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INTEGRATION OPERATORS ACTING ON DIFFERENT WEIGHTED BERGMAN SPACES INDUCED BY DOUBLING WEIGHTS

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Abstract. We completely characterize the boundedness of the Volterra-type integration operators T_g acting from the weighted Bergman spaces A^p_ω to A^q_ν for all $0<p,q<\infty$, $\omega,\nu\in\mathscr{D}$, which extending earlier results by Rättyä. Furthermore, the authors investigates the essential norms of Volterra-type operators between weighted Bergman spaces.

1. Introduction main results

Let $\mathbb{D}$ be the unit disc in the complex plane and $\mathscr{H}(\mathbb{D})$ the algebra of all analytic functions in $\mathbb{D}$. A non-negative integrable function ω on $\mathbb{D}$, defined by $\omega(z)=\omega(|z|)$ for all $z\in\mathbb{D}$, is called a radial weight. For $0<p<\infty$ and a radial weight ω, the weighted Bergman space A^p_ω consists of $f\in\mathscr{H}(\mathbb{D})$ such that

$$\|f\|^p_{A^p_\omega}=\int_{\mathbb{D}}|f(z)|^p\,\omega(z)dA(z)<\infty$$

where $dA(z)=\pi^{-1}dxdy$ is the normalized Lebesgue area measure on $\mathbb{D}$. As usual, A^p_α denotes the weighted Bergman space induced by the standard radial weight $\omega(z)=(1-|z|)^\alpha$, where $-1<\alpha<\infty$.

For a radial weight ω, write $\widehat{\omega}(z)=\int_{|z|}^1\omega(s)ds$ for all $z\in\mathbb{D}$. A radial weight ω belongs to the class $\widehat{\mathscr{D}}$ if there exists a constant $C=C(\omega)\geqslant 1$ such that $\widehat{\omega}(r)\leqslant C\widehat{\omega}(\frac{1+r}{2})$ for all $0\leqslant r<1$. Moreover, A radial weight ω belongs to the class $\check{\mathscr{D}}$ if there exist $K=K(\omega)>1$ and $C=C(\omega)>1$ such that $\widehat{\omega}(r)\geqslant C\widehat{\omega}(1-\frac{1-r}{K})$ for all $0\leqslant r<1$. The intersection $\check{\mathscr{D}}\cap\widehat{\mathscr{D}}$ is denoted by $\mathscr{D}$, and this is the class of weights that we mainly work with. For recent developments on Bergman spaces induced by weights in $\mathscr{D}$, see [17] and the reference therein.

Given $g\in\mathscr{H}(\mathbb{D})$, the Volterra-type operator T_g with symbol g is defined on $\mathscr{H}(\mathbb{D})$ as follows

$$T_gf(z)=\int_0^z f(\zeta)g'(\zeta)d\zeta,\quad z\in\mathbb{D},$$

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where $f \in \mathscr{H}(\mathbb{D})$. Similarly, we define its companion operator

$$L_g f(z) = \int_0^z f'(\zeta) g(\zeta) d\zeta, \quad z \in \mathbb{D}.$$

A natural question concerning the integration operator is how to characterize the boundedness of T_g on the holomorphic space by the symbols g. Pommerenke was probably one of the first authors to consider the operator T_g [14]. His research shows that T_g is bounded on the Hardy space H^2 if and only if g is of bounded mean oscillation, which was extended to the Hardy space H^p for $p \geqslant 1$ by Aleman and Siskakis [3]. In [4], Aleman and Cima gave descriptions of bounded operators $T_g : H^p \to H^q$ for $0 < p, q < \infty$ in term of the symbol g. When it comes to the Bergman spaces, Aleman and Siskakis [1] studied T_g on weighted Bergman spaces A^p_ω induced by a large class of weights which includes the standard weights. They obtained a necessary and sufficient condition on g for T_g which is bounded on A^p_ω with restriction that $p \geqslant 1$. Rättyä [23] generalized this result from the weighted Bergman space A^p_α into another weighted Bergman space A^q_β for $0 < p, q < \infty$ with standard weights. Along this line, Pau and Peláez [15] gave a complete characterization of bounded operators T_g mapping from A^p_ω to A^q_ω for $0 < p, q < \infty$ when the weight ω is rapidly decreasing. In a recent work of Peláez and Rättyä [22], they studied T_g acting on spaces A^p_ω induced by radial weights that are either regular or rapidly increasing. As it turns out, many of their results actually hold true for a wider class of radial weights ω satisfying the doubling property [21]. The operator T_g have been extensively studied in a variety of function spaces, see for example [5, 6, 12, 25]. For a survey of the action of T_g on some classical function spaces, including the Hardy spaces and the Bergman spaces, see for example [2, 24]. Moreover, for the higher-dimensional variant case, the generalization of the operator T_g was introduced by Hu [7]. After then, a lot of research on the this operator acting on Hardy and Bergman spaces has been done (see [11, 13, 16, 26] for example).

Inspired by the above work, we study the boundedness and compactness of Volterra-type operator T_g acting form A^p_ω to A^q_ν for $0 < p, q < \infty$. The main results of this paper are given in the following.

To state our results, for $0 < p, q < \infty$ and radial weight ω, ν, define

$$\gamma(z) = \gamma(\omega, \nu, p, q)(z) = \frac{\widehat{\nu}(z)^{\frac{1}{q}}(1-|z|)^{\frac{1}{q}}}{\widehat{\omega}(z)^{\frac{1}{p}}(1-|z|)^{\frac{1}{p}}}, \quad z \in \mathbb{D}. \tag{1}$$

The space $\mathscr{B}_\gamma$ consists of $f \in \mathscr{H}(\mathbb{D})$ such that

$$\|f\|_{\mathscr{B}_\gamma} = \sup_{z \in \mathbb{D}} |f'(z)|(1-|z|)\gamma(z) + |f(0)| < \infty. \tag{2}$$

The space $\mathscr{B}^0_\gamma$ consists of $f \in \mathscr{H}(\mathbb{D})$ such that

$$\lim_{|z| \to 1^-} |f'(z)|(1-|z|)\frac{\widehat{\nu}(z)^{\frac{1}{q}}(1-|z|)^{\frac{1}{q}}}{\widehat{\omega}(z)^{\frac{1}{p}}(1-|z|)^{\frac{1}{p}}} = 0. \tag{3}$$

We are ready to state our main results as follows.

THEOREM 1. *Let* $0 < p,q < \infty$ *and* $\omega,\ \nu \in \mathscr{D}$ *such that* $\sigma = \sigma(z) = \frac{\widehat{\nu}(z)^{\frac{p}{p-q}}}{\widehat{\omega}(z)^{\frac{q}{p-q}}} \frac{1}{1-|z|} \in \mathscr{D}$.

(a) If $q < p$, *then the following assertions are equivalent:*

(i) $T_g : A^p_\omega \to A^q_\nu$ *is bounded;*

(ii) $T_g : A^p_\omega \to A^q_\nu$ *is compact;*

(iii) $g \in A^s_\sigma$, *where* $\frac{1}{s} = \frac{1}{q} - \frac{1}{p}$.

(b) If $p \leqslant q$, *then* $T_g : A^p_\omega \to A^q_\nu$ *is bounded if and only if* $g \in \mathscr{B}_\gamma$.

(c) If $p \leqslant q$, *then* $T_g : A^p_\omega \to A^q_\nu$ *is compact if and only if* $g \in \mathscr{B}^0_\gamma$.

If $\omega(z) = (1-|z|)^\alpha$ and $\nu(z) = (1-|z|)^\beta$ for $-1 < \alpha,\beta < \infty$, then $\widehat{\omega}(z) = (1-|z|)^{\alpha+1}$ and $\widehat{\nu}(z) = (1-|z|)^{\beta+1}$ for all $z \in \mathbb{D}$. So, it is obvious that our results generalize some known results, as the special case when the involving weight ω is standard. Being precise, part (a) of Theorem 1 extends the result of Li and Stević [11] for one-dimensional case, and the results of Aleman and Siskakis are just special cases of part (b) and (c) with $p = q$ (see also the results of Rättyä [23] are just special cases of part (b) and (c)).

We need two specific tools for the Proof of Theorem 1. The first one concerns Littlewood-Paley formula for radial weights. Recall that it is well-known fact that

$$\|f\|^p_{A^p_\omega} \asymp \int_{\mathbb{D}} |f'(z)|^p (1-|z|)^p \omega(z) dA(z) + |f(0)|^p \tag{4}$$

for $f \in \mathscr{H}(\mathbb{D})$, $\omega \in \mathscr{D}$ is repeatedly used in the subsequent proofs. Since $T_g(f)(0) = 0$, (4) shows that

$$\|T_g f\|^q_{A^q_\nu} \asymp \int_{\mathbb{D}} |f(z)|^q d\mu(z),$$

where $d\mu(z) = |g'(z)|^q (1-|z|)^q \frac{\widehat{\nu}(z)}{1-|z|} dA(z)$. It means that $T_g : A^p_\omega \to A^q_\nu$ is bounded (compact) if and only if $I_d : A^p_\omega \to L^q_\mu$ is bounded (compact). Therefore, we need continuous embeddings $A^p_\omega \subset L^q_\mu$. Recall that a positive Borel measure μ on $\mathbb{D}$ is a (vanishing) q-Carleson measure for A^p_ω if the identity operator $I_d : A^p_\omega \to L^q_\mu$ is (compact) bounded. A complete characterization of such measures in the case $\omega \in \widehat{\mathscr{D}}$ can be found in [19, 20]. To complete the proof of Theorem 1, we will need a variant of the above characterization of Carleson measures in the case $\omega \in \mathscr{D}$ with pseudohyperbolic discs which can be found in [8, 9].

An analogue result of companion operator L_g state as follow, which extends the result of Li and Stević [11] for one-dimensional case.

THEOREM 2. *Let* $0 < p,q < \infty$ *and* $\omega,\ \nu \in \mathscr{D}$ *such that* $\sigma = \sigma(z) = \frac{\widehat{\nu}(z)^{\frac{p}{p-q}}}{\widehat{\omega}(z)^{\frac{q}{p-q}}} \frac{1}{1-|z|} \in \mathscr{D}$.

(a) If $q < p$, then the following assertions are equivalent:

(i) $L_g : A^p_\omega \to A^q_\nu$ is bounded;

(ii) $L_g : A^p_\omega \to A^q_\nu$ is compact;

(iii) $g \in A^s_\sigma$, where $\frac{1}{s} = \frac{1}{q} - \frac{1}{p}$.

(b) If $p \leqslant q$, then $L_g : A^p_\omega \to A^q_\nu$ is bounded if and only if $\sup_{z\in\mathbb{D}} |g(z)|\gamma(z) < \infty$.

(c) If $p \leqslant q$, then $L_g : A^p_\omega \to A^q_\nu$ is compact if and only if

$$\lim_{|a|\to 1^-} |g(a)| \frac{\widehat{\nu}(a)^{\frac{1}{q}}(1-|a|)^{\frac{1}{q}}}{\widehat{\omega}(a)^{\frac{1}{p}}(1-|a|)^{\frac{1}{p}}} = 0.$$

The Littlewood-Paley formula implies that the bounded integration operators L_g from A^p_ω to A^q_ν can be characterised once the bounded differentiation operators $Df = f'$ from A^p_ω to L^q_μ are characterised, where $d\mu(z) = |g(z)|^q(1-|z|)^q \frac{\widehat{\nu}(z)}{1-|z|} dA(z)$.

Our next goal is to give an asymptotic formula for the essential norm of $T_g : A^p_\omega \to A^q_\nu$. Recall that for two Banach spaces X and Y, the essential norm of a bounded operator $T : X \to Y$, denoted by $\|T\|_e$, is defined to be the distance from T to the set of the compact operators $K : X \to Y$, i.e.

$$\|T\|_e = \inf_K \sup_{\|f\|_X=1} \|(T-K)(f)\|_Y = \inf_K \|T-K\|.$$

The following estimates for the essential norm of $T_g : A^p_\omega \to A^q_\nu$ for the case $1 < p \leqslant q < \infty$ which is generalization of Rättyä's result [23].

THEOREM 3. *Let $1 < p \leqslant q < \infty$ and $\omega, \nu \in \mathscr{D}$. If $T_g : A^p_\omega \to A^q_\nu$ is bounded, then*

$$\|T_g\|_e \asymp \limsup_{|z|\to 1^-} |g'(z)|(1-|z|)\gamma(z).$$

The rest of the paper is organized as follows. We first give some auxiliary results needed in the proofs of the main results in next section. In Section 3, we characterize the boundedness and compactness of the integration operators. Finally, we indicate how we can get main results following from these results in Section 4.

Here and henceforth in the paper we use the notation $a \asymp b$ to indicate that the quantities a and b are proportional to each other in the sense that $c^{-1}a \leqslant b \leqslant ca$ for some inessential constant $c \geqslant 1$. The constant c may depend on p, q and ω but not on g and other variables involved. The expression $a \lesssim b$ means that there is a positive constant c such that $a \leqslant cb$.

2. Preliminaries results

Before proving the result, we present several known results which are important in the proof of main results. We start with a generalization of the Littlewood-Paley formula on A^p_ω which can be found in [17].

THEOREM 4. *Let ω be a radial weight, $0 < p < \infty$ and $k \in \mathbb{N}$. Then*

$$\|f\|^p_{A^p_\omega} \asymp \int_{\mathbb{D}} |f^{(k)}(z)|^p (1-|z|)^{kp} \omega(z) dA(z) + \sum_{j=0}^{k-1} |f^{(j)}(0)|^p, \quad f \in \mathscr{H}(\mathbb{D}),$$

if and only if $\omega \in \mathscr{D}$.

The next result shows that if $\omega \in \mathscr{D}$, then there exists a continuous and locally smooth weight that induces the same Bergman space as ω which can be found in [18].

LEMMA 1. *Let $0 < p < \infty$. We write $\tilde{\omega}(r) = \frac{\widehat{\omega}(r)}{1-r}$ for all $0 \leqslant r < 1$. Then $\|f\|_{A^p_{\tilde{\omega}}} = \|f\|_{A^p_\omega}$ for all $f \in \mathscr{H}(\mathbb{D})$ if and only if $\omega \in \mathscr{D}$.*

For $a, z \in \mathbb{D}$, let $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$. Then the pseudohyperbolic disc of center a and of radius r is $\Delta(a,r) = \{z \in \mathbb{D} : |\varphi_a(z)| < r\}$. It is well known that $\Delta(a,r)$ is an Euclidean disk centered at $(1-r^2)a/(1-r^2|a|^2)$ and of radius $(1-|a|^2)r/(1-r^2|a|^2)$. The Carleson square induced by a point $a \in \mathbb{D} \setminus \{0\}$ is the set

$$S(a) = \left\{ z \in \mathbb{D} : 1-|z| \leqslant 1-|a|, \left| \frac{\arg(a\bar{z})}{2\pi} \right| \leqslant \frac{1-|a|}{2} \right\}.$$

The next lemma is related to embedding theorem can be found in [8] and [9].

LEMMA 2. *Let $0 < p, q < \infty$ and $\omega \in \mathscr{D}$, and let μ be a positive Borel measure on $\mathbb{D}$.*

(i) If $p \leqslant q$, then there exists $r = r(\omega) \in (0,1)$ such that μ is a q-Carleson measure for A^p_ω if and only if $\sup_{a\in\mathbb{D}} \frac{\mu(\Delta(a,r))}{\omega(\Delta(a,r))^{\frac{q}{p}}} < \infty$. Moreover,

$$\|I_d\|^q_{A^p_\omega \to L^q_\mu} \asymp \sup_{a\in\mathbb{D}} \frac{\mu(\Delta(a,r))}{\omega(\Delta(a,r))^{\frac{q}{p}}}.$$

(ii) If $p \leqslant q$, then there exists $r = r(\omega) \in (0,1)$ such that the identity operator $I_d : A^p_\omega \to L^q_\mu$ is compact if and only if

$$\lim_{|a|\to 1^-} \frac{\mu(\Delta(a,r))}{\omega(\Delta(a,r))^{\frac{q}{p}}} = 0.$$

(iii) If $q < p$, then the following conditions are equivalent:

(a) $I_d : A^p_\omega \to L^q_\mu$ *is bounded;*

(b) $I_d : A^p_\omega \to L^q_\mu$ *is compact;*

(c) *the function*

$$\Theta^\omega_\mu(z) = \frac{\mu(\Delta(z,r))}{\omega(S(z))} \quad z \in \mathbb{D}\setminus\{0\},$$

belongs to $L^{\frac{p}{p-q}}_{\tilde{\omega}}$ *for some (equivalently for all)* $r \in (0,1)$. *Moreover*

$$\|I_d\|^q_{A^p_\omega \to L^q_\mu} \asymp \|\Theta^\omega_\mu\|_{L^{\frac{p}{p-q}}_{\tilde{\omega}}}.$$

The generalization version of this lemma can be found in [10].

LEMMA 3. *Let* $0 < p,q < \infty$, $\omega \in \mathscr{D}$ *and* $n \in \mathbb{N}\cup\{0\}$. *Let* μ *be a positive Borel measure on* $\mathbb{D}$.

(i) If $0 < p \leqslant q < \infty$, *then* $D^n : A^p_\omega \to L^q_\mu$ *is bounded if and only if*

$$\sup_{a\in\mathbb{D}} \frac{\mu(\Delta(a,r))}{\omega(S(a))^{\frac{q}{p}}(1-|a|)^{nq}} < \infty.$$

(ii) If $0 < p \leqslant q < \infty$, *then* $D^n : A^p_\omega \to L^q_\mu$ *is compact if and only if*

$$\lim_{|a|\to 1^-} \frac{\mu(\Delta(a,r))}{\omega(S(a))^{\frac{q}{p}}(1-|a|)^{nq}} = 0.$$

(iii) If $0 < q < p < \infty$, *then the following conditions are equivalent:*

(a) $D^n : A^p_\omega \to L^q_\mu$ *is bounded;*

(b) $D^n : A^p_\omega \to L^q_\mu$ *is compact;*

(c) *the function*

$$z \to \frac{\mu(\Delta(z,r))}{\omega(S(z))(1-|z|)^{nq}},$$

belongs to $L^{\frac{p}{p-q}}_{\tilde{\omega}}$ *for some (equivalently for all)* $r \in (0,1)$.

The pointwise estimate of functions in weighted Bergman space is a generalization of the one on the standard weighted Bergman spaces, see [28].

LEMMA 4. *Let* $\omega \in \mathscr{D}$, $0 < p < \infty$ *and* $n \in \mathbb{N}\cup\{0\}$. *Then there exists a constant* $C = C(\omega,p) > 0$ *such that*

$$|f^{(n)}(z)| \leqslant C\frac{\|f\|_{A^p_\omega}}{\omega(S(z))^{\frac{1}{p}}(1-|z|)^n}, \quad z \in \mathbb{D}.$$

3. Boundedness and compactness of integration operators

PROPOSITION 1. *Let $g \in \mathscr{H}(\mathbb{D}), 0 < p \leqslant q < \infty,\ \omega, \nu \in \mathscr{D}$. Then*

(a) $T_g : A^p_\omega \to A^q_\nu$ is bounded if and only if $g \in \mathscr{B}_{d\gamma}$.

(b) $L_g : A^p_\omega \to A^q_\nu$ is bounded if and only if $\sup_{z\in\mathbb{D}} |g(z)|\gamma(z) < \infty$.

Proof. (a) If $f \in A^p_\omega$, then by Theorem 4, Lemma 1 and Lemma 4

$$
\begin{aligned}
&\|T_g(f)\|^q_{A^q_\nu} \\
&\asymp \int_{\mathbb{D}} |f(z)|^q |g'(z)|^q (1-|z|)^q \nu(z) dA(z) \\
&\asymp \int_{\mathbb{D}} |f(z)|^q |g'(z)|^q (1-|z|)^{q-1} \widehat{\nu}(z) dA(z) \\
&\lesssim \sup_{z\in\mathbb{D}} |g'(z)|^q (1-|z|)^q \frac{\widehat{\nu}(z)(1-|z|)}{\widehat{\omega}(z)^{\frac{q}{p}}(1-|z|)^{\frac{q}{p}}} \int_{\mathbb{D}} |f(z)|^q \widehat{\omega}(z)^{\frac{q}{p}} (1-|z|)^{\frac{q}{p}} (1-|z|)^{-2} dA(z) \\
&\lesssim \sup_{z\in\mathbb{D}} |g'(z)|^q (1-|z|)^q \frac{\widehat{\nu}(z)(1-|z|)}{\widehat{\omega}(z)^{\frac{q}{p}}(1-|z|)^{\frac{q}{p}}} \\
&\quad \times \int_{\mathbb{D}} |f(z)|^p \frac{\widehat{\omega}(z)}{1-|z|} |f(z)|^{q-p} \widehat{\omega}(z)^{\frac{q}{p}-1} (1-|z|)^{\frac{q}{p}-1} dA(z) \\
&\lesssim \sup_{z\in\mathbb{D}} |g'(z)|^q (1-|z|)^q \frac{\widehat{\nu}(z)(1-|z|)}{\widehat{\omega}(z)^{\frac{q}{p}}(1-|z|)^{\frac{q}{p}}} \|f\|^{q-p}_{A^p_\omega} \int_{\mathbb{D}} |f(z)|^p \frac{\widehat{\omega}(z)}{1-|z|} dA(z) \\
&\lesssim \sup_{z\in\mathbb{D}} |g'(z)|^q (1-|z|)^q \frac{\widehat{\nu}(z)(1-|z|)}{\widehat{\omega}(z)^{\frac{q}{p}}(1-|z|)^{\frac{q}{p}}} \|f\|^q_{A^p_\omega}
\end{aligned}
$$

and therefore $T_g : A^p_\omega \to A^q_\nu$ is bounded if $g \in \mathscr{B}_{d\gamma}$.

Now assume that $T_g : A^p_\omega \to A^q_\nu$ is bounded and let $f_a(z) = \left(\frac{1-|a|^2}{1-\overline{a}z}\right)^\gamma \frac{1}{(\widehat{\omega}(a)(1-|a|))^{\frac{1}{p}}}$. Then, by Lemma 4 and Theorem 4

$$
\begin{aligned}
\frac{|g'(a)|}{(\widehat{\omega}(a)(1-|a|))^{\frac{1}{p}}} = \left|(T_g(f_a))'(a)\right| &\lesssim \frac{\|T_g(f_a)\|_{A^q_\nu}}{(\widehat{\nu}(a)(1-|a|))^{\frac{1}{q}}(1-|a|)} \\
&\lesssim \frac{\|f_a\|_{A^p_\omega}}{(\widehat{\nu}(a)(1-|a|))^{\frac{1}{q}}(1-|a|)} \\
&\lesssim \frac{1}{(\widehat{\nu}(a)(1-|a|))^{\frac{1}{q}}(1-|a|)}
\end{aligned}
$$

and it follows that $g \in \mathscr{B}_{d\gamma}$.

(b) First assume that $\sup_{z\in\mathbb{D}}|g(z)|\gamma(z)<\infty$. If $f\in A^p_\omega$, then by Theorem 4, Lemma 1 and Lemma 4

$$\begin{aligned}\|L_g(f)\|^q_{A^q_\nu} &\asymp \int_{\mathbb{D}}|f'(z)|^q|g(z)|^q(1-|z|)^q\nu(z)dA(z)\\ &\asymp \int_{\mathbb{D}}|f'(z)|^q|g(z)|^q(1-|z|)^{q-1}\widehat{\nu}(z)dA(z)\\ &\lesssim \int_{\mathbb{D}}|f'(z)|^q\widehat{\omega}(z)^{\frac{q}{p}}(1-|z|)^{\frac{q}{p}}(1-|z|)^{q-2}dA(z)\\ &= \int_{\mathbb{D}}|f'(z)|^p(1-|z|)^p\frac{\widehat{\omega}(z)}{1-|z|}|f'(z)|^{q-p}\widehat{\omega}(z)^{\frac{q}{p}-1}(1-|z|)^{\frac{q}{p}-1}(1-|z|)^{q-p}dA(z)\\ &\lesssim \|f\|^{q-p}_{A^p_\omega}\int_{\mathbb{D}}|f'(z)|^p(1-|z|)^p\frac{\widehat{\omega}(z)}{1-|z|}dA(z)\lesssim\|f\|^q_{A^p_\omega}\end{aligned}$$

Now assume that $L_g:A^p_\omega\to A^q_\nu$ is bounded and let $f_a(z)=\left(\frac{1-|a|^2}{1-\overline{a}z}\right)^\gamma\frac{1}{(\widehat{\omega}(a)(1-|a|))^{\frac{1}{p}}}$. Then, by Lemma 4 and Theorem 4

$$\begin{aligned}\frac{|g(a)|}{(\widehat{\omega}(a)(1-|a|))^{\frac{1}{p}}(1-|a|)} &= \left|(L_g(f_a))'(a)\right|\lesssim\frac{\|L_g(f_a)\|_{A^q_\nu}}{(\widehat{\nu}(a)(1-|a|))^{\frac{1}{q}}(1-|a|)}\\ &\lesssim\frac{\|f_a\|_{A^p_\omega}}{(\widehat{\nu}(a)(1-|a|))^{\frac{1}{q}}(1-|a|)}\lesssim\frac{1}{(\widehat{\nu}(a)(1-|a|))^{\frac{1}{q}}(1-|a|)}\end{aligned}$$

and it follows that $\sup_{z\in\mathbb{D}}|g(z)|\gamma(z)<\infty$. □

PROPOSITION 2. *Let $g\in\mathscr{H}(\mathbb{D}),0<p\leqslant q<\infty,\ \omega,\nu\in\mathscr{D}$. Then*

(a) $T_g:A^p_\omega\to A^q_\nu$ is compact if and only if

$$\lim_{|a|\to1^-}|g'(a)|(1-|a|)\frac{\widehat{\nu}(a)^{\frac{1}{q}}(1-|a|)^{\frac{1}{q}}}{\widehat{\omega}(a)^{\frac{1}{p}}(1-|a|)^{\frac{1}{p}}}=0; \tag{5}$$

(b) $L_g:A^p_\omega\to A^q_\nu$ is compact if and only if

$$\lim_{|a|\to1^-}|g(a)|\frac{\widehat{\nu}(a)^{\frac{1}{q}}(1-|a|)^{\frac{1}{q}}}{\widehat{\omega}(a)^{\frac{1}{p}}(1-|a|)^{\frac{1}{p}}}=0.$$

Proof. By Theorem 4 and Lemma 1, we have

$$\|T_g(f)\|^q_{A^q_\nu}\asymp\int_{\mathbb{D}}|f(z)|^q|g'(z)|^q(1-|z|)^{q-1}\widehat{\nu}(z)dA(z)\asymp\int_{\mathbb{D}}|f(z)|^qd\mu(z),$$

where $d\mu(z)=|g'(z)|^q(1-|z|)^q\frac{\widehat{\nu}(z)}{1-|z|}dA(z)$. It means that $T_g:A^p_\omega\to A^q_\nu$ is compact if and only if $I_d:A^p_\omega\to L^q_\mu$ is compact. By (b) of Lemma 2, we finally get $T_g:A^p_\omega\to A^q_\nu$ is compact if and only if

$$\lim_{|a|\to1^-}\frac{\mu(\Delta(a,r))}{(\omega(\Delta(a,r)))^{\frac{q}{p}}}=0.$$

(a) Suppose that $T_g : A^p_\omega \to A^q_\nu$ is compact. Combing the argument above, we have

$$\lim_{|a|\to 1^-} \frac{\int_{\Delta(a,r)} |g'(z)|^q (1-|z|)^q \frac{\widehat{\nu}(z)}{1-|z|} dA(z)}{(\omega(\Delta(a,r)))^{\frac{q}{p}}} = 0.$$

By subharmonicity of $|g'(a)|^q$ and $\omega,\ \nu \in \mathscr{D}$, we have

$$\lim_{|a|\to 1^-} |g'(a)|^q (1-|a|)^q \frac{\widehat{\nu}(a)(1-|a|}{\widehat{\omega}(a)^{\frac{q}{p}}(1-|a|)^{\frac{q}{p}}} \lesssim \lim_{|a|\to 1^-} \frac{\int_{\Delta(a,r)} |g'(z)|^q (1-|z|)^q \frac{\widehat{\nu}(z)}{1-|z|} dA(z)}{(\omega(\Delta(a,r)))^{\frac{q}{p}}} = 0.$$

Conversely, we assume that (5) holds. Then $\forall \varepsilon > 0,\ \exists\ R = R(\varepsilon) \in (0,1)$ such that

$$|g'(z)|^q (1-|z|)^q \frac{\widehat{\nu}(z)(1-|a|)}{\widehat{\omega}(z)^{\frac{q}{p}}(1-|z|)^{\frac{q}{p}}} < \varepsilon \quad |z| \geqslant R.$$

Let $0 < r < 1$ be arbitrary but fixed. Fix $\rho = \rho(r) \in (0,1)$ such that $|z| \geqslant R,\ \forall z \in \Delta(\xi, r)$, provided $|\xi| \geqslant \rho$. Then

$$\begin{aligned}
\frac{\int_{\Delta(\xi,r)} |g'(z)|^q (1-|z|)^q \frac{\widehat{\nu}(z)}{1-|z|} dA(z)}{\omega(\Delta(\xi,r))^{\frac{q}{p}}} &\leqslant \varepsilon \frac{\int_{\Delta(\xi,r)} \frac{\widehat{\omega}(z)^{\frac{q}{p}}(1-|z|)^{\frac{q}{p}}}{(1-|z|)^2} dA(z)}{\omega(\Delta(\xi,r))^{\frac{q}{p}}} \\
&\asymp \varepsilon \frac{\widehat{\omega}(\xi)^{\frac{q}{p}}(1-|\xi|)^{\frac{q}{p}} \int_{\Delta(\xi,r)} \frac{dA(z)}{(1-|z|)^2}}{(\widehat{\omega}(\xi)(1-|\xi|))^{\frac{q}{p}}} \\
&\asymp \varepsilon, \quad |\xi| \geqslant \rho.
\end{aligned}$$

Thus,

$$\lim_{|\xi|\to 1^-} \frac{\int_{\Delta(\xi,r)} |g'(z)|^q (1-|z|)^q \frac{\widehat{\nu}(z)}{1-|z|} dA(z)}{(\omega(\Delta(\xi,r)))^{\frac{q}{p}}} = 0.$$

We finish the proof.

(b) By Theorem 4 and Lemma 1, we have

$$\|L_g(f)\|^q_{A^q_\nu} \asymp \int_{\mathbb{D}} |f'(z)|^q |g(z)|^q (1-|z|)^{q-1} \widehat{\nu}(z) dA(z) \asymp \int_{\mathbb{D}} |f'(z)|^q d\mu(z),$$

where $d\mu(z) = |g(z)|^q (1-|z|)^q \frac{\widehat{\nu}(z)}{1-|z|} dA(z)$. It means that $L_g : A^p_\omega \to A^q_\nu$ is compact if and only if $D : A^p_\omega \to L^q_\mu$ is compact. By (ii) of Lemma 3, we finally get $L_g : A^p_\omega \to A^q_\nu$ is compact if and only if

$$\lim_{|a|\to 1^-} \frac{\mu(\Delta(a,r))}{(\omega(S(a,r)))^{\frac{q}{p}}(1-|a|)^q} = 0.$$

Firstly, we suppose that $L_g : A^p_\omega \to A^q_\nu$ is compact. Combing the argument above, we have

$$\lim_{|a|\to 1^-} \frac{\int_{\Delta(a,r)} |g(z)|^q (1-|z|)^q \frac{\widehat{\nu}(z)}{1-|z|} dA(z)}{(\omega(S(a,r)))^{\frac{q}{p}}(1-|a|)^q} = 0.$$

By subharmonicity of $|g(a)|^q$ and $\omega,\ \nu\in\mathscr{D}$, we have

$$\lim_{|a|\to 1^-}|g(a)|^q\frac{\widehat{\nu}(a)(1-|a|}{\widehat{\omega}(a)^{\frac{q}{p}}(1-|a|)^{\frac{q}{p}}}\lesssim\lim_{|a|\to 1^-}\frac{\int_{\Delta(a,r)}|g(z)|^q(1-|z|)^q\frac{\widehat{\nu}(z)}{1-|z|}dA(z)}{(\omega(S(a,r)))^{\frac{q}{p}}(1-|a|)^q}=0.$$

The reverse direction of the proof is similar to the proof in part (a) and will be omitted. We finish the proof. □

PROPOSITION 3. *Let $g\in\mathscr{H}(\mathbb{D})$, $0<q<p<\infty$, $\omega,\nu\in\mathscr{D}$ such that $\sigma=\sigma(z)=\frac{\widehat{\nu}(z)^{\frac{p}{p-q}}}{\widehat{\omega}(z)^{\frac{q}{p-q}}}\frac{1}{1-|z|}\in\mathscr{D}$. Then the following statements are equivalent.*

(a) $T_g:A^p_\omega\to A^q_\nu$ is bounded;

(b) $T_g:A^p_\omega\to A^q_\nu$ is compact;

(c) $g\in A^s_\sigma$, where $\frac{1}{s}=\frac{1}{q}-\frac{1}{p}$.

Proof. From the Theorem 4 and Lemma 1 we know that

$$\|T_gf\|^q_{A^q_\nu}\asymp\int_{\mathbb{D}}|f(z)|^q|d\mu(z),$$

where $d\mu(z)=|g'(z)|^q(1-|z|)^q\frac{\widehat{\nu}(z)}{1-|z|}dA(z)$. By (iii) of Lemma 2, we know that (a) and (b) are equivalent and both are equivalent to the following condition

$$\frac{\mu(\Delta(z,r))}{\omega(S(z))}\in L^{\frac{p}{p-q}}_{\tilde{\omega}},$$

that is,

$$\frac{\int_{\Delta(z,r)}|g'(\xi)|^q(1-|\xi|)^q\frac{\widehat{\nu}(\xi)}{1-|\xi|}dA(\xi)}{\omega(S(z))}\in L^{\frac{p}{p-q}}_{\tilde{\omega}}.$$

By the subharmonicity of $|g'(z)|^q$, we have

$$\frac{\int_{\Delta(z,r)}|g'(\xi)|^q(1-|\xi|)^q\frac{\widehat{\nu}(\xi)}{1-|\xi|}dA(\xi)}{\omega(S(z))}\gtrsim|g'(z)|^q(1-|z|)^q\frac{\widehat{\nu}(z)}{\widehat{\omega}(z)}.$$

Therefore it implies that $|g'(z)|^q(1-|z|)^q\frac{\widehat{\nu}(z)}{\widehat{\omega}(z)}\in L^{\frac{p}{p-q}}_{\tilde{\omega}}$, which is the same as

$$\int_{\mathbb{D}}|g'(z)|^{\frac{pq}{p-q}}(1-|z|)^{\frac{pq}{p-q}}\frac{\widehat{\nu}(z)^{\frac{p}{p-q}}}{\widehat{\omega}(z)^{\frac{q}{p-q}}}\frac{1}{1-|z|}dA(z)<\infty.$$

By Theorem 4, we get that $g\in A^s_\sigma$.

Conversely, if $g \in A^s_\sigma$, then by Hölder's inequality, we get

$$\begin{aligned}\|T_g f\|^q_{A^q_\nu} &\asymp \int_{\mathbb{D}} |f(z)|^q |g'(z)|^q (1-|z|)^q \frac{\widehat{\nu}(z)}{1-|z|} dA(z)\\ &\leqslant \|f\|^q_{A^p_\omega} \left(\int_{\mathbb{D}} |g'(z)|^{\frac{pq}{p-q}} (1-|z|)^{\frac{pq}{p-q}} \frac{\widehat{\nu}(z)^{\frac{p}{p-q}}}{\widehat{\omega}(z)^{\frac{q}{p-q}}} \frac{1}{1-|z|} dA(z) \right)^{1-\frac{q}{p}}\\ &\lesssim \|f\|^q_{A^p_\omega} \|g\|^q_{A^s_\sigma}. \quad \square\end{aligned}$$

PROPOSITION 4. *Let $g \in \mathscr{H}(\mathbb{D})$, $0 < q < p < \infty$, $\omega, \nu \in \mathscr{D}$ such that $\sigma = \sigma(z) = \frac{\widehat{\nu}(z)^{\frac{p}{p-q}}}{\widehat{\omega}(z)^{\frac{q}{p-q}}} \frac{1}{1-|z|} \in \mathscr{D}$. Then the following statements are equivalent.*

(a) $L_g : A^p_\omega \to A^q_\nu$ is bounded;

(b) $L_g : A^p_\omega \to A^q_\nu$ is compact;

(c) $g \in A^s_\sigma$, where $\frac{1}{s} = \frac{1}{q} - \frac{1}{p}$.

Proof. By Theorem 4 and Lemma 1, we have

$$\|L_g(f)\|^q_{A^q_\nu} \asymp \int_{\mathbb{D}} |f'(z)|^q |g(z)|^q (1-|z|)^{q-1} \widehat{\nu}(z) dA(z) \asymp \int_{\mathbb{D}} |f'(z)|^q d\mu(z),$$

where $d\mu(z) = |g(z)|^q (1-|z|)^q \frac{\widehat{\nu}(z)}{1-|z|} dA(z)$. The remainder of the proof is similar to the proof of Proposition 3 by Lemma 3, therefore is omitted. $\square$

4. Proofs of main results

Proof of Theorem 1. The theorem is an immediate consequence of (a) of Proposition 1, (a) of Proposition 2 and Proposition 3. $\square$

Proof of Theorem 2. The theorem is an immediate consequence of (b) of Proposition 1, (b) of Proposition 2 and Proposition 4. $\square$

Proof of Theorem 3. Denote $c(g) = \lim_{|z|\to 1^-} \sup |g'(z)|(1-|z|)\gamma(z)$, so that there is an $\gamma_\varepsilon \in (0,1)$. Such that $|g'(z)|(1-|z|)\gamma(z) \leqslant c(g) + \varepsilon$ for all $|z| \geqslant \gamma_\varepsilon$, where $\gamma(z) = \gamma(\omega, \nu, p, q)(z) = \frac{\widehat{\nu}(z)^{\frac{1}{q}} (1-|z|)^{\frac{1}{q}}}{\widehat{\omega}(z)^{\frac{1}{p}} (1-|z|)^{\frac{1}{p}}}$.

For an analytic function $f(z) = \sum_{k=0}^\infty a_k z^k$ in $\mathbb{D}$, let

$$T_n f(z) = \sum_{k=0}^n a_k z^k, R_n f(z) = \sum_{k=n+1}^\infty a_k z^k.$$

Since T_n is compact on A^p_ω

$$\|T_g\|_e = \|T_g(T_n + R_n)\|_e \leqslant \|T_g T_n\|_e + \|T_g R_n\|_e = \|T_g R_n\|_e \leqslant \|T_g R_n\|$$

and Lemma 1 show that

$$\begin{aligned}\|T_g\|_e^q &\leqslant \liminf_{n\to\infty}\|T_gR_n\|^q \leqslant \liminf_{n\to\infty}\sup_{\|f\|_{A^p_\omega}\leqslant 1}\|T_gR_nf\|^q_{A^q_\nu}\\ &\asymp \liminf_{n\to\infty}\sup_{\|f\|_{A^p_\omega}\leqslant 1}\int_{\mathbb{D}}|R_nf(z)|^q|g'(z)|^q(1-|z|)^q\nu(z)dA(z)\\ &\asymp \liminf_{n\to\infty}\sup_{\|f\|_{A^p_\omega}\leqslant 1}\int_{\mathbb{D}}|R_nf(z)|^q|g'(z)|^q(1-|z|)^{q-1}\widehat{\nu}(z)dA(z)\end{aligned}$$

Since $\liminf_{n\to\infty}\sup_{\|f\|_{A^p_\omega}\leqslant 1}\int_{\mathbb{D}\setminus\Delta(0,r_\varepsilon)}|R_nf(z)|^q|g'(z)|^q(1-|z|)^{q-1}\widehat{\nu}(z)dA(z)=0$ for any $r\in(0,1)$ and Lemma 4 yeild

$$\begin{aligned}\|T_g\|_e^q &\lesssim \liminf_{n\to\infty}\sup_{\|f\|_{A^p_\omega}\leqslant 1}\int_{\mathbb{D}\setminus\Delta(0,r_\varepsilon)}|R_nf(z)|^q|g'(z)|^q(1-|z|)^{q-1}\widehat{\nu}(z)dA(z)\\ &\leqslant (c(g)+\varepsilon)^q\liminf_{n\to\infty}\sup_{\|f\|_{A^p_\omega}\leqslant 1}\int_{\mathbb{D}}|R_nf(z)|^q\frac{\widehat{\omega}(z)^{\frac{q}{p}}(1-|z|)^{\frac{q}{p}}}{(1-|z|)^2}dA(z)\\ &\lesssim \|R_nf\|^q_{A^p_\omega}.\end{aligned}$$

Actually, for $1<p<\infty$ and $\omega\in\mathscr{D}$, R_n is bounded on A^p_ω. An analogue proof of this result can be found in [27]. Therefore $\|R_nf\|_{A^p_\omega}\lesssim\|f\|_{A^p_\omega}$, it follows that $\|T_g\|_e\lesssim c(g)$.

To obtain the lower bound, choose $f_a(z)=\left(\frac{1-|a|^2}{1-\overline{a}z}\right)^\gamma\frac{1}{\omega(S(a))^{\frac{1}{p}}}$ for $a\in\mathbb{D}$. Then $\|f_a\|_{A^p_\omega}\asymp 1$ and $f_a\to 0$ uniformly on compact set of $\mathbb{D}$ as $|a|\to 1^-$.

If $K:A^p_\omega\to A^q_\nu$ is compact, then

$$\begin{aligned}\|T_g-K\| &\geqslant \lim_{|a|\to 1^-}\sup\|T_g(f_a)-Kf_a\|_{A^q_\nu}\\ &\geqslant \lim_{|a|\to 1^-}\sup\|T_g(f_a)\|_{A^q_\nu}-\lim_{|a|\to 1^-}\sup\|Kf_a\|_{A^q_\nu}\\ &= \lim_{|a|\to 1^-}\sup\|T_g(f_a)\|_{A^q_\nu}\end{aligned}$$

and it follows that

$$\begin{aligned}\|T_g\|_e^q &\gtrsim \lim_{|a|\to 1^-}\sup\int_{\mathbb{D}}|f_a(z)|^q|g'(z)|^q(1-|z|)^q\nu(z)dA(z)\\ &\asymp \int_{\mathbb{D}}\left(\frac{1-|a|^2}{1-\overline{a}z}\right)^{\gamma q}|g'(z)|^q(1-|z|)^{q-1}\widehat{\nu}(z)\frac{1}{(\widehat{\omega}(a)(1-|a|))^{\frac{q}{p}}}dA(z)\end{aligned}$$

By the subharmonicity of $|g'(z)|^q$,

$$|g'(a)|^q\leqslant\frac{1}{(1-|a|)^2}\int_{\Delta(a,r)}|g'(z)|^qdA(z).$$

and since $(1-|a|)^2 \asymp (1-|z|)^2 \asymp |1-\overline{a}z|$ for $z \in \Delta(a,r)$, this yields.

$$|g'(a)|^q(1-|a|)^q \frac{\widehat{\nu}(a)}{(\widehat{\omega}(a)(1-|a|))^{\frac{q}{p}}}$$
$$\lesssim \int_{\mathbb{D}} \left(\frac{1-|a|^2}{1-\overline{a}z}\right)^{\gamma q} |g'(z)|^q(1-|z|)^{q-1} \frac{\widehat{\nu}(z)}{(\widehat{\omega}(a)(1-|a|))^{\frac{q}{p}}} dA(z).$$

It follows that $c(g) \lesssim \left\|T_g\right\|_e$. We finish the proof. □

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