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EXTENSION OF THE NOTION OF P–SYMMETRIC OPERATORS USING THE ALUTHGE TRANSFORM II

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Abstract. The class of $\tilde{P}$-symmetric pairs is introduced. Certain properties of this class of operators are obtained. Among other things, it is proved that:

(1) This class includes pairs of quasinormal operators, pairs of co-isometrics operators, pairs of partial isometries with normal squares, and all P-symmetric pairs.

(2) If A and B are two iw-hyponormal operators, then (A,B) has the Fuglede-Putnam property $(FP)_{\mathscr{C}_1(\mathscr{H})}$ if and only if $(\tilde{A},\tilde{B})$ does ($\mathscr{C}_1(\mathscr{H})$ is the ideal of trace class operators).

We also describe some classes of Hilbert space operators A, $B \in \mathscr{L}(\mathscr{H})$ for which we have $\overline{\mathscr{R}\left(\delta_{A,B}\right)}^{w^*} = \overline{\mathscr{R}\left(\delta_{\tilde{A},\tilde{B}}\right)}^{w^*}$, where $\overline{\mathscr{R}\left(\delta_{A,B}\right)}^{w^*}$ is the ultra-weak closure of the range of the generalized derivation $\delta_{A,B}$ defined on $\mathscr{L}(\mathscr{H})$ by $\delta_{A,B}(X) = AX - XB$ and $\tilde{S}$ denote the Aluthge transform of $S \in \mathscr{L}(\mathscr{H})$.

Introduction

Let $\mathscr{L}(\mathscr{H})$ denote the algebra of all bounded linear operators on an infinite dimensional complex Hilbert space H into itself. Given $A,B \in \mathscr{L}(\mathscr{H})$, we define the generalized derivation $\delta_{A,B}$ associated with (A,B) as follows

$$\delta_{A,B}(X) = AX - XB \quad \text{for all} \quad X \in \mathscr{L}(\mathscr{H}).$$

The generalized derivation $\delta_{A,B}$ was initially systematically studied by M. Rosenblum [17]. The properties of such operators (spectrums, norms, ranges, etc) have been carried out extensively by a number of authors (see for example [18, 21] and the references therein), and many of their problems remain also open.

If $A = B$, then $\delta_{A,A} = \delta_A : \mathscr{L}(\mathscr{H}) \to \mathscr{L}(\mathscr{H})$ is defined by

$$\delta_A(X) = AX - XA$$

is called the inner derivation induced by A. The concept of generalized derivations is a natural generalization of the inner derivations.

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An operator $T \in \mathscr{L}(\mathscr{H})$ has the unique polar decomposition $T = U|T|$ such that $|T| = (T^*T)^{\frac{1}{2}}$ and U is the partial isometry satisfying $\ker(U) = \ker(|T|)$. Associated with T is a related operator

$$\tilde{T} = |T|^{1/2} U |T|^{1/2},$$

called the Aluthge transform of T (see [1] for further details). In many cases, the Aluthge transform of T have the better properties than T. Recently some investigation in the operator theory have been related to relationship between operators and their Aluthge transform, see for example [2, 14] and the references cited therein.

The pair (A,B) of operators $A \in \mathscr{L}(\mathscr{H})$ and $B \in \mathscr{L}(\mathscr{H})$ is called P-symmetric if $\overline{\mathscr{R}(\delta_{A,B})}^{w^*} = \overline{\mathscr{R}(\delta_{A^*,B^*})}^{w^*}$, where $\overline{\mathscr{R}(\delta_{A,B})}^{w^*}$ denotes the ultra-weak closure of the range $\mathscr{R}(\delta_{A,B})$ of $\delta_{A,B}$ and S^* denote the adjoint of $S \in \mathscr{L}(\mathscr{H})$. M. Ech-Chad, Y. Bouhafsi and A. Zouaki [10] introduced the class of P-symmetric pairs of operators, and they gave some basic properties of those operators.

In this paper, we would like to explore this class of operators. We initiate the study of a more general class of pairs of operators (A,B) that have the following property: $BT = TA$ implies $\tilde{B}T = T\tilde{A}$ for all $T \in \mathscr{C}_1(\mathscr{H})$, where $\mathscr{C}_1(\mathscr{H})$ denote the ideal of trace class operators and $\tilde{S}$ is the Aluthge transform of $S \in \mathscr{L}(\mathscr{H})$. Such pairs of operators are termed $\tilde{P}$-symmetric. In the first part, we use different arguments to establish a characterization and some basic properties of $\tilde{P}$-symmetric pairs of operators. In the second part we give some properties concerning this class, of the same type as those established for the P-symmetric pairs of operators. This leads us to present a new generalization of P-symmetric operators introduced in [5, 6].

In order to state our results in more details we need to introduce the following notations and terminology.

Let $\mathscr{K}(\mathscr{H})$ and $\mathscr{C}_1(\mathscr{H})$ be respectively the ideal of compact operators and the ideal of trace class operators on $\mathscr{H}$. The trace function is defined on $\mathscr{C}_1(\mathscr{H})$ by

$$tr(T) = \sum_n \langle Te_n, e_n \rangle,$$

where (e_n) is any complete orthonormal system in $\mathscr{H}$. For X a linear operator acting on Banach space E, we denote by X^*, $\ker(X)$, $\ker^{\perp}(X)$, $R(X)$ and $X|M$ respectively the adjoint, the kernel, the orthogonal complement of the kernel, the range of X and the restriction of X to an invariant subspace M. Also we denote by $\sigma(X)$, $\overline{R(X)}$ and $\overline{R(X)}^{w^*}$ respectively The spectrum of X, the closure of the range of X respect to the norm topology and the ultra-weak topology. Given $\mathscr{B}$ be a Banach and $\mathscr{S}$ be a subspace of $\mathscr{B}$. By $\mathscr{B}'$ we denote the dual of $\mathscr{B}$, the set

$$\mathscr{S}^{\circ} = \{\Phi \in \mathscr{L}'(\mathscr{H}) : \Phi(x) = 0 \quad \forall x \in \mathscr{S}\},$$

denotes the annihilator of $\mathscr{S}$. For g and ω two vectors in H, we define $g \otimes \omega \in \mathscr{L}(\mathscr{H})$ as follows:

$$g \otimes \omega(x) = \langle x, \omega \rangle g \quad \text{for all} \quad x \in H.$$

Recall that a hyponormal operator $T \in \mathscr{L}(\mathscr{H})$ satisfies $T^*T \geqslant TT^*$. This class has been widely studied due to its similarities with normal operators [12]. Extensions

of hyponormality include p-hyponormal operators, defined by $(T^*T)^p \geqslant (TT^*)^p$ for $p \in]0,1]$, and log-hyponormal operators, where T is invertible and $\log|T| \geqslant \log|T^*|$. Another generalization is the class of w-hyponormal operators, satisfying $|\tilde{T}| \geqslant |T| \geqslant |\tilde{T}^*|$, with iw-hyponormal operators being the invertible ones. We denote by w_*-hyponormal the subclass of w-hyponormal operators satisfying $\ker(T) \subseteq \ker(T^*)$, noting that every iw-hyponormal operator is w_*-hyponormal. The classes of log- and w-hyponormal operators were introduced and studied in [2], where it was shown that w-hyponormal operators contains both p-hyponormal and log-hyponormal operators.

1. $\tilde{P}$-symmetric operators

DEFINITION 1. ([10]) Let $A,B \in \mathscr{L}(\mathscr{H})$. If $\overline{\mathscr{R}(\delta_{A,B})}^{w^*} = \overline{\mathscr{R}(\delta_{A^*,B^*})}^{w^*}$ we say that (A,B) is P-symmetric. We denote the set of such pairs by $\mathscr{P}(\mathscr{H})$.

DEFINITION 2. Let A,B be in $\mathscr{L}(\mathscr{H})$ and $\mathscr{J}$ be a two sided ideal of $\mathscr{L}(\mathscr{H})$. The pair (A,B) is said to possess the Fuglede-Putnam property $(FP)_{\mathscr{J}}$ if, $AT = TB$ and $T \in \mathscr{J}$ implies $A^*T = TB^*$.

THEOREM 1. ([10]) *Let $A,B \in \mathscr{L}(\mathscr{H})$. The pair (A,B) is P-symmetric if and only if (A,B) and (B,A) have $(FP)_{\mathscr{C}_1(\mathscr{H})}$ property.*

DEFINITION 3. Let $A,B \in \mathscr{L}(\mathscr{H})$. We say that (A,B) is $\tilde{P}$-symmetric if $BT = TA$ implies $\tilde{B}T = T\tilde{A}$ for every $T \in \mathscr{C}_1(\mathscr{H})$. We denote the set of such pairs by $\widetilde{\mathscr{P}}(\mathscr{H})$.

EXAMPLE 1.

1. Recall that A is quasinormal if $A(A^*A) = (A^*A)A$. It follows from [4], Proposition 1.10, that A is quasinormal if and only if $\tilde{A} = A$. Consequently, if A and B are quasinormal operators then (A,B) is $\tilde{P}$-symmetric.

2. If A and B are isometries or normal operators, then (A,B) is $\tilde{P}$-symmetric.

This is an example of trivial P-symmetric operators. We will give other examples, in the light of the following lemma:

LEMMA 1. *Let $A \in \mathscr{L}(\mathscr{H})$ be a partial isometry, then $A = A|A|$ is the polar decomposition of A and $\tilde{A} = A^*A^2$.*

Proof. Since A is a partial isometry, then A^*A is a projection, hence $(A^*A)^2 = A^*A$ and $|A| = A^*A$. We get $A = A|A|$ is the polar decomposition of A and as a result the Aluthge transform of A is given by:

$$\tilde{A} = |A|^{\frac{1}{2}}A|A|^{\frac{1}{2}} = (A^*A)A(A^*A) = A^*A^2. \quad \square$$

THEOREM 2. *The pair (A,B) is $\tilde{P}$-symmetric in the following cases:*

(i) A is co-isometric and B is a partial isometry;

(ii) A is a partial isometry and B is isometric.

Proof.

(i) By Lemma 1, the Aluthge transforms of A and B are given by $\tilde{A} = A^*A^2$ and $\tilde{B} = B^*B^2$. Let $T \in \mathscr{C}_1(\mathscr{H})$ such that $BT = TA$. Since A and B are contractions and T is compact with $BTA^* = T$, it follows from [19, Theorem 2.2] that $B^*TA = T$, which give $B^*T = TA^*$. Hence $\tilde{B}T = B^*B^2T = B^*TA^2 = TA^*A^2 = T\tilde{A}$.

(ii) The result follows by the same argument as in the first case. □

COROLLARY 1. *The pair (A,B) is $\tilde{P}$-symmetric in the following cases:*

(i) A and B are two co-isometries;

(ii) A is co-isometric and B is isometric.

LEMMA 2. ([7]) *If $A,B \in \mathscr{L}(\mathscr{H})$, then*

$$\mathscr{R}(\delta_{A,B})^0 \simeq \mathscr{R}(\delta_{A,B})^0 \cap \mathscr{K}(\mathscr{H})^0 \oplus \ker(\delta_{B,A}) \cap \mathscr{C}_1(\mathscr{H})$$

THEOREM 3. *Let $A,B \in \mathscr{L}(\mathscr{H})$ such that $\ker B \neq \{0\}$ and $\{0\} \neq \ker A^* \not\subset \ker \tilde{A}^*$ where $\tilde{A}^*$ is the adjoint operator of the Aluthge transform of A, then (A,B) is not $\tilde{P}$-symmetric.*

Proof. From the hypothesis, there exist non-zero elements f, g and ω in H such that $B(f) = 0$, $A^*(g) = 0$ and $\tilde{A}^*(g) = \omega \neq 0$. Since $\ker(B) = \ker(|B|) = \ker(|B|^{\frac{1}{2}})$, we get $\tilde{B}(f) = 0$. If $X = \|f\|^{-2}(\omega \otimes f)$ and $Y \in \mathscr{L}(\mathscr{H})$, then

$$\begin{aligned}\left\langle \left(\tilde{A}X - X\tilde{B}\right) f, g\right\rangle &= \left\langle \tilde{A}X(f), g\right\rangle - \left\langle X\tilde{B}(f), g\right\rangle \\ &= \langle X(f), w\rangle - \langle 0, g\rangle \\ &= \|\omega\|^2,\end{aligned}$$

and

$$\langle (AY - YB)f, g\rangle = \langle Yf, A^*g\rangle - \langle 0, g\rangle = 0.$$

Suppose that (A,B) is $\tilde{P}$-symmetric. It follows from Lemma 2 that $\mathscr{R}\left(\delta_{\tilde{A},\tilde{B}}\right) \subset \overline{\mathscr{R}(\delta_{A,B})}^{w^*}$. Then $\tilde{A}X - X\tilde{B} \in \overline{\mathscr{R}(\delta_{AB})}^{w^*}$. As a result, there exists a net $(Y_\alpha)_\alpha$ in $\mathscr{L}(\mathscr{H})$ such that for all x and y in H, we have

$$\langle (AY_\alpha - Y_\alpha B)x, y\rangle \longrightarrow \left\langle \left(\tilde{A}X - X\tilde{B}\right)x, y\right\rangle.$$

So $0 = \langle (AY_\alpha - Y_\alpha B)f, g\rangle \longrightarrow \left\langle \left(\tilde{A}X - X\tilde{B}\right)f, g\right\rangle = \|\omega\|^2$. Hence $\omega = 0$, which is absurd. □

EXAMPLE 2. Let $(e_n)_{n\geqslant 1}$ be an orthonormal basis of H, $H_0 = Vect\{e_1, e_2, e_3\}$ and $A_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \in \mathscr{L}(H_0)$. It is easy to see that A_0 is a partial isometry. By Lemma 1, the Aluthge transform of A_0 is given by:

$$\widetilde{A_0} = A_0^* A_0^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let $A = A_0 \oplus I$ with respect to the decomposition $H = H_0 \oplus H_0^{\perp}$, and let $B = e_1 \otimes e_1$. Since A is also a partial isometry, we have $\tilde{A} = A^* A^2 = \tilde{A_0} \oplus I$. Furthermore, we have $Be_2 = 0$, $A^*(e_2 - \sqrt{3}e_3) = 0$, and $\tilde{A}^*(e_2 - \sqrt{3}e_3) \neq 0$. Thus, by Theorem 3, the pair (A,B) is not $\tilde{P}$-symmetric.

THEOREM 4. *(A,B) is $\tilde{P}$-symmetric for every $A,B \in \mathscr{L}(\mathscr{H})$ such that (B,A) has $(FP)_{\mathscr{C}_1(\mathscr{H})}$ property, and hence $\mathscr{P}(\mathscr{H})$ is strictly included in $\tilde{\mathscr{P}}(\mathscr{H})$.*

Proof. Let $A = U_1|A|$ and $B = U_2|B|$ such that the pair (A,B) is P-symmetric. If $BT = TA$ for $T = U_3|T| \in \mathscr{C}_1(\mathscr{H})$, then $B^*T = TA^*$. Hence, by [11, Theorem 1, p. 70], we have

$$|A||T| = |T||A| \qquad (1)$$
$$U_1|T| = |T|U_1 \qquad (2)$$
$$|B|U_3 = U_3|A| \qquad (3)$$
$$U_2U_3 = U_3U_1 \qquad (4)$$

Let $\{p_n\}$ be a sequence of polynomials with no constant term such that $p_n(t) \to t^{\frac{1}{2}}$ uniformly on a certain compact set as $n \to \infty$. Then, by (1) and (3), we get

$$p_n(|A|)|T| = |T|p_n(|A|) \quad \text{and} \quad p_n(|B|)U_3 = U_3 p_n(|A|).$$

Thus $|A|^{\frac{1}{2}}|T| = |T||A|^{\frac{1}{2}}$ and $|B|^{\frac{1}{2}}U_3 = U_3|A|^{\frac{1}{2}}$. Hence, we obtain

$$U_3|A|^{\frac{1}{2}}|T| = U_3|T||A|^{\frac{1}{2}} \quad \text{so that} \quad |B|^{\frac{1}{2}}T = T|A|^{\frac{1}{2}}.$$

On the other hand, by (2) and (4), it follows that

$$U_2T = U_2U_3|T| = U_3U_1|T| = TU_1,$$

which gives

$$U_2|B|^{\frac{1}{2}}T = U_2T|A|^{\frac{1}{2}} = TU_1|A|^{\frac{1}{2}}.$$

Therefore,

$$\tilde{B}T = |B|^{\frac{1}{2}}U_2|B|^{\frac{1}{2}}T = |B|^{\frac{1}{2}}TU_1|A|^{\frac{1}{2}} = T|A|^{\frac{1}{2}}U_1|A|^{\frac{1}{2}} = T\tilde{A}.$$

Thus, (A,B) is $\tilde{P}$-symmetric. Now, we show that the inclusion is strict. Let $(e_n)_{n\geqslant 1}$ be an orthonormal basis of $\mathscr{H}$ and take $S \in \mathscr{L}(\mathscr{H})$ such that

$$Se_k = \begin{cases} 0 & \text{for } k = 1 \\ e_{k+1} & \text{for } k \geqslant 2 \end{cases}.$$

A simple calculation shows that if S is a quasinormal operator, then the pair (S,S) is trivially $\tilde{P}$-symmetric. However, (S,S) is not P-symmetric (see Theorem 1.6 in [5]). □

COROLLARY 2. *The pair (A,B) is $\tilde{P}$-symmetric in the following cases:*

(i) *A is co-isometric and B is a contraction;*

(ii) *A is a contraction and B is isometric.*

(iii) *A^* and B are either p-hyponormal or log-hyponormal.*

(iv) *A^* and B are w_*-hyponormal operators.*

Proof.

(i) Let $T \in \mathscr{C}_1(\mathscr{H})$ such that $BT = TA$. Since A and B are contractions and T is compact with $BTA^* = T$, then by [19, Theorem 2.2], we get $B^*TA = T$. Hence, $B^*T = TA^*$. Thus (B,A) has $(FP)_{\mathscr{C}_1(\mathscr{H})}$ property and Theorem 4 gives the result.

(ii) We can use the same argument as in the first case.

(iii) It follows from [20], Corollary 2, that (B,A) possesses $(FP)_{\mathscr{C}_1(\mathscr{H})}$ property. Consequently, Theorem 4 establishes that (A,B) is $\tilde{P}$-symmetric.

(iv) By using [16], Corollary 4.8, we get (B,A) has $(FP)_{\mathscr{C}_1(\mathscr{H})}$ property, and as a result, (A,B) is $\tilde{P}$-symmetric by Theorem 4. □

REMARK 1. It is easy to see that isometries and co-isometries with normal squares are quasinormal operators.

THEOREM 5. *Let $A,B \in \mathscr{L}(\mathscr{H})$ be tow partial isometries. If A^2 and B^2 are normal, then (A,B) is $\tilde{P}$-symmetric.*

Proof. Since $A^2A = AA^2$, it follows by hypothesis, Fuglede's theorem, and Lemma 1 that $A^2A^* = A^*A^2 = \tilde{A}$. Also $\ker(A^*) \subset \ker(\tilde{A})$. By the same argument, we obtain $B^2B^* = \tilde{B}$. Let $T \in \mathscr{C}_1(\mathscr{H})$ such that $BT = TA$. Then, we get $B^2T = TA^2$, that is $B^2B^*BT = TA^2A^*A$. Hence, we have

$$(\tilde{B}T - T\tilde{A})A = 0,$$

which implies that $\tilde{B}T - T\tilde{A}$ vanishes on $\overline{R(A)}$. But if $x \in \ker(A^*) \subset \ker(\tilde{A}) = \ker(A^2)$, then from $B^2T = TA^2$ we see that $Tx \in \ker(B^2) = \ker(\tilde{B})$. Hence $\tilde{B}T - T\tilde{A}$ vanish also on $\ker(A^*)$ which means

$$\tilde{B}T - T\tilde{A} = 0 \quad \text{on} \quad H = \overline{R(A)} \oplus \ker(A^*).$$

Thus (A,B) is $\tilde{P}$-symmetric. □

EXAMPLE 3. Let $H_0 = \mathscr{H} \oplus \mathscr{H}$. On H_0, we define the operators

$$A = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}.$$

Consider $T = \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix} \in \mathscr{C}_1(H_0)$ such that $C \neq 0$. It is easy to see that A and B are partial isometries, A^2 and B^2 are normal. Moreover $BT = TA$, but

$$B^*T - TA^* = \begin{pmatrix} 0 & -C \\ C & 0 \end{pmatrix} \neq 0.$$

So by Theorem 5 the pair (A,B) is $\tilde{P}$-symmetric but not P-symmetric.

COROLLARY 3. *Let $A \in \mathscr{C}_1(\mathscr{H})$ a partial isometry. If A^2 is normal, then $\tilde{A}$ is also normal.*

Proof. Since the pair (A,A) is $\tilde{P}$-symmetric by Theorem 5, we have $\tilde{A}A = A\tilde{A}$, which implies that $A^2 = A^*A^3$. Thus,

$$\tilde{A}\tilde{A}^* = A^*A^2A^{*2}A = A^{*3}A^3 = A^{*2}A^2 = A^{*2}AA^*A^2 = \tilde{A}^*\tilde{A}. \quad \square$$

REMARK 2. S. Bouali et al. proved in [9, Proposition 3.1] that the pair (A,A) is not P-symmetric when A is a nilpotent operator of order 2. But if A is nilpotent of order 2, then $\tilde{A} = 0$ by Theorem 4 in [14], which implies that the pair (A,A) is trivially $\tilde{P}$-symmetric. However, the following example shows that if $A \in \mathscr{L}(\mathscr{H})$ is nilpotent of order $n \geqslant 3$, then the pair (A,B) is not necessarily $\tilde{P}$-symmetric.

EXAMPLE 4. Let $H_0 = \mathscr{H} \oplus \mathscr{H} \oplus \mathscr{H}$, and define the operators

$$A = \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{pmatrix}.$$

Consider $T = \begin{pmatrix} 0 & C & 0 \\ 0 & 0 & C \\ 0 & 0 & 0 \end{pmatrix} \in \mathscr{C}_1(H_0)$ such that $C \neq 0$. A simple calculation shows that $A^3 = 0$, $BT = TA$, and that both A and B are partial isometries. Hence, the Aluthge

transforms of A and B are given by:

$$\tilde{A} = A^*A^2 = \begin{pmatrix} 0\,0\,0 \\ 0\,0\,I \\ 0\,0\,0 \end{pmatrix} \quad \text{and} \quad \tilde{B} = B^*B^2 = \begin{pmatrix} 0\,0\,0 \\ 0\,0\,0 \\ 0\,0\,I \end{pmatrix}.$$

Therefore $\tilde{B}T - T\tilde{A} = \begin{pmatrix} 0\,0\,-C \\ 0\,0\,\;0 \\ 0\,0\,\;0 \end{pmatrix} \neq 0$. Thus (A,B) is not $\tilde{P}$-symmetric.

THEOREM 6. *Let $A,B \in \mathscr{L}(\mathscr{H})$ be tow partial isometries such that $BA = A^2$ and $A \in \mathscr{C}_1(\mathscr{H})$. If A is nilpotent of order $n \geqslant 3$, then (A,B) is not $\tilde{P}$-symmetric.*

Proof. Suppose (A,B) is $\tilde{P}$-symmetric. Since $A \in \mathscr{C}_1(\mathscr{H})$, it follows from the hypothesis that $\tilde{B}A = A\tilde{A}$. By Lemma 1, this implies that $B^*B^2A = AA^*A^2$. Using the fact that A and B are partial isometries, we obtain

$$B^*A^3 = A^2 \quad \text{and} \quad B^*A^3A^{n-3} = A^2A^{n-3}.$$

Thus, we deduce that $A^{n-1} = 0$, which is a contradiction. □

PROPOSITION 1. Let $A,B \in \mathscr{L}(\mathscr{H})$ such that $BT = TA$ implies $|B|T = T|A|$ for every $T \in \mathscr{C}_1(\mathscr{H})$. Then (A,B) is $\tilde{P}$-symmetric.

Proof. Let $T \in \mathscr{C}_1(\mathscr{H})$ be such that $BT = TA$. Then we have $|B|^{\frac{1}{2}}BT = |B|^{\frac{1}{2}}TA$. Moreover, using the same argument as in Theorem 4, the hypothesis gives us

$$|B|^{\frac{1}{2}}T = T|A|^{\frac{1}{2}}.$$

Since $|A|^{\frac{1}{2}}A = \tilde{A}|A|^{\frac{1}{2}}$ and $|B|^{\frac{1}{2}}B = \tilde{B}|B|^{\frac{1}{2}}$, it follows that $\left(\tilde{B}T - T\tilde{A}\right)|A|^{\frac{1}{2}} = 0$. Thus $\tilde{B}T - T\tilde{A}$ vanish on $\overline{R(|A|)}$. On the other hand, if $x \in \ker(|A|) = \ker(|A|^{\frac{1}{2}})$, we obtain $|B|^{\frac{1}{2}}Tx = T|A|^{\frac{1}{2}}x = 0$. Hence, $\tilde{B}Tx = 0$, and as a result, $\tilde{B}T - T\tilde{A}$ vanishes on $\ker(|A|)$. Consequently

$$\tilde{B}T - T\tilde{A} = 0 \quad \text{on} \quad H = \overline{R(|A|)} \oplus \ker(|A|).$$

Thus (A,B) is $\tilde{P}$-symmetric. □

LEMMA 3. ([15]) *Let $A,B \in \mathscr{L}(\mathscr{H})$. Then the following assertions are equivalent:*

(i) $AT = TB$ implies $A^*T = TB^*$ for all $T \in \mathscr{C}_1(\mathscr{H})$.

(ii) If $AT = TB$, then $\overline{R(T)}$ reduces A, $(\ker T)^\perp$ reduces B, and the restrictions $A|_{\overline{R(T)}}$ and $B|_{(\ker T)^\perp}$ are unitarily equivalent normal operators.

THEOREM 7. *Let $A,B \in \mathscr{L}(\mathscr{H})$ such that A is iw-hyponormal and B is w_*-hyponormal. If $(\tilde{B},\tilde{A})$ has $(FP)_{\mathscr{C}_1(\mathscr{H})}$ property, then (B,A) also has it, and consequently, (A,B) is $\tilde{P}$-symmetric.*

Proof. Since A is invertible, we have $A|A|^{-\frac{1}{2}} = |A|^{-\frac{1}{2}}\tilde{A}$ and $|B|^{\frac{1}{2}}B = \tilde{B}|B|^{\frac{1}{2}}$. Let $T \in \mathscr{C}_1(\mathscr{H})$ such that $BT = TA$. Then $|B|^{\frac{1}{2}}BT|A|^{-\frac{1}{2}} = |B|^{\frac{1}{2}}TA|A|^{-\frac{1}{2}}$, which is equivalent to $\tilde{B}X = X\tilde{A}$ with $X = |B|^{\frac{1}{2}}T|A|^{-\frac{1}{2}} \in \mathscr{C}_1(\mathscr{H})$. So if $(\tilde{B},\tilde{A})$ has $(FP)_{\mathscr{C}_1(\mathscr{H})}$ property, Lemma 3 ensures that $\overline{R(X)}$ reduces $\tilde{B}$, $(\ker(X))^\perp$ reduces $\tilde{A}$, and $\tilde{B}|_{\overline{R(X)}}$ and $\tilde{A}|_{(\ker(X))^\perp}$ are unitarily equivalent normal operators. Therefore,

$$\tilde{A} = M \oplus R \quad \text{on} \quad H_1 = \mathscr{H} = (\ker(X))^\perp \oplus \ker(X),$$

and

$$\tilde{B} = N \oplus S \quad \text{on} \quad H_2 = \mathscr{H} = \overline{R(X)} \oplus R(X)^\perp,$$

where N and M are normal operators. Since A and B are w_*-hyponormal, it follows from [16, Lemma 4.5] that:

$$A = M \oplus R' \quad \text{on} \quad H_1 \quad \text{and} \quad B = N \oplus S' \quad \text{on} \quad H_2.$$

The operator A is invertible and so are M and R'. Since M and N are unitarily equivalent, N is also invertible. Clearly, $|A|^{-1} = |M|^{-1} \oplus |R'|^{-1}$ on H_1 and $|B| = |N| \oplus |S'|$ on H_2. We can write T and X on H_1 into H_2 as

$$X = \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}.$$

It follows from $X = |B|^{\frac{1}{2}}T|A|^{\frac{-1}{2}}$ that

$$\begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} |N|^{\frac{1}{2}}T_1|M|^{\frac{-1}{2}} & |N|^{\frac{1}{2}}T_2|R'|^{\frac{-1}{2}} \\ |S'|^{\frac{1}{2}}T_3|M|^{\frac{-1}{2}} & |S'|^{\frac{1}{2}}T_4|R'|^{\frac{-1}{2}} \end{pmatrix}.$$

Thus, we have $T_2 = 0$ and $|S'|^{\frac{1}{2}}T_3 = |S'|^{\frac{1}{2}}T_4 = 0$, implying that $S'T_3 = S'T_4 = 0$. On the other hand, $BT = TA$ gives us

$$\begin{pmatrix} NT_1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} T_1M & 0 \\ T_3M & T_4R' \end{pmatrix}.$$

From this, we deduce that $T_3 = T_4 = 0$. Consequently $T = T_1 \oplus 0$ and $NT_1 = T_1M$. Applying Fuglede-Putnam's theorem, we obtain $N^*T_1 = T_1M^*$. Hence $B^*T = TA^*$. We conclude that (B,A) has $(F,P)_{\mathscr{C}_1(\mathscr{H})}$ and (A,B) is $\tilde{P}$-symmetric by Theorem 4. □

COROLLARY 4. *If $A,B \in \mathscr{L}(\mathscr{H})$ such that A is an invertible normal operator and B is w_*-hyponormal, then $(\tilde{B},\tilde{A})$ has $(FP)_{\mathscr{C}_1(\mathscr{H})}$ property and (A,B) is $\tilde{P}$-symmetric.*

Proof. We have $A=\tilde{A}$, and since B is w_*-hyponormal, then $\tilde{B}$ is semi-hyponormal by [2, Theorem 2.4]. So if $T\in\mathscr{C}_1(\mathscr{H})$ such that $\tilde{B}T=T\tilde{A}$, it follows from [20], Corollary 2, that $\tilde{B}^*T=T\tilde{A}^*$. Consequently (A,B) is $\tilde{P}$-symmetric by Theorem 7. □

THEOREM 8. *Let $A,B\in\mathscr{L}(\mathscr{H})$ be two w_*-hyponormal operators. If $(\tilde{B},\tilde{A})$ has $(FP)_{\mathscr{C}_1(\mathscr{H})}$ property, then (A,B) is $\tilde{P}$-symmetric if and only if (B,A) has $(FP)_{\mathscr{C}_1(\mathscr{H})}$ property.*

Proof. According to Theorem 4, it suffices to show that if (A,B) is $\tilde{P}$-symmetric, then (B,A) satisfies $(FP)_{\mathscr{C}_1(\mathscr{H})}$ property. Assume that (A,B) is $\tilde{P}$-symmetric and let $T\in\mathscr{C}_1(\mathscr{H})$ be such that $BT=TA$. Then, we also have $\tilde{B}T=T\tilde{A}$. Since $(\tilde{B},\tilde{A})$ has $(FP)_{\mathscr{C}_1(\mathscr{H})}$ property, Lemma 3 ensures that $\overline{R(T)}$ reduces $\tilde{B}$ and $(\ker T)^{\perp}$ reduces $\tilde{A}$, and that the operators $\tilde{B}\big|_{\overline{R(T)}}$ and $\tilde{A}\big|_{(\ker T)^{\perp}}$ are normal and unitarily equivalent. Therefore,

$$\tilde{A}=M\oplus R \quad \text{on} \quad H_1=\mathscr{H}=(\ker T)^{\perp}\oplus\ker T,$$

and

$$\tilde{B}=N\oplus S \quad \text{on} \quad H_2=\mathscr{H}=\overline{R(T)}\oplus R(T)^{\perp},$$

where N and M are normal operators. It follows from [16], Lemma 4.5, that:

$$A=M\oplus R' \quad \text{on} \quad H_1 \quad \text{and} \quad B=N\oplus S' \quad \text{on} \quad H_2.$$

We can write T from H_1 into H_2 as $T=\begin{pmatrix} T_1 & 0\\ 0 & 0\end{pmatrix}$. Since $BT=TA$, then $NT_1=T_1M$. By the Fuglede-Putnam theorem, we obtain $N^*T_1=T_1M^*$, which implies that $B^*T=TA^*$. □

2. Ultraweak closures of derivation ranges

THEOREM 9. *Let $A,B\in\mathscr{L}(\mathscr{H})$, then the following statements are equivalent:*

1. $\overline{\mathscr{R}(\delta_{A,B})}^{w^*}=\overline{\mathscr{R}\left(\delta_{\tilde{A},\tilde{B}}\right)}^{w^*}$;
2. *(i)* $(A,B)\in\tilde{\mathscr{P}}(\mathscr{H})$ *and*
   (ii) $\tilde{B}T=T\tilde{A}$ *implies* $BT=TA$ *for all* $T\in\mathscr{C}_1(\mathscr{H})$.

Proof. Note that $\overline{\mathscr{R}(\delta_{A,B})}^{w^*}=\overline{\mathscr{R}\left(\delta_{\tilde{A},\tilde{B}}\right)}^{w^*}$ if and only if

$$\mathscr{R}(\delta_{A,B})^0\cap\mathscr{L}(\mathscr{H})'^{w^*}\simeq\mathscr{R}\left(\delta_{\tilde{A},\tilde{B}}\right)^0\cap\mathscr{L}(\mathscr{H})'^{w^*}.$$

Using Lemma 2, we get $\mathscr{R}(\delta_{A,B})^0\cap\mathscr{L}(\mathscr{H})'^{w^*}\simeq\ker(\delta_{B,A})\cap\mathscr{C}_1(\mathscr{H})$. It follows that $\overline{\mathscr{R}(\delta_{A,B})}^{w^*}=\overline{\mathscr{R}\left(\delta_{\tilde{A},\tilde{B}}\right)}^{w^*}$ if and only if $\ker(\delta_{B,A})\cap\mathscr{C}_1(\mathscr{H})=\ker\left(\delta_{\tilde{B},\tilde{A}}\right)\cap\mathscr{C}_1(\mathscr{H})$. This give the result. □

THEOREM 10. *Let $A,B \in \mathscr{L}(\mathscr{H})$ such that:*

(i) (A,B) is P-symmetric, and

(ii) $\tilde{B}T = T\tilde{A}$ implies $BT = TA$ for all $T \in \mathscr{C}_1(\mathscr{H})$.

Then $\overline{\mathscr{R}(\delta_{A,B})}^{w^*} = \overline{\mathscr{R}(\delta_{A^*,B^*})}^{w^*} = \overline{\mathscr{R}\left(\delta_{\tilde{A},\tilde{B}}\right)}^{w^*}$.

Proof. This follows immediately from Theorem 1, Theorem 4 and Theorem 9. □

LEMMA 4. *If $S = A \oplus B \in \mathscr{L}(\mathscr{H} \oplus \mathscr{H})$, then the Aluthge transform of S is given by $\tilde{S} = \tilde{A} \oplus \tilde{B}$.*

Proof. Let $A = U|A|, B = V|B|$ and $S = P|S|$ are the polar decompositions of A, B and S respectively, where P and $|S|$ act on $\mathscr{H} \oplus \mathscr{H}$ as follows: $P = U \oplus V$ and $|S| = |A| \oplus |B|$. It follows that

$$\tilde{S} = |S|^{\frac{1}{2}} P |S|^{\frac{1}{2}} = (|A|^{\frac{1}{2}} \oplus |B|^{\frac{1}{2}})(U \oplus V)(|A|^{\frac{1}{2}} \oplus |B|^{\frac{1}{2}}) = \tilde{A} \oplus \tilde{B}. \quad \square$$

COROLLARY 5. *Let $A,B \in \mathscr{L}(\mathscr{H})$ such that:*

(i) The pair (B,A) has $(FP)_{\mathscr{C}_1(\mathscr{H})}$ property, and

(ii) $\tilde{B}T = T\tilde{A}$ implies $BT = TA$ for all $T \in \mathscr{C}_1(\mathscr{H})$.

Then $\overline{\mathscr{R}(\delta_{A,B})}^{w^*} = \overline{\mathscr{R}\left(\delta_{\tilde{A},\tilde{B}}\right)}^{w^*}$ *and $(\tilde{B},\tilde{A})$ has $(FP)_{\mathscr{C}_1(\mathscr{H})}$ property.*

Proof. The first statement follows immediately from Theorem 4 and Theorem 9. Let $T \in \mathscr{C}_1(\mathscr{H})$ such that $\tilde{B}T = T\tilde{A}$, by hypothesis we have also $BT = TA$. Since (B,A) has $(FP)_{\mathscr{C}_1(\mathscr{H})}$ property, Lemma 3 ensures that

$$A = M \oplus R \quad \text{on} \quad H_1 = \mathscr{H} = (\ker T)^{\perp} \oplus \ker T,$$

and

$$B = N \oplus S \quad \text{on} \quad H_2 = \mathscr{H} = \overline{R(T)} \oplus R(T)^{\perp},$$

where N and M are normal operators. It follows from Lemma 4 that

$$\tilde{A} = M \oplus \tilde{R} \quad \text{on} \quad H_1 \quad \text{and} \quad \tilde{B} = N \oplus \tilde{S} \quad \text{on} \quad H_2.$$

The operator T can be expressed from H_1 into H_2 as $T = \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix}$. Since $BT = TA$, we get $NT_1 = T_1 M$. According to Fuglede-Putnam's theorem, we obtain $N^* T_1 = T_1 M^*$, which implies that $\tilde{B}^* T = T\tilde{A}^*$. □

PROPOSITION 2. Let $A,B \in \mathscr{L}(\mathscr{H})$. If A is iw-hyponormal and B is w_*-hyponormal such that $(\tilde{B},\tilde{A})$ has $(FP)_{\mathscr{C}_1(\mathscr{H})}$ property. Then $\overline{\mathscr{R}(\delta_{A,B})}^{w^*} = \overline{\mathscr{R}\left(\delta_{\tilde{A},\tilde{B}}\right)}^{w^*}$.

Proof. By Theorem 7, the pair (A,B) is $\tilde{P}$-symmetric. Let $T \in \mathscr{C}_1(\mathscr{H})$ be such that $\tilde{B}T = T\tilde{A}$. Since A and B are w_*-hyponormal operators, a similar argument to that used in Theorem 8 yields

$$A = M \oplus R' \quad \text{on} \quad H_1 = \mathscr{H} = (\ker T)^{\perp} \oplus \ker T,$$

and

$$B = N \oplus S' \quad \text{on} \quad H_2 = \mathscr{H} = \overline{R(T)} \oplus R(T)^{\perp}.$$

Write T on H_1 into H_2 as $T = \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix}$. Since $\tilde{B}T = T\tilde{A}$, we get $NT_1 = T_1M$, which means that $BT = TA$. Thus Theorem 9 ensures the result. □

COROLLARY 6. *If $A,B \in \mathscr{L}(\mathscr{H})$ such that A is an invertible normal operator and B is w_*-hyponormal, then $\overline{\mathscr{R}(\delta_{A,B})}^{w^*} = \overline{\mathscr{R}\left(\delta_{\tilde{A},\tilde{B}}\right)}^{w^*}$.*

Proof. It is an immediate consequence of Corollary 4 and Proposition 2. □

PROPOSITION 3. Let $A,B \in \mathscr{L}(\mathscr{H})$ be two w_*-hyponormal such that $(\tilde{B},\tilde{A})$ has $(FP)_{\mathscr{C}_1(\mathscr{H})}$ property. Then (A,B) is $\tilde{P}$-symmetric if and only if $\overline{\mathscr{R}(\delta_{A,B})}^{w^*} = \overline{\mathscr{R}\left(\delta_{\tilde{A},\tilde{B}}\right)}^{w^*}$.

Proof. By Theorem 9, it suffices to show that $\tilde{B}T = T\tilde{A}$ implies $BT = TA$ for all $T \in \mathscr{C}_1(\mathscr{H})$, which can be proven in the same way that Proposition 2. □

THEOREM 11. *Let $A,B \in \mathscr{L}(\mathscr{H})$ such that (B,A) has $(FP)_{\mathscr{C}_1(\mathscr{H})}$ property and suppose that B is invertible. Then, $(\tilde{B},\tilde{A})$ has $(FP)_{\mathscr{C}_1(\mathscr{H})}$ property, and*

$$\overline{\mathscr{R}(\delta_{A,B})}^{w^*} = \overline{\mathscr{R}(\delta_{\tilde{A},\tilde{B}})}^{w^*}.$$

Proof. By Corollary 5, it suffices to show the following property: if $\tilde{B}T = T\tilde{A}$, then $BT = TA$ for all $T \in \mathscr{C}_1(\mathscr{H})$. Let $T \in \mathscr{C}_1(\mathscr{H})$ such that $\tilde{B}T = T\tilde{A}$. Since $\tilde{A}|A|^{\frac{1}{2}} = |A|^{\frac{1}{2}}A$ and $|B|^{\frac{-1}{2}}\tilde{B} = B|B|^{\frac{-1}{2}}$, we have

$$|B|^{\frac{-1}{2}}\tilde{B}T|A|^{\frac{1}{2}} = |B|^{\frac{-1}{2}}T\tilde{A}|A|^{\frac{1}{2}},$$

and so

$$B|B|^{\frac{-1}{2}}T|A|^{\frac{1}{2}} = |B|^{\frac{-1}{2}}T|A|^{\frac{1}{2}}A.$$

Set $X = |B|^{\frac{-1}{2}}T|A|^{\frac{1}{2}}$, then $BX = XA$. Hence, by hypothesis and Lemma 3, we get

$$A = M \oplus R \quad \text{on} \quad H_1 = \mathscr{H} = (\ker X)^{\perp} \oplus \ker X,$$

and

$$B = N \oplus S \quad \text{on} \quad H_2 = \mathscr{H} = \overline{R(X)} \oplus R(X)^{\perp},$$

where N and M are unitarily equivalent normal operators. The operator B is invertible, and so are N and S. Since N and M are unitarily equivalent, M is also invertible. Furthermore, we can express X and T from H_1 into H_2 as

$$X = \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}.$$

Clearly, $|A| = |M| \oplus |R|$ on H_1 and $|B|^{-1} = |N|^{-1} \oplus |S|^{-1}$ on H_2. It follows from $X = |B|^{\frac{-1}{2}} T |A|^{\frac{1}{2}}$ that

$$\begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} |N|^{\frac{-1}{2}} T_1 |M|^{\frac{1}{2}} & |N|^{\frac{-1}{2}} T_2 |R|^{\frac{1}{2}} \\ |S|^{\frac{-1}{2}} T_3 |M|^{\frac{1}{2}} & |S|^{\frac{-1}{2}} T_4 |R|^{\frac{1}{2}} \end{pmatrix}.$$

Hence, $T_2 |R|^{\frac{1}{2}} = 0$, $T_3 = 0$, and $T_4 |R|^{\frac{1}{2}} = 0$, which gives $T_2 \tilde{R} = 0$ and $T_4 \tilde{R} = 0$. According to Theorem Lemma 4, we have $\tilde{B} = N \oplus \tilde{S}$ and $\tilde{A} = M \oplus \tilde{R}$. Then, $\tilde{B} T = T \tilde{A}$ implies that

$$\begin{pmatrix} NT_1 & NT_2 \\ 0 & \tilde{S} T_4 \end{pmatrix} = \begin{pmatrix} T_1 M & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore, $T_2 = T_4 = 0$. Thus $T = T_1 \oplus 0$ and $NT_1 = T_1 M$, which means that $BT = TA$. □

COROLLARY 7. *Let $A, B \in \mathscr{L}(\mathscr{H})$ be two iw-hyponormal operators, then (A,B) has $(FP)_{\mathscr{C}_1(\mathscr{H})}$ property if and only if $(\tilde{A}, \tilde{B})$ has it to.*

Proof. It is an immediate consequence of Theorem 7 and Theorem 11. □

COROLLARY 8. *Let $A, B \in \mathscr{L}(\mathscr{H})$ be two invertible operators. If (A,B) is P-symmetric, then so is $(\tilde{A}, \tilde{B})$.*

Proof. Since (A,B) is P-symmetric. Then, according to Theorem 1, (A,B) and (B,A) have $(FP)_{\mathscr{C}_1(\mathscr{H})}$ property. Moreover, since A and B are invertible, it follows from Theorem 11, that $(\tilde{A}, \tilde{B})$ and $(\tilde{B}, \tilde{A})$ have $(FP)_{\mathscr{C}_1(\mathscr{H})}$ property. Therefore, the result follows from Theorem 1. □

PROPOSITION 4. Let $A, B \in \mathscr{L}(\mathscr{H})$, then $\overline{\mathscr{R}(\delta_{A,B})}^{w^*} = \overline{\mathscr{R}\left(\delta_{\tilde{A},\tilde{B}}\right)}^{w^*}$ and $(\tilde{B}, \tilde{A})$ has $(FP)_{\mathscr{C}_1(\mathscr{H})}$ property, in the following cases:

1. A is a contraction and B is unitary.
2. A and B are invertible operators such that $\|A^{-1}\| \|B\| \leqslant 1$.
3. A^* is p-hyponormal or log-hyponormal and B is log-hyponormal.

4. A^* is w_*-hyponormal and B iw-hyponormal.

Proof.

1. This case follows immediately from Corollary 2 and Theorem 11.

2. It follows from [3] that (B,A) has $(FP)_{\mathscr{C}_1(\mathscr{H})}$ property. Therefore, Theorem 11 yields the result.

3. [20, Corollary 2] asserts that (B,A) has $(FP)_{\mathscr{C}_1(\mathscr{H})}$ property. Since B is invertible, Theorem 11 guarantees the result.

4. By using [16], Corollary 4.8, we deduce that (B,A) has $(FP)_{\mathscr{C}_1(\mathscr{H})}$ property. The conclusion follows from Theorem 11. □

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