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DYNAMICS OF WEIGHTED TRANSLATION OPERATORS ON HYPERGROUPS

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Abstract. In this paper, we give necessary and sufficient conditions for weighted translation operators on hypergroups to be topologically multiply recurrent, disjoint topologically transitive and simultaneously transitive. In particular, disjoint topological transitivity and simultaneous transitivity are equivalent in our case.

1. Introduction

1.1. Linear dynamics

In [13], we characterize chaotic and topologically transitive weighted translations on hypergroups, which generalizes some results on locally compact groups. Locally compact groups including the integer group $\mathbb{Z}$, the Euclidean group $\mathbb{R}$, affine groups and Heisenberg groups, are special cases of hypergroups. In this note, we will continue the study of linear dynamics on hypergroups, and tackle a number of notions that are deeper than topological transitivity, namely, topologically multiple recurrence, simultaneous transitivity and disjoint topological transitivity. Linear dynamics was studied intensely during the last five decades. We refer the interested reader to the classical monographs [2, 18] for references.

A linear operator T on a Banach space X is said to be *topologically transitive* if given two nonempty open subsets $U,V \subset X$, there is some $n \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$. If $T^n(U) \cap V \neq \emptyset$ from some n onwards, then T is called *topologically mixing*. For operators on separable Banach spaces, topological transitivity is equivalent to hypercyclicity. In addition, topological transitivity is close to, but stronger than, the notion of recurrence in topological dynamics in [14]. Indeed, T is said to be *recurrent* if for every open set $U \subset X$, there is some $n \in \mathbb{N}$ such that $T^n(U) \cap U \neq \emptyset$. However, this is not the case for topologically multiple recurrence. In [15], it is exhibited a transitive operator on $\ell^2(\mathbb{Z})$, which is not topologically multiply recurrent. In topological dynamics, an operator T is *topologically multiply recurrent* if for every positive integer N and every nonempty open set U in X, there is some $n \in \mathbb{N}$ such

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that $U \cap T^{-n}(U) \cap T^{-2n}(U) \cap \cdots \cap T^{-Nn}(U) \neq \emptyset$. The motivation to connect topological transitivity with recurrence is originally inspired by the works in [14, 15], where topologically multiply recurrent weighted shifts on $\ell^p(\mathbb{Z})$ were characterized. In fact, a wide study of the recurrence phenomenon is carried out recently (see [5, 9, 10, 17, 20] for some references).

Another generalized notion, namely, disjoint topological transitivity, was introduced by Bernal-González, Bès and Peris in [3] and [7] respectively in 2007. We recall some definitions of disjointness in [3, 7] here.

DEFINITION 1.1. Given $N \geqslant 2$, the operators $T_1, T_2, \cdots, T_N$ on a separable Banach space X are called *disjoint topologically transitive* (in short, *d-transitive*) if given nonempty open sets $U, V_1, \cdots, V_N \subset X$, there is some $n \in \mathbb{N}$ such that

$$\emptyset \neq U \cap T_1^{-n}(V_1) \cap T_2^{-n}(V_2) \cap \cdots \cap T_N^{-n}(V_N).$$

If the above condition is satisfied from some n onwards, then $T_1, T_2, \cdots, T_N$ are called *disjoint topologically mixing* (in short, *d-mixing*).

The active topic of disjointness was studied intensely by many authors. Among many important results, the characterization for weighted shifts on $\ell^p(\mathbb{Z})$ to be disjoint topologically transitive was given in [6, 7]. In [11], some sufficient and necessary conditions for weighted translations on locally compact groups to be disjoint topologically transitive were given.

Recently, in [4], Bernal-González and Jung proposed the study of simultaneous transitivity, which is a weaker notion than disjoint transitivity.

DEFINITION 1.2. Given $N \geqslant 2$, the operators $T_1, T_2, \cdots, T_N$ on a separable Banach space X are called *simultaneously transitive* (in short, *s-transitive*) if given nonempty open sets $U, V \subset X$, there is some $n \in \mathbb{N}$ such that

$$\emptyset \neq U \cap T_1^{-n}(V) \cap T_2^{-n}(V) \cap \cdots \cap T_N^{-n}(V).$$

If the above condition is satisfied from some n onwards, then $T_1, T_2, \cdots, T_N$ are said to be *simultaneously mixing* (in short, *s-mixing*).

According to the above definitions, clearly disjoint transitivity implies simultaneous transitivity. However, in [4], there exist simultaneously transitive operators, which are not disjoint transitivity. For weighted shifts on $\ell^p(\mathbb{Z})$, Bernal-González and Jung showed in [4] that simultaneous transitivity is equivalent to disjoint transitivity in this case. In this paper, we will investigate dynamics of translations on hypergroups, including these notions of topologically multiple recurrence, simultaneous transitivity and disjoint transitivity, and demonstrate the connections between them.

1.2. Hypergroups

In this subsection, we recall the definition of hypergroups for further discussion. Let K be a locally compact Hausdorff space, and let $M(K)$ be the Banach space of

regular complex Borel measures on K. Then the Banach space $C_0(K)$ of complex-valued continuous functions on K vanishing at infinity is the predual of $M(K)$. We denote by $\mathrm{supp}(\mu)$ the support of a measure $\mu \in M(K)$, and by δ_x the Dirac measure at $x \in K$.

DEFINITION 1.3. Let $(\mu, \nu) \mapsto \mu * \nu$ be a bilinear positive continuous mapping from $M(K) \times M(K)$ into $M(K)$ (called *convolution*), and let $x \mapsto x^-$ be an involutive homeomorphism on K (called *involution*) such that

1. $(M(K), +, *)$ is a complex associative algebra;
2. for all $x, y \in K$, $\delta_x * \delta_y$ is a probability measure with compact support;
3. there exists a (necessarily unique) element $e \in K$ (called *identity*) such that for all $x \in K$, $\delta_x * \delta_e = \delta_e * \delta_x = \delta_x$;
4. for all $x, y \in K$, $e \in \mathrm{supp}(\delta_x * \delta_y)$ if and only if $x = y^-$;
5. for all $x, y \in K$, $(\delta_x * \delta_y)(\check{\psi}) = (\delta_{y^-} * \delta_{x^-})(\psi)$, where $\psi \in C_0(K)$ and $\check{\psi}(t) := \psi(t^-) \quad (t \in K)$;
6. the mapping $(x, y) \mapsto \mathrm{supp}(\delta_x * \delta_y)$ from $K \times K$ into the space of all non-empty compact subsets of K, equipped with the Michael topology, is continuous (for more information about the Michael topology, see [8] for instance).

Then $K \equiv (K, *, ^-, e)$ is called a *hypergroup*.

Locally compact groups, the double coset spaces, compact hypergroups, commutative hypergroups, discrete hypergroups and nilpotent hypergroups are classical examples of hypergroups. Indeed, if K is a locally compact group, then it is a hypergroup with the convolution $\delta_x * \delta_y = \delta_{xy}$ and the inverse mapping $x \mapsto x^{-1}$ as involution. On the other hand, let H be a compact subgroup of a locally compact group G. Then the double coset space $G//H$ does not inherit a group structure from G. However, the space of regular Borel measure on $G//H$ has an algebra induced by G. For more details about hypergroups, we refer to the classical book [8] and references [16, 19, 21].

EXAMPLE 1.4. ([8, Theorem 1.1.9]) Let G be a locally compact group with identity e. Let H be a compact subgroup of G with the normalized Haar measure ω_H. Then the double coset space $G//H = \{HgH : g \in G\}$ with the quotient topology is a hypergroup with convolution

$$\delta_{Hg_1H} * \delta_{Hg_2H} = \int_H \delta_{Hg_1hg_2H} d\omega_H(h),$$

involution $(HgH)^- = Hg^-H$ and identity $H = HeH$.

Let $f : K \to \mathbb{C}$ be a Borel measurable function. For each $x, y \in K$, set

$$f(x * y) = \int_K f d(\delta_x * \delta_y),$$

if the integral exists. Given a measure $\mu \in M(K)$ and a Borel function f on K, define the convolution $f * \mu$ by

$$f * \mu(x) = \int_K f(x * y^-) d\mu(y) \qquad (x, y \in K)$$

if the integral exists. In particular, for $a \in K$, $f * \delta_a(x) = f(x * a^-) =: f^{a^-}(x)$ can be viewed as the right translation of f by a^-.

A *right Haar measure* on a hypergroup K is a nonnegative regular Borel measure λ on K such that $\lambda * \delta_x = \lambda$ for each $x \in K$. Compact hypergroups, double coset hypergroups, discrete hypergroups, nilpotent hypergroups and commutative hypergroups admit a Haar measure (see [1, 19, 22]). However, it is unclear whether every hypergroup has a Haar measure. Throughout, we assume that K is a hypergroup with a right Haar measure λ. For all $1 \leqslant p < \infty$, we denote the L^p space of K by $L^p(K)$ with respect to λ, and define

$$\|f\|_p = \left(\int_K |f|^p d\lambda \right)^{\frac{1}{p}}$$

for each $f \in L^p(K)$.

EXAMPLE 1.5. Let $G//H$ be the double coset space defined in Example 1.4, and let ω_G be a Haar measure on G. Then $G//H$ is a hypergroup, and

$$\lambda := \int_G \delta_{HgH} d\omega_G(g)$$

is a Haar measure on $G//H$ by [8, Theorem 1.5.20].

In this note, we will investigate linear dynamics of weighted translation operators defined by the weight functions and center elements of hypergroups. A bounded continuous function $w : K \to (0, \infty)$ is called a *weight* on K. Let $a \in K$, and let w be a weight on K. Then a *weighted translation operator* $T_{a,w} : L^p(K) \longrightarrow L^p(K)$ is defined by

$$T_{a,w}(f) = w \cdot (f * \delta_a) = w \cdot f^{a^-},$$

where

$$(f * \delta_a)(x) = \int_K f(x * y^-) d\delta_a(y) = f(x * a^-) = f^{a^-}(x) \qquad (x \in K,\ f \in L^p(K)).$$

If $f \in L^p(K)$, then $T_{a,w}(f) \in L^p(K)$ by [19, 3.3B].

Here we will focus on the iterates of $T_{a,w}$ only when $a \in K$ is a center element of K. Dunkl in [16, 1.6] initiated the notion of center of hypergroups, which is called the *maximum subgroup* in an equivalent definition by Jewett [19, 10.4]. The *center* $\mathrm{Ma}(K)$ of a hypergroup K is defined by

$$\mathrm{Ma}(K) = \{x \in K : \delta_x * \delta_{x^-} = \delta_e = \delta_{x^-} * \delta_x\}.$$

If K is a locally compact group, then $\mathrm{Ma}(K) = K$.

The center elements of a hypergroup admit some nice properties. Indeed, given for $x,y\in K$, $\operatorname{supp}(\delta_x*\delta_y)$ is not a singleton in general. However, if $a\in \mathrm{Ma}(K)$ and $y\in K$, then $\operatorname{supp}(\delta_a*\delta_y)$ and $\operatorname{supp}(\delta_y*\delta_a)$ are singletons by [19, 10.4B]. Hence if $a\in \mathrm{Ma}(K)$ and $n\in\mathbb{N}$, we denote the unique element of the singleton $\operatorname{supp}(\overbrace{\delta_a*\ldots*\delta_a}^{n\text{-times}})$ by a^n. Moreover, let

$$f*\delta_a^n := f*(\overbrace{\delta_a*\cdots*\delta_a}^{n\text{-times}}).$$

Then it is easy to see $f*\delta_a^n = f*\delta_{a^n}$ for $f\in L^p(K)$.

Besides, for any $x\in K$ and $f,g\in L^p(K)$, in general, the equality $(fg)^x = f^x g^x$ does not hold. On the contrary, in [13, Lemma 2.6], one has $(fg)^a = f^a g^a$ if $a\in \mathrm{Ma}(K)$. Hence, using this nice property, we get for all $n\in\mathbb{N}$ that

$$T_{a,w}^n f = w w^{a^-}\ldots w^{(a^-)^{n-1}} f^{(a^-)^n}.$$

Besides, if $w^{-1}\in L^\infty(K)$ and $a\in \mathrm{Ma}(K)$, then by [13, Lemma 2.8], the operator $T_{a,w}$ has an inverse $S_{a,w}$ defined by

$$S_{a,w}(f) = \frac{f}{w}*\delta_{a^-} \qquad (f\in L^p(K)). \tag{1}$$

Throughout, we assume $w, w^{-1}\in L^\infty(K)$.

2. Recurrence

In this section, we will provide and prove our results. As in [8, 1.2.10], define the convolution of two subsets A and B of a hypergroup K by

$$A*B := \bigcup\{\operatorname{supp}(\delta_x*\delta_y) : x\in A, y\in B\}.$$

For each $n\in\mathbb{N}$ and $x\in K$, let

$$A*\{x\}^n = (\cdots(A*\overbrace{\{x\}}^{n\text{-times}})*\cdots)*\{x\}.$$

For instance, $A*\{x\}^2 = (A*\{x\})*\{x\}$ and $A*\{x\}^3 = ((A*\{x\})*\{x\})*\{x\}$. To achieve our goal, we will apply one more property, namely, aperiodicity. Inspired by the study on aperiodic elements of a locally compact group in [12], an element $a\in \mathrm{Ma}(K)$ is called *aperiodic* in [13] if for each compact subset H of K with $\lambda(H)>0$, there exists $N\in\mathbb{N}$ such that $H\cap(H*\{a\}^n)=\emptyset$ for all $n\geqslant N$. Moreover, by [13, Corollary 3.3], the condition $H\cap(H*\{a^-\}^n)=\emptyset$ also holds for all $n\geqslant N$ if $a\in \mathrm{Ma}(K)$ is aperiodic. If K is a locally compact group, then $H\cap Ha^n=\emptyset$ and $H\cap Ha^{-n}=\emptyset$ are equivalent, which coincides with the definition of aperiodicity in [12].

Now we are ready to give the characterization of topological multiple recurrence for $T_{a,w}$ (see Theorem 2.1) below. To simplify the notations, for $a \in \mathrm{Ma}(K)$, $x \in K$ and $n \in \mathbb{N}$, we define

$$\varphi_n(x) := \prod_{j=1}^{n} w * \delta_{a^-}^j(x) = w(x*a)w(x*a^2)\cdots w(x*a^n),$$

and

$$\widetilde{\varphi}_n(x) := \left(\prod_{j=0}^{n-1} w * \delta_a^j(x)\right)^{-1} = \frac{1}{w(x)w(x*a^-)\cdots w(x*(a^-)^{n-1})}.$$

THEOREM 2.1. *Let K be a hypergroup and $a \in Ma(K)$ be aperiodic. Let w be a weight on K with $w, w^{-1} \in L^\infty(K)$. Let $1 \leqslant p < \infty$ and let $T_{a,w}$ be a weighted translation operator on $L^p(K)$. Then the following conditions are equivalent.*

(i) *$T_{a,w}$ is topologically multiply recurrent.*

(ii) *For each $N \in \mathbb{N}$, and each compact subset $H \subset K$ with $\lambda(H) > 0$, there are a sequence of Borel sets (E_k) in H, and a sequence $(n_k) \subset \mathbb{N}$ such that $\lambda(H) = \lim_{k\to\infty} \lambda(E_k)$, and for $1 \leqslant l \leqslant N$,*

$$\lim_{k\to\infty} \|\varphi_{ln_k}|_{E_k}\|_\infty = \lim_{k\to\infty} \|\widetilde{\varphi}_{ln_k}|_{E_k}\|_\infty = 0.$$

Proof. (i) $\Rightarrow$ (ii). Let $H \subset K$ be a compact set with $\lambda(H) > 0$. We denote by $\chi_H \in L^p(K)$ the characteristic function on H. By aperiodicity of a, there is some $M \in \mathbb{N}$ such that $H \cap (H * \{a\}^n) = \emptyset$ for all $n \geqslant M$.

Let $\varepsilon \in (0,1)$, and choose $0 < \delta < \frac{\varepsilon}{1+\varepsilon}$. Let $U = \{g \in L^p(K) : \|g - \chi_H\|_p < \delta^2\}$. Given some $N \in \mathbb{N}$, by the assumption of topologically multiple recurrence, there exists $m > M$ such that

$$U \cap T_{a,w}^{-m}(U) \cap T_{a,w}^{-2m}(U) \cap \cdots \cap T_{a,w}^{-Nm}(U) \neq \emptyset.$$

Hence, there exists a vector $f \in L^p(K)$ such that

$$\|f - \chi_H\|_p < \delta^2 \quad \text{and} \quad \|T_{a,w}^{lm} f - \chi_H\|_p < \delta^2$$

for $1 \leqslant l \leqslant N$. Let

$$A := \{x \in H : |f(x) - 1| \geqslant \delta\}$$

and

$$B := \{x \in K \setminus H : |f(x)| \geqslant \delta\}.$$

Then

$$|f(x)| > 1 - \delta \quad \text{for} \quad x \in H \setminus A$$

and

$$|f(x)| < \delta \quad \text{for} \quad x \in (K \setminus H) \setminus B.$$

From

$$\delta^{2p} > \|f-\chi_H\|_p^p = \int_K |f(x)-\chi_H(x)|^p d\lambda(x) \geqslant \int_A |f(x)-1|^p d\lambda(x) \geqslant \delta^p \lambda(A),$$

one has $\lambda(A) < \delta^p$, and similarly, $\lambda(B) < \delta^p$.

On the other hand, let

$$C_{l,m} := \{x \in H : |\widetilde{\varphi}_{lm}(x)^{-1} f(x*(a^-)^{lm}) - 1| \geqslant \delta\},$$

and

$$D_{l,m} := \{x \in H : |\varphi_{lm}(x) f(x)| \geqslant \delta\}.$$

Then

$$\widetilde{\varphi}_{lm}(x)^{-1} |f(x*(a^-)^{lm})| > 1-\delta \qquad (x \in H \setminus C_{l,m})$$

and

$$\varphi_{lm}(x)|f(x)| < \delta \qquad (x \in H \setminus D_{l,m}).$$

Moreover, $\lambda(C_{l,m}) < \delta^p$ because

$$\begin{aligned}\delta^{2p} &> \|T^{lm}_{a,w} f - \chi_H\|_p^p = \int_K |T^{lm}_{a,w} f(x) - \chi_H(x)|^p d\lambda(x) \\ &\geqslant \int_{C_{l,m}} |w(x)w(x*a^-)\cdots w(x*(a^-)^{lm-1}) f(x*(a^-)^{lm}) - 1|^p d\lambda(x) \\ &= \int_{C_{l,m}} |\widetilde{\varphi}_{lm}(x)^{-1} f(x*(a^-)^{lm}) - 1|^p d\lambda(x) \geqslant \delta^p \lambda(C_{l,m}).\end{aligned}$$

One can obtain $\lambda(D_{l,m}) < \delta^p$ by a similar argument.

By the properties of the sets $A, B, C_{l,m}, D_{l,m}$ and $H \cap (H * \{a^-\}^{lm}) = \emptyset$, we get

$$\widetilde{\varphi}_{lm}(x) < \frac{|f(x*(a^-)^{lm})|}{1-\delta} < \frac{\delta}{1-\delta} < \varepsilon \quad \text{for} \quad x \in H \setminus (C_{l,m} \cup (B * \{a\}^{lm}))$$

and

$$\varphi_{lm}(x) < \frac{\delta}{|f(x)|} < \frac{\delta}{1-\delta} < \varepsilon \quad \text{for} \quad x \in H \setminus (D_{l,m} \cup A).$$

Set

$$E_m := (H \setminus A) \setminus \bigcup_{l=1}^{N} ((B * \{a\}^{lm}) \cup C_{l,m} \cup D_{l,m}).$$

Then $\lambda(H \setminus E_m) < 4N\delta^p$ and $\|\varphi_{lm}|_{E_m}\|_\infty < \varepsilon$, $\|\widetilde{\varphi}_{lm}|_{E_m}\|_\infty < \varepsilon$, which implies condition (ii).

(ii) $\Rightarrow$ (i). Let U be a nonempty open subset of $L^p(K)$. Since the space $C_c(K)$ of continuous functions on K with compact support is dense in $L^p(K)$, we can choose a function $f \in C_c(K) \cap U$. Let H be the (compact) support of f. Given some N, assume

condition (ii) holds with respect to $E_k \subset H$ and the sequences $(\varphi_{ln}), (\widetilde{\varphi}_{ln})$. Then for $1 \leqslant l \leqslant N$,

$$\begin{aligned}
&\|T_{a,w}^{ln_k}(f\chi_{E_k})\|_p^p \\
&= \int_{E_k * \{a\}^{ln_k}} |w(x)w(x*a^-)\cdots w(x*(a^-)^{ln_k-1})|^p |f(x*(a^-)^{ln_k})|^p d\lambda(x) \\
&= \int_{E_k} |w(x*a^{ln_k})w(x*a^{ln_k-1})\cdots w(x*a)|^p |f(x)|^p d\lambda(x) \\
&= \int_{E_k} \varphi_{ln_k}^p(x)|f(x)|^p d\lambda(x) \leqslant \|\varphi_{ln_k}|_{E_k}\|_\infty^p \|f\|_p^p \to 0
\end{aligned}$$

as $k \to \infty$. That $\lim_{k\to\infty} \|S_{a,w}^{ln_k}(f\chi_{E_k})\|_p = 0$ can be obtained similarly. In addition,

$$\|f\chi_{E_k} - f\|_p^p = \int_{K\setminus E_k} |f(x)|^p d\lambda(x) \leqslant \lambda(K\setminus E_k)\|f\|_p^p \to 0$$

as $k \to \infty$.

For each $k \in \mathbb{N}$, let

$$v_k = f\chi_{E_k} + S_{a,w}^{n_k}(f\chi_{E_k}) + S_{a,w}^{2n_k}(f\chi_{E_k}) + \cdots + S_{a,w}^{Nn_k}(f\chi_{E_k}) \in L^p(K),$$

where $S_{a,w}$ is given by (1). Then

$$\|v_k - f\|_p \leqslant \|f\chi_{E_k} - f\|_p + \sum_{l=1}^{N} \|S_{a,w}^{ln_k}(f\chi_{E_k})\|_p$$

and

$$\begin{aligned}
\|T_{a,w}^{ln_k}v_k - f\|_p \leqslant\ & \|T_{a,w}^{ln_k}(f\chi_{E_k})\|_p + \|T_{a,w}^{(l-1)n_k}(f\chi_{E_k})\|_p + \cdots + \|T_{a,w}^{n_k}(f\chi_{E_k})\|_p \\
&+ \|f\chi_{E_k} - f\|_p + \|S_{a,w}^{n_k}(f\chi_{E_k})\|_p + \cdots + \|S_{a,w}^{(N-l)n_k}(f\chi_{E_k})\|_p.
\end{aligned}$$

Since the sequences $\{\|T_{a,w}^{ln_k}(f\chi_{E_k})\|_p\}_{k\geqslant 1}$, $\{\|f\chi_{E_k} - f\|_p\}_{k\geqslant 1}$ and $\{\|S_{a,w}^{ln_k}(f\chi_{E_k})\|_p\}_{k\geqslant 1}$ $(l \in \{1,2,\cdots,N\})$ tend to 0 $(k \to \infty)$ and $f \in U$, we get for k large enough that $v_k \in U, T_{a,w}^{n_k}v_k \in U, \cdots, T_{a,w}^{Nn_k}v_k \in U$, that is,

$$U \cap T_{a,w}^{-n_k}(U) \cap T_{a,w}^{-2n_k}(U) \cap \cdots \cap T_{a,w}^{-Nn_k}(U) \neq \emptyset,$$

as required. □

EXAMPLE 2.2. Let $G//H$ be the double coset space with the Haar measure λ on $G//H$, as defined in Example 1.4. Let w be a weight on $G//H$, and let $HaH \in \mathrm{Ma}(G//H)$ be aperiodic where $a \in G$. Then

$$\varphi_n(HxH) = \prod_{j=1}^{n} w * \delta_{Ha^{-1}H}^{j}(HxH),$$

$$\widetilde{\varphi}_n(HxH) = \left(\prod_{j=0}^{n-1} w * \delta_{HaH}^{j}(HxH)\right)^{-1}$$

and the weighted translation is

$$T_{HaH,w} f(HxH) = w(HxH) f(HxH * Ha^{-1}H)$$

where $x \in G$ and $f \in L^p(G//H)$. By Theorem 2.1, $T_{HaH,w}$ is topologically multiply recurrent if given each $N \in \mathbb{N}$, and each compact subset $C \subseteq G//H$ with $\lambda(C) > 0$, there are a sequence of Borel sets (E_k) in C and a sequence $(n_k) \subset \mathbb{N}$, such that $\lambda(C) = \lim_{k\to\infty} \lambda(E_k)$, and for $1 \leqslant l \leqslant N$, $\lim_{k\to\infty} \|\varphi_{ln_k}|_{E_k}\|_\infty = \lim_{k\to\infty} \|\widetilde{\varphi}_{ln_k}|_{E_k}\|_\infty = 0$.

Taking $N = 1$ in Theorem 2.1, we obtain the following result.

COROLLARY 2.3. *Let K be a hypergroup and $a \in Ma(K)$ be aperiodic. Let w be a weight on K with $w, w^{-1} \in L^\infty(K)$. Let $1 \leqslant p < \infty$ and let $T_{a,w}$ be a weighted translation operator on $L^p(K)$. Then the following conditions are equivalent.*

(i) *$T_{a,w}$ is recurrent.*

(ii) *For each compact subset $H \subset K$ with $\lambda(H) > 0$, there are a sequence of Borel sets (E_k) in H, and a sequence $(n_k) \subset \mathbb{N}$ such that $\lambda(H) = \lim_{k\to\infty} \lambda(E_k)$, and*

$$\lim_{k\to\infty} \|\varphi_{n_k}|_{E_k}\|_\infty = \lim_{k\to\infty} \|\widetilde{\varphi}_{n_k}|_{E_k}\|_\infty = 0.$$

Applying similar arguments to those in the proof of Theorem 2.1, one can show that topological mixing implies topological multiple recurrence.

COROLLARY 2.4. *Let K be a hypergroup and $a \in Ma(K)$ be aperiodic. Let w be a weight on K with $w, w^{-1} \in L^\infty(K)$. Let $1 \leqslant p < \infty$ and let $T_{a,w}$ be a weighted translation operator on $L^p(K)$. If $T_{a,w}$ is topologically mixing, then $T_{a,w}$ is topologically multiply recurrent on $L^p(K)$.*

Proof. Let $\varepsilon > 0$, and let H be a compact set of K. Let $U = \{g \in L^p(K) : \|g - \chi_H\|_p < \delta^2\}$ where $0 < \delta < \frac{\varepsilon}{1+\varepsilon}$. By the assumption of topological mixing, there exists some $n \in \mathbb{N}$ such that

$$T_{a,w}^n(U) \cap U \neq \emptyset$$

from n onwards. Hence, given $N \in \mathbb{N}$, there exist $f_l \in U$ and $m \in \mathbb{N}$ such that

$$\|f_l - \chi_H\| < \delta^2 \qquad \text{and} \qquad \|T_{a,w}^{lm} f_l - \chi_H\| < \delta^2$$

for $l = 1, 2, \cdots, N$. By the same arguments in the proof of Theorem 2.1, one can deduce $T_{a,w}$ is topologically multiply recurrent. □

3. Transitivity

Next, we turn our attention to give the characterizations for weighted translation operators on hypergroups to be simultaneously transitive and disjoint topologically transitive. Again, to simplify the notation, for $a_l \in \mathrm{Ma}(K)$, $x \in K$ and $l,n \in \mathbb{N}$, define

$$\varphi_{l,n}(x) := \prod_{j=1}^{n} w_l * \delta^j_{a_l^-}(x) = w_l(x * a_l) w_l(x * a_l^2) \cdots w_l(x * a_l^n),$$

and

$$\widetilde{\varphi}_{l,n}(x) := \left(\prod_{j=0}^{n-1} w_l * \delta^j_{a_l}(x) \right)^{-1} = \frac{1}{w_l(x) w_l(x * a_l^-) \cdots w_l(x * (a_l^-)^{n-1})}.$$

THEOREM 3.1. *Let K be a hypergroup, and let $1 \leqslant p < \infty$. Given some $N \geqslant 2$, let $a_l \in Ma(K)$ be aperiodic, and let w_l be a weight on K with $w_l, w_l^{-1} \in L^\infty(K)$ for $1 \leqslant l \leqslant N$. Let $T_l = T_{a_l,w_l}$ be the weighted translation on $L^p(K)$ generated by a_l and w_l. Then the following conditions are equivalent.*

(i) *$T_1, T_2, \cdots, T_N$ are simultaneously transitive.*

(ii) *$T_1, T_2, \cdots, T_N$ are disjoint topologically transitive.*

(iii) *For each compact subset $H \subset K$ with $\lambda(H) > 0$, there are a sequence of Borel sets (E_k) in H, and a sequence $(n_k) \subset \mathbb{N}$ such that $\lambda(H) = \lim_{k\to\infty} \lambda(E_k)$, and for $1 \leqslant l \leqslant N$,*

$$\lim_{k\to\infty} \|\varphi_{l,n_k}|_{E_k}\|_\infty = \lim_{k\to\infty} \|\widetilde{\varphi}_{l,n_k}|_{E_k}\|_\infty = 0,$$

and for $1 \leqslant s < l \leqslant N$,

$$\lim_{k\to\infty} \left\| (\varphi_{s,n_k} * \delta_{a_l^{n_k}}) \cdot \widetilde{\varphi}_{l,n_k}\big|_{E_k} \right\|_\infty = \lim_{k\to\infty} \left\| (\varphi_{l,n_k} * \delta_{a_s^{n_k}}) \cdot \widetilde{\varphi}_{s,n_k}\big|_{E_k} \right\|_\infty = 0.$$

Proof. (iii) $\Rightarrow$ (ii). Let U and V_l be nonempty open subsets of $L^p(K)$ for $1 \leqslant l \leqslant N$. Since the space $C_c(K)$ of continuous functions on K with compact support is dense in $L^p(K)$, choose $f \in C_c(K) \cap U$ and $g_l \in C_c(K) \cap V_l$ for each l. Let H be the union of all the compact supports of f and g_l. Assume condition (iii) holds with respect to $E_k \subset H$ and the sequences $(\varphi_{l,n}), (\widetilde{\varphi}_{l,n})$. As in the proof of Theorem 2.1, we have

$$\lim_{k\to\infty} \|f\chi_{E_k} - f\|_p = \lim_{k\to\infty} \|g_l\chi_{E_k} - g_l\|_p = 0.$$

Also, for $1 \leqslant l \leqslant N$,

$$\begin{aligned}
\|T_l^{n_k}(f\chi_{E_k})\|_p^p &= \int_{E_k * \{a_l\}^{n_k}} |w_l(x) w_l(x * a_l^-) \cdots w_l(x * (a_l^-)^{n_k - 1})|^p |f(x * (a_l^-)^{n_k})|^p d\lambda(x) \\
&= \int_{E_k} |w_l(x * a_l^{n_k}) w_l(x * a_l^{n_k - 1}) \cdots w_l(x * a_l)|^p |f(x)|^p d\lambda(x) \\
&= \int_{E_k} \varphi_{l,n_k}^p(x) |f(x)|^p d\lambda(x) \leqslant \|\varphi_{l,n_k}|_{E_k}\|_\infty^p \|f\|_p^p.
\end{aligned}$$

Hence $\|T_l^{n_k}(f\chi_{E_k})\|_p \longrightarrow 0$ as $k \longrightarrow \infty$. Let $S_l = S_{a_l,w_l}$ (see (1)).

Then $\lim_{k\to\infty}\|S_l^{n_k}(g\chi_{E_k})\|_p = 0$ because

$$\begin{aligned}\|S_l^{n_k}(g_l\chi_{E_k})\|_p^p &= \int_{E_k*\{a_l^-\}^{n_k}} \frac{1}{|w_l(x*a_l)w_l(x*a_l^2)\cdots w_l(x*a_l^{n_k})|^p}|g_l(x*a_l^{n_k})|^p d\lambda(x)\\ &\leqslant \|\widetilde{\varphi}_{l,n_k}|_{E_k}\|_\infty^p \|g_l\|_p^p.\end{aligned}$$

On the other hand, for $1 \leqslant s,l \leqslant N$ with $s \neq l$, we have

$$\begin{aligned}&\|T_l^{n_k}(S_s^{n_k}(g_s\chi_{E_k}))\|_p^p\\ &= \int_K |w_l(x)w_l(x*a_l^-)\cdots w_l(x*(a_l^-)^{n_k-1})|^p |S_s^{n_k}(g_s\chi_{E_k})(x*(a_l^-)^{n_k})|^p d\lambda(x)\\ &= \int_K \frac{|w_l(x)w_l(x*a_l^-)\cdots w_l(x*(a_l^-)^{n_k-1})|^p}{|w_s(x*(a_l^-)^{n_k}*a_s)w_s(x*(a_l^-)^{n_k}*a_s^2)\cdots w_s(x*(a_l^-)^{n_k}*a_s^{n_k})|^p}\\ &\quad\cdot|g_s\chi_{E_k}(x*(a_l^-)^{n_k}*a_s^{n_k})|^p d\lambda(x)\\ &= \int_{E_k} \frac{|w_l(x*(a_s^-)^{n_k}*a_l^{n_k})w_l(x*(a_s^-)^{n_k}*a_l^{n_k-1})\cdots w_l(x*(a_s^-)^{n_k}*a_l)|^p}{|w_s(x*(a_s^-)^{n_k-1})w_s(x*(a_s^-)^{n_k-2})\cdots w_s(x)|^p}|g_s(x)|^p d\lambda(x)\\ &= \int_{E_k} |\varphi_{l,n_k}(x*(a_s^-)^{n_k})\cdot\widetilde{\varphi}_{s,n_k}(x)|^p|g_s(x)|^p d\lambda(x)\\ &\leqslant \|(\varphi_{l,n_k}*\delta_{a_s^{n_k}})\cdot\widetilde{\varphi}_{s,n_k}|_{E_k}\|_\infty^p\|g_s\|_p^p.\end{aligned}$$

For each $k \in \mathbb{N}$, let

$$v_k = f\chi_{E_k} + S_1^{n_k}(g_1\chi_{E_k}) + S_2^{n_k}(g_2\chi_{E_k}) + \cdots + S_N^{n_k}(g_N\chi_{E_k}) \in L^p(K).$$

Then

$$\|v_k - f\|^p \leqslant \|f\chi_{E_k} - f\|_p + \sum_{l=1}^N \|S_l^{n_k}(g_l\chi_{E_k})\|_p$$

and

$$\begin{aligned}&\|T_l^{n_k}v_k - g_l\|_p\\ &\leqslant \|T_l^{n_k}(f\chi_{E_k})\|_p + \|T_l^{n_k}S_1^{n_k}(g_1\chi_{E_k})\|_p + \cdots + \|T_l^{n_k}S_{l-1}^{n_k}g_{l-1}\chi_{E_k}\|_p^p\\ &\quad + \|g_l\chi_{E_k} - g_l\|_p + \|T_l^{n_k}S_{l+1}^{n_k}(g_{l+1}\chi_{E_k})\|_p + \cdots + \|T_l^{n_k}S_N^{n_k}(g_N\chi_{E_k})\|_p.\end{aligned}$$

Hence $\lim_{k\to\infty} v_k = f$ and $\lim_{k\to\infty} T_l^{n_k}v_k = g_l$ for $l = 1,2,\cdots,N$, which implies

$$\emptyset \neq U \cap T_1^{-n_k}(V_1) \cap T_2^{-n_k}(V_2) \cap \cdots \cap T_N^{-n_k}(V_N).$$

Therefore, $T_1, T_2, \cdots, T_N$ are disjoint topologically transitive.

(ii) $\Rightarrow$ (i). By definitions, disjoint topological transitivity implies simultaneous transitivity automatically.

(i) $\Rightarrow$ (iii). Assume that $T_1, T_2, \cdots, T_N$ are simultaneously transitive. Let $H \subset K$ be a compact set with $\lambda(H) > 0$. By aperiodicity of a_l, there is some M such that $H \cap (H * \{a_l\}^n) = \emptyset$ and $H \cap (H * \{a_l^-\}^n) = \emptyset$ for all $n > M$.

Let $\chi_H \in L^p(K)$ be the characteristic function of H. Let $\varepsilon \in (0,1)$, and choose $0 < \delta < \frac{\varepsilon}{1+\varepsilon}$. By the assumption of simultaneous transitivity, there exist a vector $f \in L^p(K)$ and some $m > M$ such that

$$\|f - \chi_H\|_p < \delta^2 \quad \text{and} \quad \|T_l^m f - \chi_H\|_p < \delta^2$$

for $l = 1, 2, \ldots, N$. As in the proof of Theorem 2.1, let

$$A := \{x \in H : |f(x) - 1| \geqslant \delta\} \quad \text{and} \quad B := \{x \in K \setminus H : |f(x)| \geqslant \delta\}.$$

Then

$$|f(x)| > 1 - \delta \quad \text{for} \quad x \in H \setminus A, \quad \text{and} \quad |f(x)| < \delta \quad \text{for} \quad x \in (K \setminus H) \setminus B$$

with $\lambda(A) < \delta^p$ and $\lambda(B) < \delta^p$. On the other hand, observe that

$$\begin{aligned} \delta^{2p} > \|T_l^m f - \chi_H\|_p^p &= \int_K |T_l^m f(x) - \chi_H(x)|^p d\lambda(x) \\ &= \int_K |\widetilde{\varphi}_{l,m}(x)^{-1} f(x * (a_l^-)^m) - \chi_H(x)|^p d\lambda(x) \\ &= \int_K |\varphi_{l,m}(x) f(x) - \chi_H(x * (a_l^-)^m)|^p d\lambda(x). \end{aligned}$$

Let

$$C_{l,m} := \{x \in H : |\widetilde{\varphi}_{l,m}(x)^{-1} f(x * (a_l^-)^m) - 1| \geqslant \delta\}$$

and

$$D_{l.m} := \{x \in H : |\varphi_{l,m}(x) f(x)| \geqslant \delta\}.$$

Then

$$\widetilde{\varphi}_{l,m}(x)^{-1} |f(x * (a_l^-)^m)| > 1 - \delta \qquad (x \in H \setminus C_{l,m})$$

and

$$\varphi_{l,m}(x) |f(x)| < \delta \qquad (x \in H \setminus D_{l,m}).$$

Moreover, $\lambda(C_{l,m}) < \delta^p$ and $\lambda(D_{l,m}) < \delta^p$. Indeed, for instance, the estimate $\lambda(C_{l,m}) < \delta^p$ (that $\lambda(D_{l,m}) < \delta^p$ can be analogously proved) can be deduced by

$$\begin{aligned} \delta^{2p} &> \|T_l^m f - \chi_H\|_p^p = \int_K |T_l^m f(x) - \chi_K(x)|^p d\lambda(x) \\ &\geqslant \int_{C_{l,m}} |w_l(x) w_l(x * (a_l^-)) \cdots w_l(x * (a_l^-)^{m-1}) f(x * (a_l^-)^m) - 1|^p d\lambda(x) \geqslant \delta^p \lambda(C_{l,m}). \end{aligned}$$

Hence, making use of $H \cap (H * \{a_l\}^n) = \emptyset$ and $H \cap (H * \{a_l^-\}^n) = \emptyset$, we arrive at

$$\widetilde{\varphi}_{l,m}(x) < \frac{|f(x * (a_l^-)^m)|}{1 - \delta} < \frac{\delta}{1 - \delta} < \varepsilon \qquad \text{on} \qquad H \setminus (C_{l,m} \cup (B * \{a_l\}^m))$$

and

$$\varphi_{l,m}(x) < \frac{\delta}{|f(x)|} < \frac{\delta}{1 - \delta} < \varepsilon \qquad \text{on} \qquad H \setminus (D_{l,m} \cup A).$$

In the next step, we will demonstrate the other two weight conditions for $1 \leqslant s < l \leqslant N$. Let

$$F_{s,m} := \{x \in K \setminus H : |\widetilde{\varphi}_{s,m}(x)^{-1} f(x * (a_s^-)^m)| \geqslant \delta\}$$

for $1 \leqslant s < l \leqslant N$. Then

$$|w_s(x)w_s(x * a_s^-) \cdots w_s(x * (a_s^-)^{m-1}) f(x * (a_s^-)^m)| < \delta \quad \text{on } (H * \{a_l^-\}^m * \{a_s\}^m) \setminus F_{s,m} \subset K \setminus H$$

which says

$$|w_s(x * (a_l^-)^m * a_s^m) \cdots w_s(x * (a_l^-)^m * a_s) f(x * (a_l^-)^m)| < \delta \text{ on } H \setminus (F_{s,m} * \{a_l\}^m * \{a_s^-\}^m).$$

Also, $\lambda(F_{s,m}) < \delta^p$ because

$$\begin{aligned} \delta^{2p} &> \|T_s^m f - \chi_H\|_p^p \geqslant \int_{F_{s,m}} |T_s^m f(x)|^p d\lambda(x) \\ &= \int_{F_{s,m}} |w_s(x)w_s(x * a_s^-) \cdots w_s(x * (a_s^-)^{m-1}) f(x * (a_s^-)^m)|^p d\lambda(x) \geqslant \delta^p \lambda(F_{s,m}). \end{aligned}$$

On the other hand, by the definition of $C_{l,m}$, we get

$$|w_l(x)w_l(x * a_l^-) \cdots w_l(x * (a_l^-)^{m-1}) f(x * (a_l^-)^m) - 1| < \delta \quad \text{on} \quad H \setminus C_{l,m}.$$

Combining all these, for $1 \leqslant s < l \leqslant N$ and $x \in H \setminus ((F_{s,m} * \{a_l\}^m * \{a_s^-\}^m) \cup C_{l,m})$, we obtain

$$\begin{aligned} &\frac{w_s(x * (a_l^-)^m * a_s^m) \cdots w_s(x * (a_l^-)^m * a_s)}{w_l(x)w_l(x * a_l^-) \cdots w_l(x * (a_l^-)^{m-1})} \\ &= \frac{w_s(x * (a_l^-)^m * a_s^m) \cdots w_s(x * (a_l^-)^m * a_s) |f(x * (a_l^-)^m)|}{w_l(x)w_l(x * a_l^-) \cdots w_l(x * (a_l^-)^{m-1}) |f(x * (a_l^-)^m)|} < \frac{\delta}{1 - \delta} < \varepsilon. \end{aligned}$$

Therefore,

$$\varphi_{s,m}(x * (a_l^-)^m) \cdot \widetilde{\varphi}_{l,m}(x) < \varepsilon \quad \text{on} \quad H \setminus ((F_{s,m} * \{a_l\}^m * \{a_s^-\}^m) \cup C_{l,m}).$$

Similarly, one has

$$|w_l(x)w_l(x * a_l^-) \cdots w_l(x * (a_l^-)^{m-1}) f(x * (a_l^-)^m)| < \delta \quad \text{on} \quad (H * \{a_s^-\}^m * \{a_l\}^m) \setminus F_{l,m}.$$

That is,

$$|w_l(x * (a_s^-)^m * a_l^m) \cdots w_l(x * (a_s^-)^m * a_l) f(x * (a_l^-)^m)| < \delta \quad \text{on} \quad H \setminus (F_{l,m} * \{a_s\}^m * \{a_l^-\}^m).$$

Hence, for $1 \leqslant s < l \leqslant N$ and $x \in H \setminus ((F_{l,m} * \{a_s\}^m * \{a_l^-\}^m) \cup C_{s,m})$, we get

$$\begin{aligned} &\frac{w_l(x * (a_s^-)^m * a_l^m) \cdots w_l(x * (a_s^-)^m * a_l)}{w_s(x)w_s(x * a_s^-) \cdots w_s(x * (a_s^-)^{m-1})} \\ &= \frac{w_l(x * (a_s^-)^m * a_l^m) \cdots w_l(x * (a_s^-)^m * a_l) |f(x * (a_s^-)^m)|}{w_s(x)w_s(x * a_s^-) \cdots w_s(x * (a_s^-)^{m-1}) |f(x * (a_s^-)^m)|} < \frac{\delta}{1 - \delta} < \varepsilon, \end{aligned}$$

which implies

$$\varphi_{l,m}(x*(a_s^-)^m)\cdot\widetilde{\varphi}_{s,m}(x)<\varepsilon \quad \text{on} \quad H\setminus((F_{l,m}*\{a_s\}^m*\{a_l^-\}^m)\cup C_{s,m}).$$

Finally, put

$$\begin{aligned}E_m=(H\setminus A)\setminus\bigcup_{1\leqslant l\leqslant N}((B*\{a_l\}^m)\cup C_{l,m}\cup D_{l,m})\\ \setminus\bigcup_{1\leqslant s<l\leqslant N}((F_{s,m}*\{a_l\}^m*\{a_s^-\}^m)\cup(F_{l,m}*\{a_s\}^m*\{a_l^-\}^m)).\end{aligned}$$

Then

$$\lambda(H\setminus E_m)<6N^2\delta^p,\qquad \|\varphi_{l,m}|_{E_m}\|_\infty<\varepsilon,\qquad \|\widetilde{\varphi}_{l,m}|_{E_m}\|_\infty<\varepsilon,$$

and

$$\left\|(\varphi_{s,m}*\delta_{a_l^m})\cdot\widetilde{\varphi}_{l,m}\big|_{E_m}\right\|_\infty<\varepsilon,\qquad \left\|(\varphi_{l,m}*\delta_{a_s^m})\cdot\widetilde{\varphi}_{s,m}\big|_{E_m}\right\|_\infty<\varepsilon,$$

which proves condition (iii). □

By replacing the subsequence (n_k) with the full sequence (n) in the proof of Theorem 3.1, we can demonstrate that simultaneous mixing is equivalent to disjoint topological mixing. That is, we obtain the following.

COROLLARY 3.2. *Let K be a hypergroup, and let $1\leqslant p<\infty$. Given some $N\geqslant 2$, let $a_l\in Ma(K)$ be aperiodic, and let w_l be a weight on K with $w_l,w_l^{-1}\in L^\infty(K)$ for $1\leqslant l\leqslant N$. Let $T_l=T_{a_l,w_l}$ be the weighted translation on $L^p(K)$ generated by a_l and w_l. Then the following conditions are equivalent.*

(i) *$T_1,T_2,\cdots,T_N$ are simultaneously mixing.*

(ii) *$T_1,T_2,\cdots,T_N$ are disjoint topologically mixing.*

(iii) *For each compact subset $H\subset K$ with $\lambda(H)>0$, there exists a sequence of Borel sets (E_n) in H such that $\lambda(H)=\lim_{n\to\infty}\lambda(E_n)$,*

$$\lim_{n\to\infty}\|\varphi_{l,n}|_{E_n}\|_\infty=\lim_{n\to\infty}\|\widetilde{\varphi}_{l,n}|_{E_n}\|_\infty=0$$

for $1\leqslant l\leqslant N$, and

$$\lim_{n\to\infty}\left\|(\varphi_{s,n}*\delta_{a_l^n})\cdot\widetilde{\varphi}_{l,n}\big|_{E_n}\right\|_\infty=\lim_{n\to\infty}\left\|(\varphi_{l,n}*\delta_{a_s^n})\cdot\widetilde{\varphi}_{s,n}\big|_{E_n}\right\|_\infty=0$$

for $1\leqslant s<l\leqslant N$.

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REFERENCES

[1] M. AMINI AND C-H. CHU, *Harmonic functions on hypergroups*, J. Funct. Anal. **261** (2011) 1835–1864.

[2] F. BAYART AND É. MATHERON, *Dynamics of linear operators*, Cambridge Tracts in Math. **179**, Cambridge University Press, Cambridge, 2009.

[3] L. BERNAL-GONZÁLEZ, *Disjoint hypercyclic operators*, Studia Math. **182** (2007) 113–131.

[4] L. BERNAL-GONZÁLEZ AND A. JUNG, *Simultaneous universality*, J. Approx. Theory **232** (2018) 43–65.

[5] N. C. BERNARDES JR. AND A. PERIS, *On shadowing and chain recurrence in linear dynamics*, Adv. Math. **441** (2024) Article 109539.

[6] J. BÈS, Ö. MARTIN AND R. SANDERS, *Weighted shifts and disjoint hypercyclicity*, J. Operator Theory **72** (2014) 15–40.

[7] J. BÈS AND A. PERIS, *Disjointness in hypercyclicity*, J. Math. Anal. Appl. **336** (2007) 297–315.

[8] W. R. BLOOM AND H. HEYER, *Harmonic Analysis of Probability Measures on Hypergroups*, De Kruyter, Berlin, 1995.

[9] A. BONILLA, K-G. GROSSE-ERDMANN, A. LÓPEZ-MARTÍNEZ AND A. PERIS, *Frequently recurrent operators*, J. Funct. Anal. **283** (2022) Article 109713.

[10] R. CARDECCIA AND S. MURO, *Arithmetic progressions and chaos in linear dynamics*, Integr. Equ. Oper. Theory **94** (2022) Article 11.

[11] C-C. CHEN, *Disjoint hypercyclic weighted translations on groups*, Banach J. Math. Anal. **11** (2017) 459–476.

[12] C-C. CHEN AND C-H. CHU, *Hypercyclic weighted translations on groups*, Proc. Amer. Math. Soc. **139** (2011) 2839–2846.

[13] C-C. CHEN AND S. M. TABATABAIE, *Chaotic operators on hypergroups*, Oper. Matrices **12** (2018) 143–156.

[14] G. COSTAKIS, A. MANOUSSOS AND I. PARISSIS, *Recurrent linear operators*, Complex Anal. Oper. Th. **8** (2014) 1601–1643.

[15] G. COSTAKIS AND I. PARISSIS, *Szemerèdi's theorem, frequent hypercyclicity and multiple recurrence*, Math. Scand. **110** (2012) 251–272.

[16] C. F. DUNKL, *The measure algebra of a locally compact hypergroup*, Trans. Amer. Math. Soc. **179** (1973) 331–348.

[17] S. GRIVAUX AND A. LÓPEZ-MARTÍNEZ, *Recurrence properties for linear dynamical systems: an approach via invariant measures*, J. Math. Pures Appl. **169** (2023) 155–188.

[18] K.-G. GROSSE-ERDMANN AND A. PERIS, *Linear Chaos*, Universitext, Springer, 2011.

[19] R. I. JEWETT, *Spaces with an abstract convolution of measures*, Adv. Math. **18** (1975) 1–101.

[20] A. LÓPEZ-MARTÍNEZ AND Q. MENET, *Two remarks on the set of recurrent vectors*, J. Math. Anal. Appl. **541** (2025) Article 128686.

[21] R. SPECTOR, *Aperçu de la théorie des hypergroups*, in: Analyse Harmonique sur les Kroups de Lie, 643–673, Lec. Notes Math. Ser., **497**, Springer, 1975.

[22] R. SPECTOR, *Measures invariantes sur les hypergroups*, Trans. Amer. Math. Soc. **239** (1978) 147–165.

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