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REFINEMENTS OF AUDENAERT INTERPOLATION INEQUALITY

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Abstract. In this note, we present several refinements of Audenaert's inequality and his generalized results using log-convexity of some functions depending on special cases of unitarily invariant norms, the trace norm and the operator norm. At the same time, we provide a modification to an alternative proof, obtained by Al-Khlyleh and Alrimawi, of the generalized Audenaert's inequality, but for the operator norm.

1. Introduction

Let $\mathbf{M}_n$ be the set of $n\times n$ complex matrices. Recall that a norm $\|\cdot\|$ on $\mathbf{M}_n$ is unitarily invariant if $\|UAV\| = \|A\|$ for any $A\in\mathbf{M}_n$ and all unitary matrices $U,V\in\mathbf{M}_n$. The trace and operator norms, special cases of unitarily invariant norms, are defined as $\|A\|_1 = \sum_{i=1}^n s_i(A)$ and $\|A\|_\infty = s_1(A)$, respectively. Let $x=(x_1,x_2,\ldots,x_n)$, $y=(y_1,y_2,\ldots,y_n)\in\mathbf{R}^n$. We rearrange the components of x in nonincreasing order as $x_{[1]}\geqslant x_{[2]}\geqslant\cdots\geqslant x_{[n]}$. If

$$\sum_{i=1}^k x_{[i]}\leqslant\sum_{i=1}^k y_{[i]},\quad k=1,2,\ldots,n,$$

we say that x is *weakly majorized* by y, denoted by $x\prec_w y$. We say that x is *majorized* by y, denoted by $x\prec y$, if further the two sums are equal at $k=n$. Moreover, if $x_i,y_i>0$ $(i=1,\ldots,n)$ and

$$\prod_{i=1}^k x_{[i]}\leqslant\prod_{i=1}^k y_{[i]},\quad k=1,2,\ldots,n,$$

we say that x is *weakly log majorized* by y, denoted by $x\prec_{wlog} y$. If $x\prec_{wlog} y$ and $\prod_{i=1}^n x_i=\prod_{i=1}^n y_i$, then we say that x is *log majorized* by y, denoted by $x\prec_{log} y$. It is well known that weak log-majorization implies weak majorization. Note that the Fan dominance theorem [5, p. 93] asserts that $\|A\|\leqslant\|B\|$ if and only if $s(A)\prec_w s(B)$.

In [4], Audenaert obtained that if $A,B\in\mathbf{M}_n$ and $t\in[0,1]$, then

$$\|AB^*\|^2\leqslant\|tA^*A+(1-t)B^*B\|\|(1-t)A^*A+tB^*B\|. \tag{1}$$

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The inequality (1) interpolates between the arithmetic-geometric mean inequality ($t = 1/2$) and the Cauchy-Schwarz inequality ($t = 0$ or $t = 1$). Lin [8] presented an alternative proof of (1).

The author also proved a generalization of the inequality (1) in [4] which can be stated as follows: let $A, B \in \mathbf{M}_n$ and let $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, $r \geqslant 0$ and $t \in [0,1]$. Then

$$\||AB^*|^r\| \leqslant \|(tA^*A + (1-t)B^*B)^{\frac{rp}{2}}\|^{\frac{1}{p}} \|((1-t)A^*A + tB^*B)^{\frac{rq}{2}}\|^{\frac{1}{q}}. \tag{2}$$

When $p = q = 2$ and $t = 0$ or $t = 1$ in (2), it becomes matrix version of the Cauchy-Schwarz inequality.

Zou and Jiang [11] gave another generalization of (1): If $A, X, B \in \mathbf{M}_n$ and $t \in [0,1]$, then

$$\|AXB^*\|^2 \leqslant \|tA^*AX + (1-t)XB^*B\| \|(1-t)A^*AX + tXB^*B\|. \tag{3}$$

Al-Khlyleh and Kittaneh [2] proved a further interpolating inequality for unitarily invariant norms: If $A, X, B \in \mathbf{M}_n$ such that X is positive semidefinite and $t \in [0,1]$, then

$$\|AXB^*\|^2 \leqslant \|X\|_\infty^2 \|tA^*A + (1-t)B^*B\| \|(1-t)A^*A + tB^*B\|. \tag{4}$$

For more results on interpolation between the arithmetic-geometric mean inequality and the Cauchy- Schwarz inequality for matrices, see [3, 9, 10].

Al-Khlyleh and Alrimawi [1] provided an alternative proof of (2). However, there is a gap in their proof. Authors of [1, (21)] get

$$\left\| \left((\alpha AA^* + (1-\alpha)BB^*)^{\frac{rp}{2}} \right)^{\frac{2}{p}} \right\| \leqslant \left\| \left((\alpha AA^* + (1-\alpha)BB^*)^{\frac{rp}{2}} \right)^2 \right\|^{\frac{1}{p}}, \tag{5}$$

for all $A, B \in \mathbf{M}_n$, $r \geqslant 0$, $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, and for all $\alpha \in [0,1]$. However, such an inequality does not hold. Actually, by the property of operator monotone function $f(t) = t^{\frac{1}{p}}$ on $[0, \infty)$, we have $\|f(|A|)\| \geqslant f(\|A\|)$ for any unitarily invariant norm and $A \in \mathbf{M}_n$. See [6, Theorem 5]. So, the inequality (5) is reversed. Actually, inequality (5) is true for the operator norm $\|\cdot\|_\infty$. More specifically, the equality holds in (5) for the operator norm $\|\cdot\|_\infty$. Thus, the alternative proof (2) given by Al-Khlyleh and Alrimawi [1] holds only for the operator norm $\|\cdot\|_\infty$, but not for the general class of unitarily invariant norms, and the proof can be rewritten with a slight modification of replacing every unitarily invariant norm $\|\cdot\|$ by the operator norm $\|\cdot\|_\infty$, and replacing every s_j by s_1.

In this paper, we present the refinements of inequalities (1), (3) and (4) by the property of log-convex function depending on the trace norm $\|\cdot\|_1$.

2. Main results

Before presenting the main theorems, we list some preliminary results for later convenience. A refinement of the trace version of Young's inequality can be stated as follows.

LEMMA 2.1. (See [7, Theorem 3.2]) *Let $A,B \in \mathbf{M}_n$ be positive semidefinite. If $0 \leqslant v \leqslant 1$, then*

$$\mathrm{tr}|A^v B^{1-v}| + r_0(\sqrt{\mathrm{tr}A} - \sqrt{\mathrm{tr}B})^2 \leqslant \mathrm{tr}(vA + (1-v)B), \tag{6}$$

where $r_0 = \min\{v, 1-v\}$.

REMARK 1. (6) implies that

$$\|A^v B^{1-v}\|_1 + r_0(\sqrt{\|A\|_1} - \sqrt{\|B\|_1})^2 \leqslant \|vA + (1-v)B\|_1. \tag{7}$$

DEFINITION 1. Let $M_1, M_2 \in \mathbf{M}_n$ such that M_1 and M_2 are positive semidefinite. For trace norm $\|.\|_1$, we define the mapping $v \longmapsto F[M_1,M_2](v)$ as follows,

$$F[M_1,M_2](v) = \|M_1^v M_2^{1-v}\|_1 \|M_1^{1-v} M_2^v\|_1, \quad v \in [0,1].$$

PROPOSITION 2.2. *Let $M_1, M_2 \in \mathbf{M}_n$ such that M_1 and M_2 are positive semidefinite. Then the function $F[M_1,M_2](\cdot)$ has the following properties:*

(1) *$F[M_1,M_2](\cdot)$ is log-convex (and hence convex) on $[0,1]$,*

(2) *$F[M_1,M_2](\cdot)$ is decreasing on $[0,1/2]$, increasing on $[1/2,1]$ and attains its minimum at $1/2$,*

(3) *for $v_1, v_2, \alpha \in [0,1]$*

$$\begin{aligned} F[M_1,M_2](\alpha v_1 + (1-\alpha)v_2) &\leqslant (F[M_1,M_2](v_1))^{\alpha} (F[M_1,M_2](v_2))^{1-\alpha} \\ &\leqslant \alpha F[M_1,M_2](v_1) + (1-\alpha)F[M_1,M_2](v_2). \end{aligned}$$

To refine the inequality (1) for the trace norm, we first give the following theorem.

THEOREM 2.3. *Let $M_1, M_2 \in \mathbf{M}_n$ such that M_1 and M_2 are positive semidefinite. Then for $v \in [0,1]$,*

$$\begin{aligned} \|M_1^{1/2} M_2^{1/2}\|_1^2 &= F[M_1,M_2](1/2) \\ &\leqslant F[M_1,M_2](v) \\ &\leqslant (\|vM_1 + (1-v)M_2\|_1 - r_0(\|M_1\|_1^{\frac{1}{2}} - \|M_2\|_1^{\frac{1}{2}})^2) \\ &\quad \times (\|(1-v)M_1 + vM_2\|_1 - r_0(\|M_1\|_1^{\frac{1}{2}} - \|M_2\|_1^{\frac{1}{2}})^2), \end{aligned}$$

where $r_0 = \min\{v, 1-v\}$.

Proof. By Proposition 2.2, the function $F[M_1,M_2](\cdot)$ is decreasing on $[0,1/2]$, increasing on $[1/2,1]$ and attains its minimum at $1/2$. Therefore,

$$\|M_1^{1/2}M_2^{1/2}\|_1^2 = F[M_1,M_2](1/2) \leqslant F[M_1,M_2](v).$$

Now we prove the second inequality. By inequality (7), we have

$$\|M_1^v M_2^{1-v}\|_1 \leqslant \|vM_1+(1-v)M_2\|_1 - r_0(\|M_1\|_1^{\frac{1}{2}} - \|M_2\|_1^{\frac{1}{2}})^2,$$

and

$$\|M_1^{1-v} M_2^{v}\|_1 \leqslant \|(1-v)M_1+vM_2\|_1 - r_0(\|M_1\|_1^{\frac{1}{2}} - \|M_2\|_1^{\frac{1}{2}})^2.$$

Thus,

$$\begin{aligned} F[M_1,M_2](v) &= \|M_1^v M_2^{1-v}\|_1 \|M_1^{1-v} M_2^{v}\|_1 \\ &\leqslant (\|vM_1+(1-v)M_2\|_1 - r_0(\|M_1\|_1^{\frac{1}{2}} - \|M_2\|_1^{\frac{1}{2}})^2) \\ &\quad \times (\|(1-v)M_1+vM_2\|_1 - r_0(\|M_1\|_1^{\frac{1}{2}} - \|M_2\|_1^{\frac{1}{2}})^2). \quad \square \end{aligned}$$

Using Theorem 2.3, we can get the following improvement of the Audenaert inequality (1) for the trace norm.

THEOREM 2.4. *Let $A,B \in \mathbf{M}_n$. Then for $v \in [0,1]$,*

$$\begin{aligned} \|AB^*\|_1^2 &= F[A^*A,B^*B](1/2) \\ &\leqslant F[A^*A,B^*B](v) \\ &\leqslant [\|vA^*A+(1-v)B^*B\|_1 - r_0(\|A^*A\|_1^{\frac{1}{2}} - \|B^*B\|_1^{\frac{1}{2}})^2] \\ &\quad \times [\|(1-v)A^*A+vB^*B\|_1 - r_0(\|A^*A\|_1^{\frac{1}{2}} - \|B^*B\|_1^{\frac{1}{2}})^2], \end{aligned}$$

where $r_0 = \min\{v,1-v\}$.

Proof. Let $A = U|A|, B = V|B|$ be the polar decompositions of A and B with unitary matrices U and V. Thus,

$$\begin{aligned} \|AB^*\|_1^2 &= \|U|A||B|V^*\|_1^2 \\ &= \||A||B|\|_1^2 \\ &= \|(A^*A)^{1/2}(B^*B)^{1/2}\|_1^2 \\ &= F[A^*A,B^*B](1/2). \end{aligned}$$

This proves the first equality. To prove the remaining inequalities, suppose $M_1 = A^*A$, $M_2 = B^*B$. By Theorem 2.3, we have

$$\begin{aligned} F[A^*A,B^*B](1/2) &\leqslant F[A^*A,B^*B](v) \\ &\leqslant (\|vA^*A+(1-v)B^*B\|_1 - r_0(\|A^*A\|_1^{\frac{1}{2}} - \|B^*B\|_1^{\frac{1}{2}})^2) \\ &\quad \times (\|(1-v)A^*A+vB^*B\|_1 - r_0(\|A^*A\|_1^{\frac{1}{2}} - \|B^*B\|_1^{\frac{1}{2}})^2). \quad \square \end{aligned}$$

Next we will give the refinements of inequalities (3) and (4) for the trace norm $\|\cdot\|_1$. The following lemma will be important in our proofs.

LEMMA 2.5. *Let $M_1, M_2, X \in \mathbf{M}_n$ be positive semidefinite and $v \in [0,1]$. Then*

$$\begin{aligned} &F[X^{\frac{1}{2}}M_1X^{\frac{1}{2}}, X^{\frac{1}{2}}M_2X^{\frac{1}{2}}](v) \\ &\quad\leqslant (\|X\|_\infty\|vM_1+(1-v)M_2\|_1 - r_0(\|X^{\frac{1}{2}}M_1X^{\frac{1}{2}}\|_1^{\frac{1}{2}} - \|X^{\frac{1}{2}}M_2X^{\frac{1}{2}}\|_1^{\frac{1}{2}})^2) \\ &\qquad\times (\|X\|_\infty\|(1-v)M_1+vM_2\|_1 - r_0(\|X^{\frac{1}{2}}M_1X^{\frac{1}{2}}\|_1^{\frac{1}{2}} - \|X^{\frac{1}{2}}M_2X^{\frac{1}{2}}\|_1^{\frac{1}{2}})^2), \end{aligned} \tag{8}$$

and

$$\begin{aligned} &F[X^{\frac{1}{2}}M_1X^{\frac{1}{2}}, X^{\frac{1}{2}}M_2X^{\frac{1}{2}}](v) \\ &\quad\leqslant (\|vM_1X+(1-v)XM_2\|_1 - r_0(\|X^{\frac{1}{2}}M_1X^{\frac{1}{2}}\|_1^{\frac{1}{2}} - \|X^{\frac{1}{2}}M_2X^{\frac{1}{2}}\|_1^{\frac{1}{2}})^2) \\ &\qquad\times (\|(1-v)M_1X+vXM_2\|_1 - r_0(\|X^{\frac{1}{2}}M_1X^{\frac{1}{2}}\|_1^{\frac{1}{2}} - \|X^{\frac{1}{2}}M_2X^{\frac{1}{2}}\|_1^{\frac{1}{2}})^2), \end{aligned} \tag{9}$$

where $r_0 = \min\{v, 1-v\}$.

Proof. Since $X^{\frac{1}{2}}M_1X^{\frac{1}{2}}, X^{\frac{1}{2}}M_2X^{\frac{1}{2}}$ are positive semidefinite, Theorem 2.3 implies that

$$\begin{aligned} &F[X^{\frac{1}{2}}M_1X^{\frac{1}{2}}, X^{\frac{1}{2}}M_2X^{\frac{1}{2}}](v) \\ &\quad\leqslant (\|vX^{\frac{1}{2}}M_1X^{\frac{1}{2}}+(1-v)X^{\frac{1}{2}}M_2X^{\frac{1}{2}}\|_1 - r_0(\|X^{\frac{1}{2}}M_1X^{\frac{1}{2}}\|_1^{\frac{1}{2}} - \|X^{\frac{1}{2}}M_2X^{\frac{1}{2}}\|_1^{\frac{1}{2}})^2) \\ &\qquad\times (\|(1-v)X^{\frac{1}{2}}M_1X^{\frac{1}{2}}+vX^{\frac{1}{2}}M_2X^{\frac{1}{2}}\|_1 - r_0(\|X^{\frac{1}{2}}M_1X^{\frac{1}{2}}\|_1^{\frac{1}{2}} - \|X^{\frac{1}{2}}M_2X^{\frac{1}{2}}\|_1^{\frac{1}{2}})^2) \\ &\quad= (\|X^{\frac{1}{2}}(vM_1+(1-v)M_2)X^{\frac{1}{2}}\|_1 - r_0(\|X^{\frac{1}{2}}M_1X^{\frac{1}{2}}\|_1^{\frac{1}{2}} - \|X^{\frac{1}{2}}M_2X^{\frac{1}{2}}\|_1^{\frac{1}{2}})^2) \\ &\qquad\times (\|X^{\frac{1}{2}}((1-v)M_1+vM_2)X^{\frac{1}{2}}\|_1 - r_0(\|X^{\frac{1}{2}}M_1X^{\frac{1}{2}}\|_1^{\frac{1}{2}} - \|X^{\frac{1}{2}}M_2X^{\frac{1}{2}}\|_1^{\frac{1}{2}})^2) \\ &\quad\leqslant (\|X\|_\infty\|vM_1+(1-v)M_2\|_1 - r_0(\|X^{\frac{1}{2}}M_1X^{\frac{1}{2}}\|_1^{\frac{1}{2}} - \|X^{\frac{1}{2}}M_2X^{\frac{1}{2}}\|_1^{\frac{1}{2}})^2) \\ &\qquad\times (\|X\|_\infty\|(1-v)M_1+vM_2\|_1 - r_0(\|X^{\frac{1}{2}}M_1X^{\frac{1}{2}}\|_1^{\frac{1}{2}} - \|X^{\frac{1}{2}}M_2X^{\frac{1}{2}}\|_1^{\frac{1}{2}})^2). \end{aligned} \tag{10}$$

In the last inequality, we used the fact that if $X, Y, Z \in \mathbf{M}_n$, then $\|XYZ\| \leqslant \|X\|_\infty\|Z\|_\infty\|Y\|$. This proves the inequality (8).

To prove the inequality (9), we recall that if $X, Y \in \mathbf{M}_n$ such that XY is Hermitian, then

$$\|XY\| \leqslant \|\Re(YX)\| \leqslant \|YX\|,$$

where $\Re A = \frac{1}{2}(A + A^*)$, see [5, Proposition IX.1.2].

So,

$$\begin{aligned}\|X^{\frac{1}{2}}(vM_1+(1-v)M_2)X^{\frac{1}{2}}\|_1 &\leqslant \|\Re((vM_1+(1-v)M_2)X)\|_1\\ &= \|\Re(vM_1X+(1-v)M_2X)\|_1\\ &= \|\Re(vM_1X+(1-v)XM_2)\|_1\\ &\leqslant \|vM_1X+(1-v)XM_2\|_1.\end{aligned} \tag{11}$$

Similarly,

$$\|X^{\frac{1}{2}}((1-v)M_1+vM_2)X^{\frac{1}{2}}\|_1 \leqslant \|(1-v)M_1X+vXM_2\|_1. \tag{12}$$

Therefore, the inequalities (10), (11) and (12) yield (9). □

The following theorem represents an improvement of the inequality (4) for the trace norm $\|\cdot\|_1$.

THEOREM 2.6. *Let $A,X,B\in\mathbf{M}_n$ such that X is positive semidefinite. Then there exists a log-convex function $f(v)$ on the interval $[0,1]$ such that*

$$\begin{aligned}\|AXB^*\|_1^2 &= f\left(\frac{1}{2}\right)\\ &\leqslant f(v)\\ &\leqslant (\|X\|_\infty\|vA^*A+(1-v)B^*B\|_1 - r_0(\||AX^{\frac{1}{2}}|^2\|_1^{\frac{1}{2}} - \||BX^{\frac{1}{2}}|^2\|_1^{\frac{1}{2}})^2)\\ &\quad\times(\|X\|_\infty\|(1-v)A^*A+vB^*B\|_1 - r_0(\||AX^{\frac{1}{2}}|^2\|_1^{\frac{1}{2}} - \||BX^{\frac{1}{2}}|^2\|_1^{\frac{1}{2}})^2),\end{aligned}$$

where $r_0=\min\{v,1-v\}$.

Proof. Let $f(v)=F[|AX^{\frac{1}{2}}|^2,|BX^{\frac{1}{2}}|^2](v)$. Then by Theorem 2.4, we have

$$\begin{aligned}f\left(\frac{1}{2}\right) &= F[|AX^{\frac{1}{2}}|^2,|BX^{\frac{1}{2}}|^2]\left(\frac{1}{2}\right)\\ &= F[(AX^{\frac{1}{2}})^*(AX^{\frac{1}{2}}),(BX^{\frac{1}{2}})^*(BX^{\frac{1}{2}})]\left(\frac{1}{2}\right)\\ &= \|(AX^{\frac{1}{2}})(BX^{\frac{1}{2}})^*\|_1^2\\ &= \|AXB^*\|_1^2.\end{aligned}$$

By Proposition 2.2, the function $f(v)$ is log-convex on [0,1] and attains its minimum at $\frac{1}{2}$, so $f(\frac{1}{2})\leqslant f(v)$. The last inequality can be derived from the inequality (8) as

follows.

$$\begin{aligned}
f(v) &= F[|AX^{\frac{1}{2}}|^2, |BX^{\frac{1}{2}}|^2](v) \\
&= F[X^{\frac{1}{2}}A^*AX^{\frac{1}{2}}, X^{\frac{1}{2}}B^*BX^{\frac{1}{2}}](v) \\
&\leqslant (\|X\|_\infty \|vA^*A + (1-v)B^*B\|_1 - r_0(\|X^{\frac{1}{2}}A^*AX^{\frac{1}{2}}\|_1^{\frac{1}{2}} - \|X^{\frac{1}{2}}B^*BX^{\frac{1}{2}}\|_1^{\frac{1}{2}})^2) \\
&\quad \times (\|X\|_\infty \|(1-v)A^*A + vB^*B\|_1 - r_0(\|X^{\frac{1}{2}}A^*AX^{\frac{1}{2}}\|_1^{\frac{1}{2}} - \|X^{\frac{1}{2}}B^*BX^{\frac{1}{2}}\|_1^{\frac{1}{2}})^2) \\
&= (\|X\|_\infty \|vA^*A + (1-v)B^*B\|_1 - r_0(\||AX^{\frac{1}{2}}|^2\|_1^{\frac{1}{2}} - \||BX^{\frac{1}{2}}|^2\|_1^{\frac{1}{2}})^2) \\
&\quad \times (\|X\|_\infty \|(1-v)A^*A + vB^*B\|_1 - r_0(\||AX^{\frac{1}{2}}|^2\|_1^{\frac{1}{2}} - \||BX^{\frac{1}{2}}|^2\|_1^{\frac{1}{2}})^2). \quad \square
\end{aligned}$$

Using the inequality (9), we refine the inequality (3) for the trace norm $\|\cdot\|_1$ as follows.

THEOREM 2.7. *Let $A, X, B \in \mathbf{M}_n$. Then there exists a unitary matrix U and a log-convex function $g(v)$ on the interval $[0,1]$ such that*

$$\begin{aligned}
\|AXB^*\|_1^2 &= g\left(\frac{1}{2}\right) \\
&\leqslant g(v) \\
&\leqslant [\||vA^*AX + (1-v)XB^*B\|_1 - r_0(\||AU|X|^{\frac{1}{2}}|^2\|_1^{\frac{1}{2}} - \||B|X|^{\frac{1}{2}}|^2\|_1^{\frac{1}{2}})^2] \\
&\quad \times [\||(1-v)A^*AX + vXB^*B\|_1 - r_0(\||AU|X|^{\frac{1}{2}}|^2\|_1^{\frac{1}{2}} - \||B|X|^{\frac{1}{2}}|^2\|_1^{\frac{1}{2}})^2],
\end{aligned}$$

where $r_0 = \min\{v, 1-v\}$.

Proof. Let $X = U|X|$ be the polar decomposition of X with a unitary matrix U. Let $A_1 = AU|X|^{1/2}, B_1 = B|X|^{1/2}$. Let $g(v) = F[A_1^*A_1, B_1^*B_1](v)$. Then by Theorem 2.4, we have

$$\begin{aligned}
g\left(\frac{1}{2}\right) &= F[A_1^*A_1, B_1^*B_1]\left(\frac{1}{2}\right) \\
&= \|A_1B_1^*\|_1^2 \\
&= \|(AU|X|^{1/2})(B|X|^{1/2})^*\|_1^2 \\
&= \|A(U|X|)B^*\|_1^2 \\
&= \|AXB^*\|_1^2.
\end{aligned}$$

By Proposition 2.2, the function $g(v)$ is log-convex on [0,1] and attains its minimum at $\frac{1}{2}$, so $g(\frac{1}{2}) \leqslant g(v)$. The last inequality can be derived from the inequality (9) as

follows.

$$
\begin{aligned}
g(v) &= F[A_1^*A_1, B_1^*B_1](v)\\
&= F[|X|^{\frac{1}{2}}(AU)^*(AU)|X|^{\frac{1}{2}}, |X|^{\frac{1}{2}}(B^*B)|X|^{\frac{1}{2}}](v)\\
&\leqslant [\||v(AU)^*(AU)|X| + (1-v)|X|(B^*B)\|_1 - r_0(\||X|^{\frac{1}{2}}(AU)^*(AU)|X|^{\frac{1}{2}}\|_1^{\frac{1}{2}}\\
&\quad -\||X|^{\frac{1}{2}}(B^*B)|X|^{\frac{1}{2}}\|_1^{\frac{1}{2}})^2]\\
&\quad \times[\|(1-v)(AU)^*(AU)|X| + v|X|(B^*B)\|_1 - r_0(\||X|^{\frac{1}{2}}(AU)^*(AU)|X|^{\frac{1}{2}}\|_1^{\frac{1}{2}}\\
&\quad -\||X|^{\frac{1}{2}}(B^*B)|X|^{\frac{1}{2}}\|_1^{\frac{1}{2}})^2]\\
&= [\|U^*(vA^*A(U|X|) + (1-v)(U|X|)(B^*B))\|_1 - r_0(\||AU|X|^{\frac{1}{2}}|^2\|_1^{\frac{1}{2}}\\
&\quad -\||B|X|^{\frac{1}{2}}|^2\|_1^{\frac{1}{2}})^2]\\
&\quad \times[\|U^*((1-v)A^*A(U|X|) + v(U|X|)(B^*B))\|_1 - r_0(\||AU|X|^{\frac{1}{2}}|^2\|_1^{\frac{1}{2}}\\
&\quad -\||B|X|^{\frac{1}{2}}|^2\|_1^{\frac{1}{2}})^2]\\
&= [\|vA^*AX + (1-v)XB^*B\|_1 - r_0(\||AU|X|^{\frac{1}{2}}|^2\|_1^{\frac{1}{2}} - \||B|X|^{\frac{1}{2}}|^2\|_1^{\frac{1}{2}})^2]\\
&\quad \times[\|(1-v)A^*AX + vXB^*B\|_1 - r_0(\||AU|X|^{\frac{1}{2}}|^2\|_1^{\frac{1}{2}} - \||B|X|^{\frac{1}{2}}|^2\|_1^{\frac{1}{2}})^2]. \quad \square
\end{aligned}
$$

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REFERENCES

[1] M. AL-KHLYLEH, F. ALRIMAWI, *Further interpolation inequalities related to arithmetic-geometric mean, Cauchy-Schwarz and Höler inequalities for unitarily invariant norm*, Math. Inequal. Appl. **23** (2020) 1135–1143.

[2] M. AL-KHLYLEH, F. KITTANEH, *Inequalities related to a recent result of Audenaert*, Linear and Multilinear Algebra. **65** (2017) 922–929.

[3] A. AL-NATOOR, O. HIRZALLAH, F. KITTANEH, *Interpolating inequalities for functions of positive semidefinite matrices*, Banach J. Math. Anal. **12** (2018) 955–969.

[4] K. M. R. AUDENAERT, *Interpolating between the arithmetic-geometric mean and Cauchy-Schwarz matrix norm inequalities*, Oper Matrices. **9** (2015) 475–479.
[5] R. BHATIA, *Matrix analysis*, Springer-Verlag, New York, 1997.
[6] F. HIAI, X. ZHAN, *Inequalities involving unitarily invariant norms and operator monotone functions*, Linear Algebra Appl. **341** (2002) 151–169.
[7] F. KITTANEH, Y. MANASRAH, *Improved Young and Heinz inequalities for matrices*, J. Math. Anal. Appl. **361** (2010) 262–269.
[8] M. LIN, *Remarks on two recent results of Audenaert*, Linear Algebra Appl. **489** (2016) 24–29.
[9] J. YANG, L. LU, *New proofs on two recent inequalities for unitarily invariant norms*, J. Inequal. Appl., (2020) 2020:133, doi:10.1186/s13660-020-02402-z.
[10] J. YANG, S. MA, *Several unitarily invariant norm inequalities for matrices*, Ann. Funct. Anal., (2025) 16:15, doi:10.1007/s43034-025-00407-7.
[11] L. ZOU, Y. JIANG, *A note on interpolating between the arithmetic-geometric mean and Cauchy-Schwarz matrix norm inequalities*, J. Math. Inequal. **10** (2016) 1119–1122.

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