic functions on $\mathbb{D}$.
3. dA is the normalized Lebesgue measure on $\mathbb{D}$.
4. ν and μ are normal weights in the sense of Shields and Williams.
5. $\widetilde{\nu}$ is the associate weight of ν.
6. ω is a positive, radial, continuous function on $\mathbb{D}$ such that
$$\int_0^1 \omega(r)\,dr < \infty.$$
7. g is a function in $\mathscr{H}(\mathbb{D})$.
8. ϕ is an analytic self-map of $\mathbb{D}$.
9. T_g is the integral operator.

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10. S_g is the companion integral operator.
11. T_g^ϕ is the Volterra composition operator.
12. S_g^ϕ is the companion Volterra composition operator.
13. A^1_ω is the weighted Bergman space with weight ω.
14. $\mathscr{H}^\infty_v$ is the weighted Banach space.
15. $\mathscr{H}^0_v$ is the little weighted Banach space.
16. $\mathscr{B}_v$ is the weighted Bloch space.
17. $\mathscr{B}^0_v$ is the little weighted Bloch space.
18. $N_W(\mathbb{D})$ is the set of all normal weights on $\mathbb{D}$.

1. Introduction

Denote by $\mathscr{H}(\mathbb{D})$ the class of all analytic functions on $\mathbb{D}$, where $\mathbb{D}$ is the open unit disk in the complex plane $\mathbb{C}$. For $g \in \mathscr{H}(\mathbb{D})$, Ch. Pommerenke [27] introduced an integral operator $T_g : \mathscr{H}(\mathbb{D}) \to \mathscr{H}(\mathbb{D})$ which maps every $f \in \mathscr{H}(\mathbb{D})$ to a function vanishing at 0, defined by

$$T_g f(z) = \int_0^z f(\zeta)\, dg(\zeta) = \int_0^z f(\zeta) g'(\zeta)\, d\zeta, \quad z \in \mathbb{D}.$$

Another integral operator, generally known as a companion operator of T_g, was defined by Yoneda [32] as

$$S_g f(z) = \int_0^z f'(\zeta) g(\zeta)\, d\zeta, \quad z \in \mathbb{D}.$$

The operators T_g and S_g can be viewed as cousins of Hankel and Toeplitz operators and have appeared in the literature under various names such as Volterra operators, Cesàro operators, Riemann-Stieltjes operators, and integration operators. They arise in various branches of physics, mechanics, complex analysis, functional analysis, operator theory, and other areas of mathematics.

For example, the sum of T_g and S_g is a multiplication operator induced by g. These operators are related to resolvent operators induced by semigroups of composition operators and also arise as Berezin transforms in the study of weighted composition operators.

For any $g \in \mathscr{H}(\mathbb{D})$ and ϕ an analytic self-map of $\mathbb{D}$, the Volterra composition operators T_g^ϕ and S_g^ϕ are defined on $\mathscr{H}(\mathbb{D})$ by

$$(T_g^\phi f)(z) = \int_0^z f(\phi(\zeta)) g'(\zeta)\, d\zeta, \quad z \in \mathbb{D},$$

and

$$(S_g^\phi f)(z) = \int_0^z f'(\phi(\zeta)) g(\zeta)\, d\zeta, \quad z \in \mathbb{D}.$$

Recently, these types of operators have been studied on spaces of analytic functions by several authors; see, for example, [1]–[6], [10], [15], [19]–[21] and references therein. Motivated by the results in [9] and [16], this paper characterizes nuclear Volterra composition operators acting between Bloch and weighted-type spaces of analytic functions, under the assumption that the weights involved are normal.

Recall that nuclear operators constitute an important subclass of compact operators that play a fundamental role in functional analysis. They extend the notions of trace and determinant – originally defined for finite-dimensional matrices – to infinite-dimensional settings. This extension is essential, as many problems in spectral theory and quantum physics involve operators on infinite-dimensional spaces. By enabling the calculation of traces and determinants in such contexts, nuclear operators offer powerful tools for analyzing spectral properties, describing quantum states, and formulating physical theories rigorously. Their use leads to a deeper understanding of complex systems beyond finite-dimensional approximations and generalizes the classical concept of trace-class operators from Hilbert spaces to Banach spaces.

In the context of partial differential equations (PDEs) and integral equations, nuclear operators frequently arise as integral operators with smooth or rapidly decaying kernels. This appearance helps develop effective solution methods and facilitates spectral analysis.

In this paper, we establish conditions under which Volterra composition operators qualify as nuclear. Volterra operators are instrumental in modeling systems exhibiting memory and hereditary effects, where the current state depends not only on present inputs but also on the entire past history. Such operators commonly occur in areas including viscoelasticity, population dynamics, and control theory. Conversely, composition operators often emerge from changes of variables or transformations characteristic of dynamical systems, capturing how functions evolve under these dynamics or coordinate changes.

Investigating the nuclearity of Volterra composition operators yields significant insights into their structure and behavior, with important consequences. For instance, nuclearity enables the design and rigorous analysis of approximation schemes by ensuring these operators can be approximated closely by finite-rank operators, a feature critical in numerical computations and simulations. Furthermore, nuclearity contributes to stability analyses of iterative methods by controlling the propagation of small perturbations through successive operator applications. Finally, these results impact regularity theory in nonlinear analysis by providing a solid framework to handle complex integral and functional equations, where enhanced control over operator properties yields stronger and more precise regularity conclusions. Collectively, these findings deepen our understanding of Volterra composition operators and broaden their applicability in both mathematical theory and physical applications.

Throughout this paper, the symbol C denotes a positive constant whose value may change at different occurrences. The notation $A \lesssim B$ means that there exists a constant $C > 0$ such that $A \leqslant CB$, and $A \asymp B$ means that there are positive constants C, D such that $CB \leqslant A \leqslant DB$.

2. Preliminaries

In this section, we define some spaces of analytic functions and recall auxiliary results that will be used in the proof of the main results of this paper.

Recall that a radial weight v is a positive, continuous, non-increasing function on $\mathbb{D}$, satisfying $v(z) = v(|z|)$ for all $z \in \mathbb{D}$. A radial weight v is said to be normal in the sense of Shields and Williams if it satisfies the following two conditions:

(**I**) There exists $\beta > 0$ such that the function $r \mapsto \dfrac{v(r)}{(1-r^2)^\beta}$ is almost increasing; equivalently,

$$\inf_n \frac{v\left(1-2^{-(n+1)}\right)}{v\left(1-2^{-n}\right)} > 0.$$

(**II**) There exists $\gamma > 0$ such that the function $r \mapsto \dfrac{v(r)}{(1-r^2)^\gamma}$ is almost decreasing; equivalently, there exists $k \in \mathbb{N}$ such that

$$\limsup_n \frac{v\left(1-2^{-(n+k)}\right)}{v\left(1-2^{-n}\right)} < 1.$$

It is clear that the standard weights $v_\alpha(z) = (1-|z|^2)^\alpha$, with $\alpha > 0$, are examples of normal weights. Throughout this paper, we denote the set of all normal weights on $\mathbb{D}$ by $N_W(\mathbb{D})$. For further details on normal weights, see [6], [14], and [29].

The weighted Banach spaces $\mathscr{H}_v^\infty$ and $\mathscr{H}_v^0$ are defined as follows:

$$\mathscr{H}_v^\infty = \left\{ f \in \mathscr{H}(\mathbb{D}) : \|f\|_{\mathscr{H}_v^\infty} = \sup_{z\in\mathbb{D}} v(z)|f(z)| < \infty \right\},$$

and

$$\mathscr{H}_v^0 = \left\{ f \in \mathscr{H}(\mathbb{D}) : \lim_{|z|\to 1^-} v(z)|f(z)| = 0 \right\}.$$

The weighted Bloch space $\mathscr{B}_v$ consists of all functions $f \in \mathscr{H}(\mathbb{D})$ such that

$$\|f\|_v := \sup_{z\in\mathbb{D}} v(z)|f'(z)| < \infty.$$

Note that $\|\cdot\|_v$ defines only a semi-norm on $\mathscr{B}_v$; however, a norm can be introduced by

$$\|f\|_{\mathscr{B}_v} := |f(0)| + \sup_{z\in\mathbb{D}} v(z)|f'(z)|,$$

with respect to which $\mathscr{B}_v$ becomes a Banach space. Similarly, the little weighted Bloch space $\mathscr{B}_v^0$ is the closed subspace consisting of functions $f \in \mathscr{B}_v$ satisfying

$$\lim_{|z|\to 1^-} v(z)|f'(z)| = 0.$$

If $\lim_{|z|\to 1^-} v(z) = 0$, the weight v is called typical. For typical weights, $\mathscr{H}^\infty_v$ is the bidual of $\mathscr{H}^0_v$. In particular, every normal weight is also typical. Furthermore, if v satisfies condition **(I)**, then condition **(II)** holds if and only if

$$\mathscr{H}^0_v = \mathscr{B}^0_{v(z)(1-|z|^2)},$$

see [24]. Specifically, if $v \in N_W(\mathbb{D})$, then

$$\mathscr{H}^0_v = \mathscr{B}^0_{v(z)(1-|z|^2)},$$

and by duality,

$$\mathscr{H}^\infty_v = \mathscr{B}_{v(z)(1-|z|^2)},$$

see [22]. Moreover, for $v \in N_W(\mathbb{D})$, the space $\mathscr{H}^0_v$ is isomorphic to c_0, and $\mathscr{H}^\infty_v$ is isomorphic to ℓ_∞, see [9], [22]–[25], and [30]. To present our principal findings, it is necessary to introduce and discuss certain properties of the associated weights. For further details, the reader may consult [9], [14], [20], and the references cited therein. Recall that for a given weight v, its associated weight $\widetilde{v}$ is defined by

$$\widetilde{v}(z) = \left(\sup\left\{|f(z)| : f \in \mathscr{H}^\infty_v, \|f\|_{\mathscr{H}^\infty_v} \leqslant 1\right\}\right)^{-1} = \frac{1}{\|\delta_z\|_{\mathscr{H}^\infty_v \to \mathbb{C}}},$$

where $\delta_z : \mathscr{H}^\infty_v \to \mathbb{C}$ denotes the evaluation functional defined by $\delta_z(f) = f(z)$. Furthermore, the weight v and its associated weight $\widetilde{v}$ satisfy the following properties (see [7]):

(P1) The function $\widetilde{v}$ is itself a weight, and it holds that $v \leqslant \widetilde{v}$, with the norms satisfying $\|\cdot\|_{\mathscr{H}^\infty_v} = \|\cdot\|_{\mathscr{H}^\infty_{\widetilde{v}}}$.

(P2) For every $\zeta \in \mathbb{D}$, there exists a function f_ζ in the closed unit ball $B_{\mathscr{H}^\infty_v}$ of $\mathscr{H}^\infty_v$ such that

$$|f_\zeta(\zeta)| = \frac{1}{\widetilde{v}(\zeta)}.$$

(P3) If v is typical, then $\widetilde{v}$ is also typical, and in this case,

$$\widetilde{v}(z) = \left(\sup\left\{|f(z)| : f \in \mathscr{H}^0_v, \|f\|_{\mathscr{H}^\infty_v} \leqslant 1\right\}\right)^{-1} = \frac{1}{\|\delta_z\|_{\mathscr{H}^0_v \to \mathbb{C}}}.$$

(P4) If v satisfies condition **(I)**, then v is essential. That is, there exists a constant $C \geqslant 1$ such that for all $z \in \mathbb{D}$,

$$v(z) \leqslant \widetilde{v}(z) \leqslant C v(z).$$

Since all weights considered in this work satisfy property **(I)**, the associated weight $\widetilde{v}$ can be replaced by v throughout the paper.

Next, we recall a duality pairing between the space $\mathscr{H}^\infty_v$ and a weighted Bergman space, originally established by Shields and Williams in [29]. Let v be a typical weight

and let ω be a positive, radial, continuous function on $\mathbb{D}$ such that $\int_0^1 \omega(r)\,dr < \infty$. Define A^1_ω as the subspace of $H(\mathbb{D})$ consisting of all functions f for which

$$\|f\|_{A^1_\omega} = \int_{\mathbb{D}} |f(z)|\,\omega(z)\,dA(z) < \infty,$$

where dA is the normalized Lebesgue measure on $\mathbb{D}$.

For a normal weight v and a function ω as above, the pair (v,ω) is called a *normal pair* if

$$v(r)\omega(r) = (1-r^2)^\alpha, \quad 0 \leqslant r < 1 \tag{1}$$

for some $\alpha > \beta - 1$, where $\beta > 0$ is defined as in condition **(I)**.

Given a normal pair (v,ω), the following bilinear form defines a pairing between $\mathscr{H}^\infty_v$ and A^1_ω (see [29, p. 292]):

$$(f,g) = \int_{\mathbb{D}} f(z)g(\overline{z})(1-|z|^2)^\alpha dA(z) = \int_{\mathbb{D}} f(\overline{z})g(z)(1-|z|^2)^\alpha dA(z). \tag{2}$$

This pairing induces a duality relationship $(\mathscr{H}^0_v)' = A^1_\omega$, as established in [29, Theorem 2, p. 296]. Consequently, we also have $(\mathscr{H}^0_v)'' = \mathscr{H}^\infty_v$.

Moreover, if $K_\zeta(z) = \frac{\alpha+1}{(1-\zeta z)^{\alpha+2}}$ with $\alpha > -1$ denotes the kernel function, then for a normal pair (v,ω), it holds that $K_\zeta \in \mathscr{H}^0_v \cap A^1_\omega$, and

$$g(\zeta) = (K_\zeta, g) \quad \text{for all } g \in A^1_\omega, \quad \text{and} \quad f(\zeta) = (f, K_\zeta) \quad \text{for all } f \in \mathscr{H}^\infty_v,$$

see Lemma 10 in [29].

Recall that a bounded linear operator T between Banach spaces X and Y is called:

(a) *Nuclear* if there exist sequences $(x'_k) \subset X'$ and $(y_k) \subset Y$ such that

$$\sum_k \|x'_k\|_{X'}\|y_k\|_Y < \infty \quad \text{and} \quad T = \sum_{k=1}^{\infty} x'_k \otimes y_k,$$

where $x'_k \otimes y_k : X \to Y$ is defined by $x \mapsto x'_k(x)y_k$.

(b) *Absolutely summing* if there exists a constant $C > 0$ such that for every finite sequence $(x_i)_{i=1}^n \subset X$,

$$\sum_{i=1}^n \|Tx_i\|_Y \leqslant C \sup_{\|x'\|_{X'} \leqslant 1} \sum_{i=1}^n |x'(x_i)| = C \sup_{|\eta_i| \leqslant 1} \left\| \sum_{i=1}^n \eta_i x_i \right\|_X.$$

It is well-known that every nuclear operator is compact and absolutely summing. Furthermore, if X is isomorphic to c_0 or ℓ_∞, then a linear operator $T : X \to Y$ is absolutely summing if and only if it is nuclear; see [16].

3. Boundedness and compactness of Volterra composition operators

In this section, we characterize boundedness and compactness of Volterra composition operators. We also prove some auxiliary results and some lemmas that are used in the proofs of the main results of this paper.

Recall that for $\mathscr{F} \subset Y^X$, the set of all functions from a set X into a topological space Y, the pointwise topology τ_p on $\mathscr{F}$ is equal to the subspace topology that it inherits from the product space $\prod_{x\in X} Y_x$, where Y_x is a copy of Y, when $\mathscr{F}$ is identified as a subset of this Cartesian product via the canonical inclusion

$$\mathscr{F} \longrightarrow \prod_{x\in X} Y_x$$

defined by $f \longmapsto \{f(x)\}_{x\in X}$.

Let $(X_\lambda, \tau_\lambda)$ be an arbitrary family of topological spaces. Then the weakest topology on

$$\prod_{\lambda\in\Lambda} X_\lambda = \left\{ f | f : \Lambda \longrightarrow \bigcup_{\lambda\in\Lambda} X_\lambda \text{ such that } f(\lambda) \in X_\lambda \text{ for each } \lambda \in \Lambda \right\}$$

with respect to which each projection map $p_\lambda : \prod_{\lambda\in\Lambda} X_\lambda \longrightarrow X_\lambda$ defined as $p_\lambda(f) = f_\lambda$ is continuous, where f_λ is the λ th component of f. The subbsae and the base for the product topology are given respectively as

$$\mathscr{S} = \left\{ p_\lambda^{-1}(G_\lambda) : G_\lambda \text{ is open in } X_\lambda \text{ for each } \lambda \in \Lambda \right\}$$

and

$$\mathscr{B} = \left\{ \bigcap_{i=1}^{n} p_\lambda^{-1}(G_{\lambda_i}) : G_{\lambda_i} \text{ are open in } X_{\lambda_i}, \lambda_i \in \Lambda, i = 1,2,\cdots n,\ n \in \mathbb{N} \right\}.$$

A net $\{x_\lambda\}_{\lambda\in\Lambda}$ converges to x in the product space $\prod_{\lambda\in\Lambda} X_\lambda$ if and only if $p_\lambda(x_\lambda) \longrightarrow p_\lambda(x)$ for every projection map $p_\lambda,\ \lambda \in \Lambda$.

Let X' be the dual of a normed linear space X. Then

$$X' = \{x' \in X' : x' : X \to \mathbb{C} \text{ is a bounded linear functional}\}.$$

Then $X' \subset \mathbb{C}^X$. Recall that the w^*-topology on X' is the smallest topology on X' such that each member in the family $\mathscr{F} = \{\Phi_x : x \in X\} \subset X''$ is continuous, where for each $x \in X$, $\Phi_x : X' \to \mathbb{C}$ is the evaluation map defined as $\Phi_x(x') = x'(x)$ for all $x' \in X'$. A net $\{x'_\lambda\}_{\lambda\in\Lambda}$ converges to x' in X' if and only if $x'_\lambda(x) \longrightarrow x'(x)$ for every $x \in X$.

Next we prove that the pointwise topology τ_p is weaker than the w^*-topology on X'. Clearly, $x'(x) \in \mathbb{C}$ for all $x \in X$. For each $x \in X$, let $Y_x = \mathbb{C}$ equipped with the usual topology. Consider the Cartesian product

$$\prod_{x\in X} \mathbb{C} = \prod_{x\in X} Y_x = \left\{ f | f : X \to \bigcup_{x\in\Lambda} Y_x \text{ such that } f(x) = x'(x) \in Y_x = \mathbb{C} \text{ for all } x \in X \right\}$$

with the product topology. To each $x' \in X$, we associate $f_{x'} \in \prod_{x\in\Lambda} Y_x$ given by $f_{x'}(x) = x'(x)$.

Define a map $\widetilde{\Lambda} : X' \to \prod_{x\in X} Y_x$ as $\widetilde{\Lambda}(x') = f_{x'}$ for all $x' \in X'$. Clearly, $\widetilde{\Lambda}$ is an embedding map. Claim that the τ_p is weaker that the w^*-topology on X'. To settle the claim we need to show that every net which is convergent with respect to the w^*-topology on X' is convergent with respect to τ_p.

For let x'_λ be a net in X' which converges to x' in X' with respect to the w^*-topology. This is equivalent to the statement that $x'_\lambda(x) \to x'(x)$ for each $x \in X$. But then

$$p_x(f_{x'_\lambda}) = x'_\lambda(x) \to x'(x) = p_x(f_{x'})$$

for each $x \in X$, which in turns implies that $f_{x'_\lambda}(x) \to f_{x'_\lambda}(x)$ for each $x \in X$. Thus by the definition of the convergence of a net in the product topology, we have that $f_{x'_\lambda}$ is a net in $\prod_{x\in X} \mathbb{C}$ which converges to $f_{x'}$ with respect to τ_p. Hence τ_p is weaker that the w^*-topology on X'.

Next, we present some lemmas which will be used in the proof of the main results of this paper. The next two lemmas can easily be obtained by modifying the arguments in Lemma 1 of [6] and Lemma 2.12 of [20].

LEMMA 1. *Suppose that $\nu, \mu \in N_W(\mathbb{D})$, $g \in H(\mathbb{D})$ and ϕ a holomorphic self-map of $\mathbb{D}$ such that $T_g^\phi : \mathscr{H}_\nu^0 \longrightarrow \mathscr{H}_\mu^0$ is bounded. Then $(T_g^\phi)''$ and T_g^ϕ agree on $\mathscr{H}_\nu^\infty$ and therefore, $T_g^\phi : \mathscr{H}_\nu^\infty \longrightarrow \mathscr{H}_\mu^\infty$ is also bounded.*

Proof. The boundedness of $T_g^\phi : \mathscr{H}_\nu^0 \longrightarrow \mathscr{H}_\mu^0$ implies that $(T_g^\phi)' : (H_\mu^0)' \longrightarrow (\mathscr{H}_\nu^0)'$ and $(T_g^\phi)'' : \mathscr{H}_\nu^\infty \longrightarrow \mathscr{H}_\mu^\infty$ are also bounded. It is well known that the span of point evaluation functionals $\Lambda_z \in (H_\mu^0)'$ defined by $\Lambda_z f = f(z)$ on H_μ^0 is dense in $(H_\mu^0)'$. Thus, we have

$$\langle (T_g^\phi)'(\Lambda_z), f\rangle = \langle \Lambda_z, T_g^\phi(f)\rangle = \int_0^z f(\phi)(\zeta) g'(\zeta) d\zeta. \tag{3}$$

Again, for $f \in H_\nu^\infty$, we have that

$$\langle (T_g^\phi)''(f), \Lambda_z\rangle = \langle f, (T_g^\phi)'(\Lambda_z)\rangle. \tag{4}$$

Also the functions f_r defined by $f_r(z) = f(rz)$ converges to f as r tends to 1 in the w^*-topology. Thus

$$\langle f_r, h^*\rangle \longrightarrow \langle f, h^*\rangle \quad \text{for all } h^* \in (H_\nu^0)'.$$

Further, note that $f_r \longrightarrow f$ uniformly on compact subsets in $\mathbb{D}$, and so from (3) and

(4), we obtain

$$\begin{aligned}\langle (T_g^\phi)''(f), \Lambda_z\rangle &= \langle f, (T_g^\phi)'(\Lambda_z)\rangle \\ &= \lim_{r\to 1}\langle f_r, (T_g^\phi)'(\Lambda_z)\rangle \\ &= \lim_{r\to 1}\langle T_g^\phi(f_r), \Lambda_z\rangle \\ &= \lim_{r\to 1}\int_0^z f_r(\phi(\zeta))g'(\zeta)d\zeta \\ &= \int_0^z f(\phi(\zeta))g'(\zeta)d\zeta \\ &= \langle T_g^\phi(f), \Lambda_z\rangle.\end{aligned}$$

Hence, it follows that $(T_g^\phi)'' = T_g^\phi$. $\square$

LEMMA 2. *Suppose that $\nu, \mu \in N_W(\mathbb{D})$, $g \in H(\mathbb{D})$ and ϕ a holomorphic self-map of $\mathbb{D}$ such that $S_g^\phi : \mathscr{H}_\nu^0 \longrightarrow \mathscr{H}_\mu^0$ is bounded. Then $(S_g^\phi)''$ and S_g^ϕ agree on $\mathscr{H}_\nu^\infty$ and therefore $S_g^\phi : \mathscr{H}_\nu^\infty \longrightarrow \mathscr{H}_\mu^\infty$ is bounded.*

Proof. Proceeding as in the Lemma 1, we have

$$\begin{aligned}\langle (S_g^\phi)''(f), \Lambda_z\rangle &= \langle f, (S_g^\phi)'(\Lambda_z)\rangle \\ &= \lim_{r\to 1}\langle f_r, (S_g^\phi)'(\Lambda_z)\rangle \\ &= \lim_{r\to 1}\langle S_g^\phi(f_r), \Lambda_z\rangle \\ &= \lim_{r\to 1}\int_0^z f_r'(\phi(\zeta))g(\zeta)d\zeta\end{aligned}$$

Since $f_r'(\phi) \longrightarrow f'(\phi)$ locally uniformly in $\mathbb{D}$, so we obtain

$$\begin{aligned}\lim_{r\to 1}\int_0^z f_r'(\phi)(\zeta)g(\zeta)d\zeta &= \int_0^z f'(\phi)(\zeta)g(\zeta)d\zeta \\ &= \langle S_g^\phi(f), \Lambda_z\rangle.\end{aligned}$$

Hence, it follows that $(S_g^\phi)'' = S_g^\phi$. $\square$

Next we characterize bounded Volterra composition operators between weighted spaces of analytic functions. The arguments are standard and may possibly appear in literatures in a more general sense, so we give an outline proof for completeness.

PROPOSITION 1. *Let $\nu, \mu \in N_W(\mathbb{D})$, $g \in H(\mathbb{D})$ and ϕ a self-map of $\mathbb{D}$. Then*

(i) The following are equivalent.

(a) $T_g^\phi : \mathscr{H}_\nu^\infty \longrightarrow \mathscr{H}_\mu^\infty$ is bounded.

(b) $T_g^\phi : \mathscr{H}_\nu^0 \longrightarrow \mathscr{H}_\mu^\infty$ *is bounded.*

(c) g, ν, μ *and* ϕ *satisfy the following condition:*

$$M_1 = \sup_{z\in\mathbb{D}}(1-|z|^2)\frac{\mu(z)}{\nu(\phi(z))}|g'(z)| < \infty. \tag{5}$$

Moreover, $T_g^\phi : \mathscr{H}_\nu^0 \longrightarrow \mathscr{H}_\mu^0$ *is bounded if and only if* $g \in \mathscr{H}^0_{\mu(z)}$ *and* (5) *holds.*

(ii) *The following are equivalent.*

(d) $S_g^\phi : \mathscr{H}_\nu^\infty \longrightarrow \mathscr{H}_\mu^\infty$ *is bounded.*

(e) $S_g^\phi : \mathscr{H}_\nu^0 \longrightarrow \mathscr{H}_\mu^\infty$ *is bounded.*

(f) g, ν, μ *and* ϕ *satisfy the following condition:*

$$\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)}{(1-|\phi(z)|^2)}\frac{\mu(z)}{\nu(\phi(z))}|g(z)| < \infty. \tag{6}$$

Moreover, $S_g^\phi : \mathscr{H}_\nu^0 \longrightarrow \mathscr{H}_\mu^0$ *is bounded if and only if* $g \in \mathscr{H}^0_{(1-|z|^2)\mu(z)}$ *and* (6) *holds.*

Proof. (i) Since μ is a normal weight, $\mathscr{H}_{\mu(z)} = \mathscr{B}_{(1-|z|^2)\mu(z)}$. Thus boundedness of $T_g^\phi : \mathscr{H}_\nu^\infty \longrightarrow \mathscr{H}_\mu^\infty$ is equivalent to the boundedness of $M_{g'}C_\phi : \mathscr{H}_\nu^\infty \longrightarrow \mathscr{H}^\infty_{(1-|z|^2)\mu(z)}$.

(a) $\Leftrightarrow$ (c). By Proposition 3.1 in [26], we have that (a) $\Leftrightarrow$ (c).

It is obvious that (a) $\Rightarrow$ (b). Thus to complete the proof, we need to show that (b) $\Rightarrow$ (c).

(b) $\Rightarrow$ (c). Suppose that $M_{g'}C_\phi : \mathscr{H}_\nu^0 \longrightarrow \mathscr{H}^\infty_{(1-|z|^2)\mu(z)}$ is bounded. For $\zeta \in \mathbb{D}$, let $f_{\phi(\zeta)}$ be a function that satisfies the property (**P2**) and consider the function

$$h_\zeta(z) = \frac{1-|\phi(\zeta)|^2}{1-\overline{\phi(\zeta)}z}f_{\phi(\zeta)}(z) \tag{7}$$

Since $f_{\phi(\zeta)} \in \mathscr{H}_\nu^\infty$ for each $\zeta \in \mathbb{D}$, $\sup_{\zeta\in\mathbb{D}}\|f_{\phi(\zeta)}\|_{\mathscr{H}_\nu^\infty} \lesssim 1$ and $\dfrac{1-|\phi(\zeta)|^2}{|1-\overline{\phi(\zeta)}z|} \lesssim 1$, so we have $\sup_{\zeta\in\mathbb{D}}\|h_\zeta\|_{\mathscr{H}_\nu^\infty} \lesssim 1$. We claim that $h_\zeta \in \mathscr{H}_\nu^0$ for any $\zeta \in \mathbb{D}$. Let $\zeta \in \mathbb{D}$ be fixed. Then we can find $\delta_0 \in [0,1)$ such that $|\phi(\zeta)| < \delta_0$.

Since $\frac{\nu(r)}{(1-r^2)^\gamma}$ is almost decreasing, where γ is defined as in (**II**), so for $z \in \mathbb{D}$ such that $|z| > |\phi(\zeta)|$, we have

$$\frac{\nu(|z|)}{(1-|z|^2)^\gamma} \lesssim \frac{\nu(|\phi(\zeta)|)}{(1-|\phi(\zeta)|^2)^\gamma}.$$

That is,

$$\frac{\nu(z)}{\nu(\phi(\zeta))} \lesssim \frac{(1-|z|^2)^\gamma}{(1-|\phi(\zeta)|^2)^\gamma}.$$

Thus, by Lemma 3.1 in [31], we have

$$\begin{aligned}
\nu(z)|h_\zeta(z)| &= \nu(z)\frac{1-|\phi(\zeta)|^2}{|1-\overline{\phi(\zeta)}z|}|f_{\phi(\zeta)}(z)| \\
&\lesssim \frac{\nu(z)}{\nu(\phi(\zeta))}\frac{1-|\phi(\zeta)|^2}{|1-\overline{\phi(\zeta)}z|} \\
&\lesssim \frac{(1-|z|^2)^\gamma}{(1-|\phi(\zeta)|^2)^\gamma}\frac{1-|\phi(\zeta)|^2}{|1-\overline{\phi(\zeta)}z|} \\
&\lesssim \frac{(1-|z|^2)^\gamma}{(1-\delta_0^2)^{\gamma+1}} \to 0 \text{ as } |z| \to 1^-.
\end{aligned}$$

This settles the claim. Moreover,

$$h_\zeta(\phi(\zeta)) \asymp \frac{1}{\nu(\phi(\zeta))}.$$

Thus by the boundedness of $M_{g'}C_\phi : \mathscr{H}^0_\nu \longrightarrow \mathscr{H}^\infty_{(1-|z|^2)\mu(z)}$, we have that

$$(1-|\zeta|^2)\mu(\zeta)|g'(\zeta)||h_\zeta(\phi(\zeta))| \leqslant \|M_{g'}C_\phi h_\zeta\|_{\mathscr{H}^\infty_{(1-|z|^2)\mu(z)}} \lesssim \|h_\zeta\|_{\mathscr{H}^\infty_\nu} \lesssim 1.$$

Taking supremum over $\zeta \in \mathbb{D}$, we get (5).

By Proposition 3.2 in [26], we have that $T^\phi_g : \mathscr{H}^0_\nu \longrightarrow \mathscr{H}^0_\mu$ is bounded if and only if $g \in \mathscr{B}^0_{(1-|z|^2)\mu(z)}$ and (5) holds. This completes the proof of (i).

(ii) The boundedness of $S^\phi_g : \mathscr{H}^\infty_\nu \longrightarrow \mathscr{H}^\infty_\mu$ is equivalent to the boundedness of $M_gC_\phi D : \mathscr{H}^\infty_\nu \longrightarrow \mathscr{H}^\infty_{(1-|z|^2)\mu(z)}$, where D is the differentiation operator. We write an outline proof of $(d) \Leftrightarrow (f)$ and $(e) \Leftrightarrow (f)$. The details of all other cases are omitted.

$(d) \Leftrightarrow (f)$. By Theorem 7 in [26], we have that $(d) \Leftrightarrow (f)$.

$(e) \Rightarrow (f)$. Suppose that $M_gC_\phi D : \mathscr{H}^0_\nu \longrightarrow \mathscr{H}^\infty_{(1-|z|^2)\mu(z)}$ is bounded. Let $\zeta \in \mathbb{D}$ be fixed and

$$\tilde{h}_\zeta(z) = \left(\frac{1-|\phi(\zeta)|^2}{1-\overline{\phi(\zeta)}z} - \frac{(1-|\phi(\zeta)|^2)^2}{(1-\overline{\phi(\zeta)}z)^2}\right) f_{\phi(\zeta)}(z)$$

for every $z \in \mathbb{D}$, where $f_{\phi(\zeta)}$ is a function in the unit ball of $\mathscr{H}^\infty_\nu$ such that $f_{\phi(\zeta)}(\phi(\zeta)) \asymp \frac{1}{\nu(\phi(\zeta))}$. Using the fact that

$$\left|\frac{1-|\phi(\zeta)|^2}{1-\overline{\phi(\zeta)}z} - \frac{(1-|\phi(\zeta)|^2)^2}{(1-\overline{\phi(\zeta)}z)^2}\right| \lesssim 1,$$

we have $\|\tilde{h}_\zeta\|_{\mathscr{H}^\infty_\nu} \lesssim 1$ for every $\zeta \in \mathbb{D}$. Moreover, $\tilde{h}_\zeta(\phi(\zeta)) = 0$ and

$$\tilde{h}'_\zeta(z) = \left(\frac{1-|\phi(\zeta)|^2}{(1-\overline{\phi(\zeta)}z)^2} - \frac{2(1-|\phi(\zeta)|^2)^2}{(1-\overline{\phi(\zeta)}z)^3}\right)\overline{\phi(\zeta)} f_{\phi(\zeta)}(z) + \left(\frac{1-|\phi(\zeta)|^2}{1-\overline{\phi(\zeta)}z} - \frac{(1-|\phi(\zeta)|^2)^2}{(1-\overline{\phi(\zeta)}z)^2}\right) f'_{\phi(\zeta)}(z)$$

for every $z \in \mathbb{D}$. Thus

$$\tilde{h}'_\zeta(\phi(\zeta)) \asymp -\frac{\overline{\phi(\zeta)}}{(1-|\phi(\zeta)|^2)\nu(\phi(\zeta))}.$$

Once again, proceeding as in case (i), we can easily show that $\tilde{h}_\zeta \in \mathscr{H}^0_\nu$. Thus by the boundedness of $M_g C_\phi D : \mathscr{H}^0_\nu \longrightarrow \mathscr{H}^\infty_{(1-|z|^2)\mu(z)}$, we have that

$$(1-|\zeta|^2)\mu(\zeta)|g(\zeta)||\tilde{h}'_\zeta(\phi(\zeta))| \leqslant \|M_g C_\phi D(\tilde{h}_\zeta)\|_{\mathscr{H}^\infty_{(1-|z|^2)\mu(z)}} \lesssim \|\tilde{h}_\zeta\|_{\mathscr{H}^\infty_\nu} \lesssim 1.$$

Taking supremum over $\zeta \in \mathbb{D}$, we get

$$\sup_{\zeta\in\mathbb{D}} \frac{(1-|\zeta|^2)\mu(\zeta)}{(1-|\phi(\zeta)|^2)\nu(\phi(\zeta))}|\phi(\zeta)||g(\zeta)| < \infty.$$

Thus

$$\sup_{|\phi(\zeta)|>1/2} \frac{(1-|\zeta|^2)\mu(\zeta)}{(1-|\phi(\zeta)|^2)\nu(\phi(\zeta))}|g(\zeta)| < \infty. \tag{8}$$

Let $f_0(z) = z$ for all $z \in \mathbb{D}$. Since ν is a typical weight, so for any given $\varepsilon > 0$, we can find $r_0 \in (0,1)$ such that $\nu(z) < \varepsilon$ for $|z| > r_0$, Thus

$$\begin{aligned}\|f\|_{H^\infty_\nu} &= \sup_{z\in\mathbb{D}} \nu(z)|f(z)| \\ &\leqslant \sup_{|z|\leqslant r_0} \nu(z) + \sup_{|z|>r_0} \nu(z) \\ &\leqslant M + \varepsilon, \quad \text{where} \quad M = \sup_{|z|\leqslant r_0} \nu(z).\end{aligned}$$

Hence, $\|f_0\|_{H^\infty_\nu} \lesssim 1$ and $f_0 \in H^0_\nu$ also. Therefore, by the boundedness of $M_g C_\phi D : \mathscr{H}^0_\nu \longrightarrow \mathscr{H}_{(1-|z|^2)\mu(z)}$, we have

$$(1-|\zeta|^2)\mu(\zeta)|g(\zeta)| \lesssim \|f_0\|_{H^\infty_\nu}. \tag{9}$$

By (9), we have $(1-|\zeta|^2)\mu(z)|g(\zeta)| \lesssim 1$. Thus, by the fact that a continuous function on a compact set attains its bound, we have

$$\begin{aligned}\sup_{|\phi(\zeta)|\leqslant 1/2} \frac{(1-|\zeta|^2)\mu(\zeta)}{\nu(\phi(\zeta))(1-|\phi(\zeta)|^2}|g(\zeta)| &\lesssim \sup_{|w|\leqslant 1/2} \frac{1}{\nu(w)(1-|w|^2)} \\ &< \infty\end{aligned} \tag{10}$$

Therefore, by (8) and (10), we have

$$\begin{aligned}\sup_{\zeta\in\mathbb{D}}\frac{(1-|\zeta|^2)\mu(\zeta)}{(1-|\phi(\zeta)|^2)\nu(\phi(\zeta))}|g(\zeta)| &\leqslant \sup_{|\phi(\zeta)|\leqslant 1/2}\frac{(1-|\zeta|^2)\mu(\zeta)}{(1-|\phi(\zeta)|^2)\nu(\phi(\zeta))}|g(\zeta)| \\ &\quad + \sup_{|\phi(\zeta)|>1/2}\frac{(1-|\zeta|^2)\mu(\zeta)}{(1-|\phi(\zeta)|^2)\nu(\phi(\zeta))}|g(\zeta)| \\ &< \infty\end{aligned}$$

This completes the proof. □

PROPOSITION 2. *Let $\nu,\mu\in N_W(\mathbb{D})$. If $g\in H(\mathbb{D})$ and ϕ a self-map of $\mathbb{D}$ such that $T_g^\phi:\mathscr{H}_\nu^0\longrightarrow\mathscr{H}_\mu^\infty$ is bounded, then*

(*i*) *The following are equivalent.*

- (*a*) $T_g^\phi:\mathscr{H}_\nu^0\longrightarrow\mathscr{H}_\mu^\infty$ *is compact.*
- (*b*) $T_g^\phi:\mathscr{H}_\nu^0\longrightarrow\mathscr{H}_\mu^\infty$ *is weakly compact.*
- (*c*) *The bi-transpose $(T_g^\phi)''$ of $T_g^\phi:\mathscr{H}_\nu^0\longrightarrow\mathscr{H}_\mu^\infty$ satisfies $(T_g^\phi)''=T_g^\phi$.*

(*ii*) *If $g\in\mathscr{H}_\mu^0$, then $T_g^\phi:\mathscr{H}_\nu^0\to\mathscr{H}_\mu^0$ is bounded and the bi-transpose $(T_g^\phi)''$ of T_g^ϕ satisfies $(T_g^\phi)''=T_g^\phi$. Moreover, $T_g^\phi:\mathscr{H}_\nu^0\to\mathscr{H}_\mu^0$ is compact if and only if $T_g^\phi(H_\nu^\infty)\subset\mathscr{H}_\mu^0$.*

Proof. (*i*) (*a*) $\Leftrightarrow$ (*b*) Since $\mathscr{H}_\mu^\infty=\mathscr{B}_{(1-|z|^2)\mu(z)}$, so $T_g^\phi f\in\mathscr{H}_\mu^\infty$ if and only if $M_{g'}C_\phi f\in\mathscr{H}^\infty_{(1-|z|^2)\mu}$. Moreover, $\|T_g^\phi f\|_{\mathscr{H}_\mu^\infty}\asymp\|M_{g'}C_\phi f\|_{\mathscr{H}^\infty_{(1-|z|^2)\mu}}$. Thus compactness (and/or weak compactness) of $T_g^\phi:\mathscr{H}_\nu^0\longrightarrow\mathscr{H}_\mu^\infty$ is equivalent to compactness (and/or weak compactness) of $M_{g'}C_\phi:\mathscr{H}_\nu^0\longrightarrow\mathscr{H}^\infty_{(1-|z|^2)\mu}$. By Proposition 1 in [8], $M_{g'}C_\phi f$ is compact if and only if it is weakly compact. Therefore, it follows that (*a*) and (*b*) are equivalent.

(*a*) $\Rightarrow$ (*c*) Assume that (*a*) holds. Then by Gantmacher-Nakamura's theorem, see Theorem 5.5 in [11], the bi-transpose $(T_g^\phi)''$ of $T_g^\phi:\mathscr{H}_\nu^0\longrightarrow\mathscr{H}_\mu^\infty$ acts continuously from $\mathscr{H}_\nu^\infty$ with w^*-topology to $\mathscr{H}_\mu^\infty$ with w^*-topology. Since pointwise topology τ_p is weaker than w^*-topology on $\mathscr{H}_\nu^\infty$ and $\mathscr{H}_\mu^\infty$, so $(T_g^\phi)''$ and T_g^ϕ are both w^* and τ_p continuous and by Lemma 1, $(T_g^\phi)''$ and T_g^ϕ agree in $\mathscr{H}_\nu^0$ which is w^*-dense in $\mathscr{H}_\nu^\infty$. Hence, (*c*) holds.

(*c*) $\Rightarrow$ (*b*) Assume that the bi-transpose $(T_g^\phi)''$ of $T_g^\phi:\mathscr{H}_\nu^0\longrightarrow\mathscr{H}_\mu^\infty$ satisfies $(T_g^\phi)''=T_g^\phi$. Then by Proposition 1 it holds that $(T_g^\phi)''$ maps $\mathscr{H}_\nu^\infty$) to $\mathscr{H}_\mu^\infty$. Therefore, $(T_g^\phi)''(\mathscr{H}_\nu^\infty)\subset\mathscr{H}_\mu^\infty$. Using the fact that a bounded linear operator T acting from a Banach space X to a Banach space Y is weakly compact if and only if $T''(X'')\subset Y$, see p. 482 in [13], we have that $T_g^\phi:\mathscr{H}_\nu^0\longrightarrow\mathscr{H}_\mu^\infty$ is weakly compact.

(*ii*) If $g \in \mathscr{H}^0_{\mu(z)}$, then by Proposition 1, $T_g^\phi : \mathscr{H}^0_\nu \to \mathscr{H}^0_\mu$ is bounded. Moreover, by Lemma 1, we have that $(T_g^\phi)'' = T_g^\phi$. Finally, by Gantmacher-Nakamura's theorem, it holds that the compactness of $T_g^\phi : \mathscr{H}^0_\nu \to \mathscr{H}^0_\mu$ is equivalent to $T_g^\phi(H^\infty_\nu) \subset \mathscr{H}^0_\mu$. This completes the proof. □

PROPOSITION 3. *Let $\nu, \mu \in N_W(\mathbb{D})$. If $g \in H(\mathbb{D})$ and ϕ a self-map of $\mathbb{D}$ such that $S_g^\phi : \mathscr{H}^0_\nu \longrightarrow \mathscr{H}^\infty_\mu$ is bounded, then*

(*i*) *The following are equivalent.*

(*a*) $S_g^\phi : \mathscr{H}^0_\nu \longrightarrow \mathscr{H}^\infty_\mu$ *is compact.*

(*b*) $S_g^\phi : \mathscr{H}^0_\nu \longrightarrow \mathscr{H}^\infty_\mu$ *is weakly compact.*

(*c*) *The bi-transpose $(S_g^\phi)''$ of $S_g^\phi : \mathscr{H}^0_\nu \longrightarrow \mathscr{H}^\infty_\mu$ satisfies $(S_g^\phi)'' = S_g^\phi$.*

(*ii*) *If $g \in \mathscr{H}^0_{(1-|z|^2)\mu(z)}$, then $S_g^\phi : \mathscr{H}^0_\nu \to \mathscr{H}^0_\mu$ is bounded and the bi-transpose $(S_g^\phi)''$ of S_g^ϕ satisfies $(S_g^\phi)'' = S_g^\phi$. Moreover, $S_g^\phi : \mathscr{H}^0_\nu \to \mathscr{H}^0_\mu$ is compact if and only if $S_g^\phi(H^\infty_\nu) \subset \mathscr{H}^0_\mu$.*

Proof. (i) Proceeding as in Proposition 2, $S_g^\phi f \in H^\infty_\mu$ if and only if $M_{g'}C_\phi Df \in H^\infty_{(1-|z|^2)\mu(z)}$ with comparable norm. Thus compactness (and/or weak compactness) of $T_g^\phi : \mathscr{H}^0_\nu \longrightarrow \mathscr{H}^\infty_\mu$ is equivalent to compactness (and/or weak compactness) of $M_{g'}C_\phi D : \mathscr{H}^0_\nu \longrightarrow \mathscr{H}^\infty_\mu$ which is further equivalent to compactness (and/or weak compactness) of $M_{g'}C_\phi : \mathscr{H}^0_{(1-|z|^2)\nu(z)} \longrightarrow \mathscr{H}^\infty_\mu$. Thus the first part of Proposition can now be settled similarly to the proof of the first part of the Proposition 2.

(ii) If $g \in \mathscr{H}^0_{(1-|z|^2)\mu(z)}$, then by Proposition 2, $S_g^\phi : \mathscr{H}^0_\nu \to \mathscr{H}^0_\mu$ is bounded. Moreover, by Lemma 114, we can easily prove that $(S_g^\phi)'' = S_g^\phi$. The last part can be settled by Gantmacher-Nakamura's theorem as in the second part of the Proposition 2. □

Finally, we recall Piestsch's theorem which plays an important role in the proof of the main results of this paper. For more about Piestsch's theorem, we refer the readers to Theorem 2.12 of [12].

THEOREM 1. (Pietsch's Theorem) *Let E and F be Banach spaces and $1 \leqslant p < +\infty$. A bounded linear operator $T : E \longrightarrow F$ is absolutely p-summing if and only if there is a constant C and a regular Borel probability measure ρ on closed unit ball B_{E^*} of E^* with the weak*-topology $\sigma(E^*, E)$ such that for every $x \in E$,*

$$\|T(x)\| \leqslant C \left(\int_{B_{E^*}} |\varsigma(x)|^p d\rho(\varsigma) \right)^{1/p}.$$

4. Nuclear Volterra composition operators between weighted Banach spaces

In this section, we characterize nuclear Volterra composition operators between weighted Banach spaces of analytic functions.

THEOREM 2. *Let $\nu,\mu \in N_W(\mathbb{D})$, $\alpha > -1$ and ω a weight function such that $\{\nu,\omega\}$ is a normal pair and (1) holds. If $g \in H(\mathbb{D})$ and ϕ a self-map of $\mathbb{D}$ be such that $T_g^\phi : \mathscr{H}_\nu^\infty \longrightarrow \mathscr{H}_\mu^\infty$ is bounded, then the following statements are equivalent.*

(a) $T_g^\phi : \mathscr{H}_\nu^0 \longrightarrow \mathscr{H}_\mu^\infty$ is nuclear.

(b) $T_g^\phi : \mathscr{H}_\nu^\infty \longrightarrow \mathscr{H}_\mu^\infty$ is nuclear.

(c) g, ν, μ and ϕ satisfy the following condition:

$$M_\alpha = \int_{\mathbb{D}} \left[\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)\mu(z)|g'(z)|}{|1-\overline{\zeta}\phi(z)|^{\alpha+2}}\right] \frac{(1-|\zeta|^2)^\alpha}{\nu(\zeta)} dA(\zeta) < \infty.$$

Proof. $(c) \Rightarrow (a)$ Since $\mathscr{H}_\nu^0 \simeq c_0$ see [9], so $T_g^\phi : \mathscr{H}_\nu^0 \longrightarrow \mathscr{H}_\mu^\infty$ is nuclear if and only if $T_g^\phi : \mathscr{H}_\nu^0 \longrightarrow \mathscr{H}_\mu^\infty$ is absolutely summing. Thus to complete the proof of the implication $(c) \Rightarrow (a)$, we need to prove that $T_g^\phi : \mathscr{H}_\nu^0 \longrightarrow \mathscr{H}_\mu^\infty$ is absolutely summing. First note that if μ is normal, $\mathscr{H}^\infty_{\mu(z)} = \mathscr{B}_{\mu(z)(1-|z|^2)}$, $\|T_g^\phi(f)\|_{\mathscr{H}_\mu^\infty} \asymp \|T_g^\phi(f)\|_{\mathscr{B}_{\mu(z)(1-|z|^2)}}$ for any $f \in H_\nu^0$ and $T_g^\phi(f)$ vanishes at origin for any $f \in H(\mathbb{D})$. Now polynomials are dense in $\mathscr{H}_\nu^0$, so it is sufficient to consider polynomials $f_1, f_2, \cdots, f_N$ in the definition of absolutely summing. Now for any $C > 1$, we can select z_i, $i = 1,2,\cdots N$ such that

$$\begin{aligned}
\sum_{i=1}^N \|T_g^\phi f_i\|_{\mathscr{H}_\mu^\infty} &\asymp \sum_{i=1}^N \left(\left|(T_g^\phi f_i)(0)\right| + \sup_{z\in\mathbb{D}} (1-|z|^2)\mu(z)\left|(T_g^\phi f_i)'(z)\right| \right) \\
&= \sum_{i=1}^N \sup_{z\in\mathbb{D}} (1-|z|^2)\mu(z)\left|f_i(\phi(z))g'(z)\right| \\
&\leqslant C \sum_{i=1}^N (1-|z_i|^2)\mu(z_i)\left|f_i(\phi(z_i))g'(z_i)\right| \qquad (11)
\end{aligned}$$

Using the duality pairing in (2), we have that

$$\left|f_i(\phi(z_i))\right| = \left|(f_i, K_{\phi(z_i)})\right| = \left|(\alpha+1)\int_{\mathbb{D}} f_i(\zeta) \frac{(1-|\zeta|^2)^\alpha}{|1-\overline{\zeta}\phi(z_i)|^{\alpha+2}} dA(\zeta)\right|.$$

Thus from (11), we have that

$$\sum_{i=1}^{N}\|T_g^{\phi}f_i\|_{\mathscr{H}_{\mu}^{\infty}} \leqslant C\left(\sup_{w\in\mathbb{D}}\sum_{i=1}^{N}|f_i(w)|\nu(w)\right) \times\int_{\mathbb{D}}\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)\mu(z)|g'(z)|}{|1-\overline{\zeta}\phi(z)|^{\alpha+2}}\frac{(1-|\zeta|^2)^{\alpha}}{\nu(\zeta)}dA(\zeta) = C\left(\sup_{w\in\mathbb{D}}\sum_{i=1}^{N}|f_i(w)|\nu(w)\right)M_{\alpha} \tag{12}$$

Using the equality

$$\sup_{w\in\mathbb{D}}\sum_{i=1}^{N}|f_i(w)|\nu(w) = \sup_{|\eta_i|=1}\sup_{w\in\mathbb{D}}\left|\sum_{i=1}^{N}\eta_i f_i(w)\right|\nu(w) = \sup_{|\eta_i|=1}\left\|\sum_{i=1}^{N}\eta_i f_i\right\|_{\mathscr{H}_{\mu}^{\infty}}$$

in (12), we have that

$$\sum_{i=1}^{N}\|T_g^{\phi}f_i\|_{\mathscr{H}_{\mu}^{\infty}} \leqslant C\sup_{|\eta_i|=1}\left\|\sum_{i=1}^{N}\eta_i f_i\right\|_{\mathscr{H}_{\mu}^{\infty}}.$$

Since $C>0$ is arbitrary, so $T_g^{\phi}:\mathscr{H}_{\nu}^{0}\longrightarrow\mathscr{H}_{\mu}^{\infty}$ is absolutely summing.

$(a)\Rightarrow(c)$ Since every nuclear operator is absolutely summing, therefore, $T_g^{\phi}:\mathscr{H}_{\nu}^{0}\longrightarrow\mathscr{H}_{\mu}^{\infty}$ is absolutely summing. By Theorem 1, there is a probability Borel measure ρ on $\sigma(A_{\omega}^{1},\mathscr{H}_{\nu}^{0})$-compact unit ball $\mathbb{B}_1$ of A_{ω}^{1}, where $\sigma(A_{\omega}^{1},\mathscr{H}_{\nu}^{0})$ is the weak*-topology on $(\mathscr{H}_{\nu}^{0})' = A_{\omega}^{1}$, such that

$$\|T_g^{\phi}f\|_{\mathscr{H}_{\mu}^{\infty}} \leqslant C\int_{\mathbb{B}_1}|\xi(f)|d\rho(\xi) \tag{13}$$

for every $f\in\mathscr{H}_{\nu}^{0}$ and some constant $C>0$ independent of f. For every $\zeta\in\mathbb{D}$, we have that $K_{\zeta}\in\mathscr{H}_{\nu}^{0}$. Therefore,

$$(\alpha+1)\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)\mu(z)|g'(z)|}{|1-\zeta\phi(z)|^{\alpha+2}} \leqslant \|T_g^{\phi}K_{\zeta}\|_{\mathscr{B}_{\mu(z)(1-|z|^2)}} \leqslant C\|T_g^{\phi}K_{\zeta}\|_{\mathscr{H}_{\mu(z)}^{\infty}}. \tag{14}$$

Replacing f in (13) by K_{ζ}, and then using (14), we have that

$$\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)\mu(z)|g'(z)|}{|1-\zeta\phi(z)|^{\alpha+2}} \leqslant \frac{C}{\alpha+1}\int_{\mathbb{B}_1}|\xi(K_{\zeta})|d\rho(\xi).$$

Integrating over all $\zeta\in\mathbb{D}$ with respect to ωdA, using (1) and then applying Fubini's theorem, we get

$$\int_{\mathbb{D}}\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)\mu(z)|g'(z)|}{|1-\overline{\zeta}\phi(z)|^{\alpha+2}}\frac{(1-|\zeta|^2)^{\alpha}}{\nu(\zeta)}dA(\zeta) \leqslant \frac{C}{\alpha+1}\int_{\mathbb{B}_1}\int_{\mathbb{D}}|\xi(K_{\zeta})|\omega(\zeta)dA(\zeta)d\rho(\xi).$$

Now $K_\zeta \in \mathscr{H}^0_\nu$ and ξ in A^1_ω, so $\xi(K_\zeta) = \langle K_\zeta, \xi\rangle = \xi(\zeta)$. Thus using the fact that ρ is a probability Borel measure, we get

$$\begin{aligned}\int_{\mathbb{D}} \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)\mu(z)|g'(z)|}{|1-\overline{\zeta}\phi(z)|^{\alpha+2}} \frac{(1-|\zeta|^2)^\alpha}{\nu(\zeta)} dA(\zeta) &\leqslant \frac{C}{\alpha+1} \sup_{\xi\in\mathbb{B}_1} \int_{\mathbb{D}} |\xi(\zeta)|\omega(\zeta)dA(\zeta) \\ &= \frac{C}{\alpha+1} \sup_{\xi\in\mathbb{B}_1} \|\xi\|_{A^1_\omega} = \frac{C}{\alpha+1}.\end{aligned}$$

This completes the proof of the implication $(a) \Rightarrow (c)$.

$(a) \Rightarrow (b)$ Since the transpose of a nuclear operator is nuclear, see 3.1.8 in [15], so by an application of Proposition 2(i), it follows that $(a) \Rightarrow (b)$.

$(b) \Rightarrow (a)$ Recall that class of nuclear operators forms an operator ideal, that is, nuclear operators are a class of operators which do not fix a copy of l^∞, (see [8] and [9]), and $\mathscr{H}^0_\nu$ is a closed subspace of $\mathscr{H}^\infty_\nu$, so it is obvious that $(b) \Rightarrow (a)$. This completes the proof. □

THEOREM 3. *Let $\nu, \mu \in N_W(\mathbb{D})$, $\alpha > -1$ and ω a weight function such that $\{\nu, \omega\}$ is a normal pair and (1) holds. If $g \in \mathscr{B}^0_{(1-|z|^2)\mu(z)}$ and ϕ a self-map of $\mathbb{D}$ such that $T^\phi_g : \mathscr{H}^\infty_\nu \longrightarrow \mathscr{H}^\infty_\mu$ is bounded, then the following statements are equivalent.*

(a) $T^\phi_g : \mathscr{H}^0_\nu \longrightarrow \mathscr{H}^\infty_\mu$ *is nuclear.*

(b) $T^\phi_g : \mathscr{H}^\infty_\nu \longrightarrow \mathscr{H}^\infty_\mu$ *is nuclear.*

(c) $T^\phi_g : \mathscr{H}^0_\nu \longrightarrow \mathscr{H}^0_\mu$ *is nuclear.*

(d) $T^\phi_g : \mathscr{H}^\infty_\nu \longrightarrow \mathscr{H}^0_\mu$ *is nuclear.*

(e) For g, ϕ and ν satisfy the following condition:

$$M_\alpha = \int_{\mathbb{D}} \left[\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)\mu(z)|g'(z)|}{|1-\overline{\zeta}\phi(z)|^{\alpha+2}}\right] \frac{(1-|\zeta|^2)^\alpha}{\nu(\zeta)} dA(\zeta) < \infty.$$

Proof. By Theorem 1, we have that $(a) \Leftrightarrow (b) \Leftrightarrow (e)$. To complete the proof it is sufficient to prove that $(a) \Rightarrow (c) \Rightarrow (d) \Rightarrow (a)$.

$(a) \Rightarrow (c)$ Suppose that (a) holds. Then by Proposition 2(ii) $T^\phi_g : \mathscr{H}^0_\nu \longrightarrow \mathscr{H}^0_\mu$ is bounded. Moreover, $T^\phi_g : \mathscr{H}^0_\nu \longrightarrow \mathscr{H}^\infty_\mu$ is absolutely summing if and only if $T^\phi_g : \mathscr{H}^0_\nu \longrightarrow \mathscr{H}^0_\mu$. Thus the implication $(a) \Rightarrow (d)$ follows.

$(c) \Rightarrow (d)$ Using Proposition 2(ii) and the fact that the transpose of a nuclear operator is nuclear the implication follows.

$(d) \Rightarrow (a)$. The proof follows as the proof of implication $(b) \Rightarrow (a)$ in Theorem 1. This completes the proof. □

We identify several corollaries that provide valuable insights into specific cases of the main theorems. To avoid repetition, in all these corollaries, let $\nu, \mu \in N_W(\mathbb{D})$, $\alpha > -1$, and ω be a weight function such that $\{\nu, \omega\}$ forms a normal pair and condition (1) is satisfied.

COROLLARY 1. *If $g \in H(\mathbb{D})$ and ϕ is a self-map of $\mathbb{D}$ such that the operator $T_g^\phi : \mathscr{H}_\nu^\infty \longrightarrow \mathscr{B}_\mu$ is bounded, then the following statements are equivalent:*

(a) $T_g^\phi : \mathscr{H}_\nu^0 \longrightarrow \mathscr{B}_\mu$ is nuclear.

(b) $T_g^\phi : \mathscr{H}_\nu^\infty \longrightarrow \mathscr{B}_\mu$ is nuclear.

(c) The functions g, ϕ, and the weight ν satisfy the integral condition

$$\int_{\mathbb{D}} \left[\sup_{z\in\mathbb{D}} \frac{\mu(z)|g'(z)|}{|1-\overline{\zeta}\phi(z)|^{\alpha+2}} \right] \frac{(1-|\zeta|^2)^\alpha}{\nu(\zeta)} dA(\zeta) < \infty.$$

Furthermore, if $g \in \mathscr{B}_\mu^0$ and $T_g^\phi : \mathscr{H}_\nu^\infty \to \mathscr{B}_\mu$ is bounded, then the conditions (a), (b), and (c) are equivalent to the nuclearity of

$$T_g^\phi : \mathscr{H}_\nu^0 \longrightarrow \mathscr{B}_\mu^0,$$

which is further equivalent to the nuclearity of

$$T_g^\phi : \mathscr{H}_\nu^\infty \longrightarrow \mathscr{B}_\mu^0.$$

COROLLARY 2. *Let ϕ be a self-map of $\mathbb{D}$ such that the composition operator $C_\phi : \mathscr{H}_\nu^\infty \longrightarrow \mathscr{B}_\mu$ is bounded. Then the following statements are equivalent:*

(a) $C_\phi : \mathscr{H}_\nu^0 \longrightarrow \mathscr{B}_\mu$ is nuclear.

(b) $C_\phi : \mathscr{H}_\nu^\infty \longrightarrow \mathscr{B}_\mu$ is nuclear.

(c) The pair ϕ and ν satisfy the integral condition

$$\int_{\mathbb{D}} \left[\sup_{z\in\mathbb{D}} \frac{\mu(z)|\phi'(z)|}{|1-\overline{\zeta}\phi(z)|^{\alpha+2}} \right] \frac{(1-|\zeta|^2)^\alpha}{\nu(\zeta)} dA(\zeta) < \infty.$$

Furthermore, if $\phi \in \mathscr{B}_\mu^0$ and $C_\phi : \mathscr{H}_\nu^\infty \to \mathscr{B}_\mu$ is bounded, then the conditions (a), (b), and (c) above are equivalent to the nuclearity of

$$C_\phi : \mathscr{H}_\nu^0 \longrightarrow \mathscr{B}_\mu^0,$$

which is further equivalent to the nuclearity of

$$C_\phi : \mathscr{H}_\nu^\infty \longrightarrow \mathscr{B}_\mu^0.$$

COROLLARY 3. *If $g \in H(\mathbb{D})$ is such that the operator $T_g : \mathscr{H}^\infty_\nu \longrightarrow \mathscr{B}_\mu$ is bounded, then the following statements are equivalent:*

(a) $T_g : \mathscr{H}^0_\nu \longrightarrow \mathscr{B}_\mu$ *is nuclear.*

(b) $T_g : \mathscr{H}^\infty_\nu \longrightarrow \mathscr{B}_\mu$ *is nuclear.*

(c) The functions g and ν satisfy the integral condition

$$\int_{\mathbb{D}} \left[\sup_{z\in\mathbb{D}} \frac{\mu(z)|g'(z)|}{|1-\overline{\zeta}z|^{\alpha+2}} \right] \frac{(1-|\zeta|^2)^\alpha}{\nu(\zeta)} dA(\zeta) < \infty.$$

Furthermore, if $g \in \mathscr{B}^0_\mu$ and $T_g : \mathscr{H}^\infty_\nu \to \mathscr{B}_\mu$ is bounded, then the conditions (a), (b), and (c) are equivalent to the nuclearity of

$$T_g : \mathscr{H}^0_\nu \longrightarrow \mathscr{B}^0_\mu,$$

which is further equivalent to the nuclearity of

$$T_g : \mathscr{H}^\infty_\nu \longrightarrow \mathscr{B}^0_\mu.$$

THEOREM 4. *Let $\nu, \mu \in N_W(\mathbb{D})$, $\alpha > -1$ and ω a weight function such that $\{\nu, \omega\}$ is a normal pair and (1) holds. If $g \in H(\mathbb{D})$ and ϕ a self-map of $\mathbb{D}$ such that $S^\phi_g : \mathscr{H}^\infty_\nu \longrightarrow \mathscr{H}^\infty_\mu$ is bounded, then the following statements are equivalent.*

(a) $S^\phi_g : \mathscr{H}^0_\nu \longrightarrow \mathscr{H}^0_\mu$ *is nuclear.*

(b) $S^\phi_g : \mathscr{H}^\infty_\nu \longrightarrow \mathscr{H}^\infty_\mu$ *is nuclear.*

(c) g, ν, μ and ϕ satisfy the following condition:

$$N_\alpha = \int_{\mathbb{D}} \left[\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)\mu(z)|g(z)|}{|1-\overline{\zeta}\phi(z)|^{\alpha+3}} \right] \frac{(1-|\zeta|^2)^\alpha}{\nu(\zeta)} dA(\zeta) < \infty.$$

Proof. That $(a) \Leftrightarrow (b)$ can be proved, proceeding as in the proof of Theorem 2. Thus to complete the proof, we need to prove that $(a) \Leftrightarrow (c)$.

$(c) \Rightarrow (a)$. Proceeding as in the proof of $(c) \Rightarrow (a)$ of Theorem 2, for any $C > 1$, we can select z_i, $i = 1, 2, \cdots N$ such that

$$\sum_{i=1}^N \|T^\phi_g f_i\|_{\mathscr{H}^\infty_\mu} \leqslant C \sum_{i=1}^N (1-|z_i|^2)\mu(z_i)\left|f_i(\phi(z_i))g(z_i)\right| \tag{15}$$

and

$$f_i(w) = \langle f_i, K_w \rangle = (\alpha+1) \int_{\mathbb{D}} f_i(\zeta) \frac{(1-|\zeta|^2)^\alpha}{(1-\overline{\zeta}w)^{\alpha+2}} dA(\zeta).$$

Differentiating with respect to w, we have that

$$f_i'(w) = (\alpha+1)(\alpha+2)\int_{\mathbb{D}} f_i(\zeta)\frac{\overline{\zeta}(1-|\zeta|^2)^\alpha}{(1-\overline{\zeta}w)^{\alpha+3}}dA(\zeta).$$

Thus

$$\left|f_i'(\phi(z_i))\right| \leqslant (\alpha+1)(\alpha+2)\int_{\mathbb{D}} |f_i(\zeta)|\frac{\overline{\zeta}(1-|\zeta|^2)^\alpha}{|1-\overline{\zeta}\phi(z_i)|^{\alpha+3}}dA(\zeta). \tag{16}$$

Using (15) and (16), we have that

$$\begin{aligned}\sum_{i=1}^N \|T_g^\phi f_i\|_{\mathscr{H}^\infty_\mu} &\leqslant C(\alpha+1)(\alpha+2)\left(\sup_{w\in\mathbb{D}}\sum_{i=1}^N \left|f_i(w)\right|\nu(w)\right)\\ &\quad\times\int_{\mathbb{D}}\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)\mu(z)\left|g'(z)\right|}{\left|1-\overline{\zeta}\phi(z)\right|^{\alpha+3}}\frac{(1-|\zeta|^2)^\alpha}{\nu(\zeta)}dA(\zeta)\\ &= C(\alpha+1)(\alpha+2)\left(\sup_{w\in\mathbb{D}}\sum_{i=1}^N \left|f_i(w)\right|\nu(w)\right)N_\alpha\end{aligned} \tag{17}$$

The proof can now be completed proceeding as in the Theorem 2. We omit the details.

$(a)\Rightarrow(c)$ Once again, proceeding as in the proof of Theorem 2, there exists a probability Borel measure ρ on $\sigma(A^1_\omega,\mathscr{H}^0_\mu)$-compact unit ball $\mathbb{B}_1$ of A^1_ω and ξ in A^1_ω such that

$$\|S_g^\phi f\|_{\mathscr{H}^\infty_\mu} \leqslant C\int_{\mathbb{B}_1}|\xi(f)|d\rho(\xi) \tag{18}$$

for every $f\in\mathscr{H}^0_\mu$ and some constant $C>0$ independent of f. For every $\zeta\in\mathbb{D}$, we have that $K_\zeta\in\mathscr{H}^0_\mu$.

$$(\alpha+1)(\alpha+2)\sup_{z\in\mathbb{D}}\frac{|\zeta|(1-|z|^2)\mu(z)|g(z)|}{|1-\overline{\zeta}\phi(z)|^{\alpha+3}} \leqslant \|S_g^\phi K_\zeta\|_{\mathscr{B}_{\mu(z)(1-|z|^2)}} \leqslant C\|S_g^\phi K_\zeta\|_{\mathscr{H}^\infty_{\mu(z)}}. \tag{19}$$

Replacing f in (18) by K_ζ, and then using (19), we have that

$$\sup_{z\in\mathbb{D}}\frac{|\zeta|(1-|z|^2)\mu(z)|g(z)|}{|1-\overline{\zeta}\phi(z)|^{\alpha+3}} \leqslant \frac{C}{(\alpha+1)(\alpha+2)}\int_{\mathbb{B}_1}|\xi(K_\zeta)|d\rho(\xi). \tag{20}$$

Integrating (20) over $\mathbb{D}$ with respect to ωdA and proceeding as in the Theorem 2, we get

$$\int_{\mathbb{D}}\sup_{z\in\mathbb{D}}\frac{|\zeta|(1-|z|^2)\mu(z)|g(z)|}{|1-\overline{\zeta}\phi(z)|^{\alpha+3}}\frac{(1-|\zeta|^2)^\alpha}{\nu(\zeta)}dA(\zeta) \leqslant \frac{C}{(\alpha+1)(\alpha+2)}. \tag{21}$$

Now if $|\zeta| > 1/2$, then by (21), we have that

$$\int_{|\zeta|>1/2} \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)\mu(z)|g(z)|}{|1-\overline{\zeta}\phi(z)|^{\alpha+3}} \frac{(1-|\zeta|^2)^\alpha}{\nu(\zeta)} dA(\zeta) \leqslant \frac{2C}{(\alpha+1)(\alpha+2)}. \tag{22}$$

Further, if $|\zeta| \leqslant 1/2$, then we have that

$$\int_{|\zeta|\leqslant 1/2} \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)\mu(z)|g(z)|}{|1-\overline{\zeta}\phi(z)|^{\alpha+3}} \frac{(1-|\zeta|^2)^\alpha}{\nu(\zeta)} dA(\zeta) \leqslant \frac{2^{\alpha+1}\|g\|_{\mathscr{H}^\infty_\mu}}{\nu(1/2)}. \tag{23}$$

Thus

$$\begin{aligned} N_\alpha &\leqslant \int_{|\zeta|\leqslant 1/2} \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)\mu(z)|g(z)|}{|1-\overline{\zeta}\phi(z)|^{\alpha+3}} \frac{(1-|\zeta|^2)^\alpha}{\nu(\zeta)} dA(\zeta) \\ &\quad + \int_{|\zeta|>1/2} \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)\mu(z)|g(z)|}{|1-\overline{\zeta}\phi(z)|^{\alpha+3}} \frac{(1-|\zeta|^2)^\alpha}{\nu(\zeta)} dA(\zeta) \\ &\leqslant \frac{2^{\alpha+1}\|g\|_{\mathscr{H}^\infty_\mu}}{\nu(1/2)} + \frac{2C}{(\alpha+1)(\alpha+2)}. \end{aligned}$$

This completes the proof. □

We once again identify several corollaries that provide valuable insights into specific cases of the main theorem of this section under the following assumptions: let $\nu, \mu \in N_W(\mathbb{D})$, $\alpha > -1$, and let ω be a weight function such that the pair $\{\nu, \omega\}$ forms a normal pair and condition (1) is satisfied.

COROLLARY 4. *If $g \in \mathscr{H}^0_{(1-|z|^2)\mu(z)}$ and ϕ an analytic self-map of $\mathbb{D}$ such that $S^\phi_g : \mathscr{H}^\infty_\nu \longrightarrow \mathscr{H}^\infty_\mu$ is bounded, then the conditions (a), (b), and (c) in Theorem 4 are equivalent to the nuclearity of the operator $S^\phi_g : \mathscr{H}^\infty_\nu \longrightarrow \mathscr{H}^0_\mu$ which is further equivalent to the nuclearity of $S^\phi_g : \mathscr{H}^0_\nu \longrightarrow \mathscr{H}^0_\mu$.*

COROLLARY 5. *If $g \in H(\mathbb{D})$and ϕ an analytic self-map of $\mathbb{D}$ such that $S^\phi_g : \mathscr{H}^\infty_\nu \longrightarrow B_\mu$ is bounded, then the following statements are equivalent.*

(a) $S^\phi_g : \mathscr{H}^0_\nu \longrightarrow \mathscr{B}_\mu$ is nuclear.

(b) $S^\phi_g : \mathscr{H}^\infty_\nu \longrightarrow \mathscr{B}_\mu$ is nuclear.

(c) For g, ϕ and ν satisfy the following condition:

$$\int_{\mathbb{D}} \left[\sup_{z\in\mathbb{D}} \frac{\mu(z)|g(z)|}{|1-\overline{\zeta}\phi(z)|^{\alpha+3}} \right] \frac{(1-|\zeta|^2)^\alpha}{\nu(\zeta)} dA(\zeta) < \infty.$$

If $g \in \mathscr{B}^0_\mu$ and ϕ an analytic self-map of $\mathbb{D}$ such that $S^\phi_g : \mathscr{H}^\infty_\nu \longrightarrow \mathscr{B}_\mu$ is bounded, then the conditions (a), (b), and (c) above are equivalent to the nuclearity of the operator $S^\phi_g : \mathscr{H}^\infty_\nu \longrightarrow \mathscr{B}^0_\mu$ which is further equivalent to the nuclearity of $S^\phi_g : \mathscr{H}^0_\nu \longrightarrow \mathscr{B}^0_\mu$.

We can also obtain the following corollaries.

COROLLARY 6. *If ϕ is an analytic self-map of $\mathbb{D}$ such that $C_\phi D : \mathscr{H}^\infty_\nu \longrightarrow B_\mu$ is bounded, then the following statements are equivalent.*

(a) $C_\phi D : \mathscr{H}^0_\nu \longrightarrow \mathscr{B}_\mu$ is nuclear.

(b) $C_\phi D : \mathscr{H}^\infty_\nu \longrightarrow \mathscr{B}_\mu$ is nuclear.

(c) ϕ and ν satisfy the following condition:

$$\int_{\mathbb{D}} \left[\sup_{z\in\mathbb{D}} \frac{\mu(z)|\phi'(z)|}{|1-\overline{\zeta}\phi(z)|^{\alpha+3}} \right] \frac{(1-|\zeta|^2)^\alpha}{\nu(\zeta)} dA(\zeta) < \infty.$$

If ϕ an analytic self-map of $\mathbb{D}$ such that $\phi \in \mathscr{H}^0_\mu$ and $C_\phi D : \mathscr{H}^\infty_\nu \longrightarrow \mathscr{B}_\mu$ is bounded, then the conditions (a), (b), and (c) above are equivalent to the nuclearity of the operator $C_\phi D : \mathscr{H}^\infty_\nu \longrightarrow \mathscr{B}^0_\mu$ which is further equivalent to the nuclearity of $C_\phi D : \mathscr{H}^0_\nu \longrightarrow \mathscr{B}^0_\mu$.

COROLLARY 7. *Let $g \in H(\mathbb{D})$ be such that $S_g : \mathscr{H}^\infty_\nu \longrightarrow B_\mu$ is bounded, then the following statements are equivalent.*

(a) $S_g : \mathscr{H}^0_\nu \longrightarrow \mathscr{B}_\mu$ is nuclear.

(b) $S_g : \mathscr{H}^\infty_\nu \longrightarrow \mathscr{B}_\mu$ is nuclear.

(c) For g and ν satisfy the following condition:

$$\int_{\mathbb{D}} \left[\sup_{z\in\mathbb{D}} \frac{\mu(z)|g'(z)|}{|1-\overline{\zeta}z|^{\alpha+3}} \right] \frac{(1-|\zeta|^2)^\alpha}{\nu(\zeta)} dA(\zeta) < \infty.$$

If $g \in \mathscr{B}^0_{\mu(z)}$ be such that $S_g : \mathscr{H}^\infty_\nu \longrightarrow \mathscr{B}_\mu$ is bounded, then the conditions (a), (b), and (c) above are equivalent to the nuclearity of the operator $S_g : \mathscr{H}^\infty_\nu \longrightarrow \mathscr{B}^0_\mu$ which is further equivalent to the nuclearity of $S_g : \mathscr{H}^0_\nu \longrightarrow \mathscr{B}^0_\mu$.

5. Nuclear T^ϕ_g and S^ϕ_g between Bloch spaces of order β

Finally, we apply our results to T^ϕ_g and S^ϕ_g acting between weighted Banach spaces and Bloch spaces of order β, where $\beta > 0$. Recall that if ν is a classical weight, that is $\nu(z) = (1-|z|^2)^\beta$, then the spaces $\mathscr{H}^\infty_\nu$, $\mathscr{H}^0_\nu$, $\mathscr{B}_\nu$ and $\mathscr{B}^0_\nu$ reduces respectively to weighted and Bloch spaces of order β, denoted by $\mathscr{H}^\infty_\beta$, $\mathscr{H}^0_\beta$, $\mathscr{B}_\beta$ and $\mathscr{B}^0_\beta$. Moreover, for $\beta > 1$, we have that $\mathscr{H}^0_{\beta-1} = \mathscr{B}^0_\beta$ and $\mathscr{H}^\infty_{\beta-1} = \mathscr{B}^\infty_\beta$ with equivalent norm. Thus, we have the following corollary.

COROLLARY 8. *Let $\beta > 1$, $\gamma > 0$, $g \in H(\mathbb{D})$ and ϕ an analytic self-map of $\mathbb{D}$ such that $T^\phi_g : \mathscr{B}_\beta \longrightarrow \mathscr{B}_\gamma$ is bounded. Then the following statements are equivalent.*

(a) $T_g^\phi : \mathscr{B}_\beta^0 \longrightarrow \mathscr{B}_\gamma$ *is nuclear.*

(b) $T_g^\phi : \mathscr{B}_\beta \longrightarrow \mathscr{B}_\gamma$ *is nuclear.*

(c) g, β, γ *and* ϕ *satisfy the following condition:*

$$P_\alpha = \int_{\mathbb{D}} \left[\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\gamma |g'(z)|}{|1-\overline{\zeta}\phi(z)|^{\alpha+2}} \right] (1-|\zeta|^2)^{\alpha-\beta+1} dA(\zeta) < \infty.$$

The case when $\beta = 1$ is not covered in the above corollary.

However, the argument in the proof of the Theorem 2.2 in [16] can be modified to settle this case also. We consider it in the next theorem.

THEOREM 5. *Let* $g \in H(\mathbb{D})$ *and* ϕ *an analytic self-map of* $\mathbb{D}$ *such that* $T_g^\phi : \mathscr{B} \longrightarrow \mathscr{B}$ *is bounded. Then the following statements are equivalent.*

(a) $T_g^\phi : \mathscr{B}^0 \longrightarrow \mathscr{B}$ *is nuclear.*

(b) g *and* ϕ *satisfy the following condition:*

$$M = \int_{\mathbb{D}} \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)|\phi'(z)|}{|1-\overline{w}\phi(z)|^3} |g(z)| dA(w) < \infty.$$

Proof. $(b) \Rightarrow (a)$ Since $c_0 \simeq \mathscr{B}^0$ and polynomials are dense in $\mathscr{B}^0$. Moreover, by Corollary 1.5 in [17], we have that $f \in \mathscr{B}$, then we have that

$$f'(z) = 2\int_{\mathbb{D}} \frac{1-|w|^2}{(1-z\overline{w})^3} f'(w) dA(w). \tag{24}$$

Consider polynomials $p_1, p_2, \cdots, p_N$ in the definition of absolutely summing and then using (24), we have

$$\sum_{i=1}^N \|T_g^\phi p_i\|_{\mathscr{B}} \leqslant C \left(\sup_{w\in\mathbb{D}} \sum_{i=1}^N \left|p_i'(w)\right|(1-|w|^2) \right) \int_{\mathbb{D}} \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)\left|\phi'(z)\right|}{\left|1-\overline{w}\phi(z)\right|^3} \left|g(z)\right| dA(w).$$

The proof can be settled proceeding as in the proof of the Theorem 2.

$(a) \Rightarrow (b)$ By Theorem 1, there exists a probability Borel measure ρ on $\sigma((\mathscr{B}^0)^*, \mathscr{B})$-compact unit ball B' of $(\mathscr{B}^0)^* = A^1$ such that

$$\|T_g^\phi f\|_{\mathscr{B}} \leqslant C \int_{B'} |\xi(f)| d\rho(\xi)$$

for every $f \in \mathscr{B}_0$ and some constant $C > 0$ independent of f. Next for every $w \in \mathbb{D}$, we consider $f_w(z) = 1/(1-\overline{w}z)^2$ which lies in $\mathscr{B}_0 \cap H^\infty$. By duality and reproducing property of f_w in the Bergman space $(\mathscr{B}_0)^* = A^1$, we have

$$\xi(f_w) = \langle f_w, h\rangle = \int_{\mathbb{D}} h(z)\overline{f_w(z)} dA(z) = \int_{\mathbb{D}} \frac{h(z)}{(1-\overline{z}w)^2} dA(z) = h(w).$$

Also,

$$\sup_{z\in\mathbb{D}} \frac{2|w|(1-|z|^2)|\phi'(z)||g(z)|}{|1-\bar{w}\phi(z)|^3} = \|T_g^\phi f_w\|_{A^1} \leqslant \|T_g^\phi f_w\|_{\mathscr{B}}$$

Now proceeding as in the proof of Theorem 4, we can easily see that (b) holds. This completes the proof. □

Similarly, we have the following corollary.

COROLLARY 9. *Let $\beta > 1$, $\gamma > 0$, $g \in H(\mathbb{D})$ and ϕ an analytic self-map of $\mathbb{D}$ such that $S_g^\phi : \mathscr{B}_\beta \longrightarrow \mathscr{B}_\gamma$ is bounded. Then the following statements are equivalent.*

(a) $S_g^\phi : \mathscr{B}_\beta^0 \longrightarrow \mathscr{B}_\gamma$ is nuclear.

(b) $S_g^\phi : \mathscr{B}_\beta \longrightarrow \mathscr{B}_\gamma$ is nuclear.

(c) For g, ϕ and ν satisfy the following condition:

$$Q_\alpha = \int_{\mathbb{D}} \left[\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\gamma |g(z)|}{|1-\bar{\zeta}\phi(z)|^{\alpha+3}}\right](1-|\zeta|^2)^{\alpha-\beta+1} dA(\zeta) < \infty.$$

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