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MAPS PRESERVING THE LOCAL SPECTRUM OF GENERALIZED PRODUCT OF OPERATORS

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Abstract. Let $\mathscr{X}$ and $\mathscr{Y}$ be two infinite dimensional Banach spaces and $\mathscr{B}(\mathscr{X})$ (resp. $\mathscr{B}(\mathscr{Y})$) be the algebra of bounded linear operators on $\mathscr{X}$ (resp. $\mathscr{Y}$). For $T \in \mathscr{B}(\mathscr{X})$ and $x \in \mathscr{X}$, $\sigma_T(x)$ denotes the local spectrum of T at x. Fix an integer $k \geqslant 2$, and let $A_1 * A_2 * \ldots * A_k$ stand for a generalized product of any k operators $A_1, A_2, \ldots, A_k \in \mathscr{B}(\mathscr{X})$. Given two nonzero vectors $x_0 \in \mathscr{X}$ and $y_0 \in \mathscr{Y}$, in this paper, we characterize all surjective maps $\varphi : \mathscr{B}(\mathscr{X}) \to \mathscr{B}(\mathscr{Y})$ which satisfy

$$\sigma_{A_1 * A_2 * \ldots * A_k}(x_0) = \sigma_{\varphi(A_1) * \varphi(A_2) * \ldots * \varphi(A_k)}(y_0)$$

for any $A_1, A_2, \ldots, A_k \in \mathscr{B}(\mathscr{X})$.

1. Introduction

Throughout this paper, $\mathscr{X}$ and $\mathscr{Y}$ denote two infinite dimensional Banach spaces. Let $\mathscr{B}(\mathscr{X})$ and $\mathscr{B}(\mathscr{Y})$ be the algebras of all bounded linear operators on $\mathscr{X}$ and $\mathscr{Y}$, respectively. We use $\mathscr{X}^*$ to denote the dual space of $\mathscr{X}$. For $T \in \mathscr{B}(\mathscr{X})$, T^* stands for the adjoint operator of T. $\mathscr{F}(\mathscr{X})$ denotes the set of all finite rank operators in $\mathscr{B}(\mathscr{X})$. Given a positive integer n, $\mathscr{F}_n(\mathscr{X})$ denotes the set of all operators of rank at most n. For nonzero $f \in \mathscr{X}^*$ and nonzero $x \in \mathscr{X}$, $x \otimes f$ stands for the rank one operator on $\mathscr{X}$, which is defined by $(x \otimes f)z = f(z)x$ for any $z \in \mathscr{X}$. Note that every nonzero operator in $\mathscr{F}_1(\mathscr{X})$ can be written in this form. Especially, the trivial operator 0 in $\mathscr{F}_1(\mathscr{X})$ can be written in the form $x \otimes f$ if and only if $x = 0$ or $f = 0$. Let $\mathscr{N}_1(\mathscr{X})$ denote the set of all nilpotent operators in $\mathscr{F}_1(\mathscr{X})$. It is obvious that $x \otimes f \in \mathscr{N}_1(\mathscr{X})$ if and only if $f(x) = 0$. For $T \in \mathscr{B}(\mathscr{X})$ and $x \in \mathscr{X}$, the local resolvent set of T at x, denoted by $\rho_T(x)$, is the union of all open subsets U of $\mathbb{C}$ for which there is an analytic function $\phi : U \to \mathscr{X}$ such that $(T - \lambda I)\phi(\lambda) = x$ for any $\lambda \in U$. The local spectrum of T at x is defined by $\sigma_T(x) := \mathbb{C} \setminus \rho_T(x)$, and is obviously a closed subset of $\sigma(T)$. The properties of local spectrum have close connections with single valued extension property. Recall that $T \in \mathscr{B}(\mathscr{X})$ is said to have single valued extension property, SVEP for short, at $\lambda_0 \in \mathbb{C}$, if for every open set $U \subseteq \mathbb{C}$ centered at λ_0, the only analytic solution $f(\cdot) : U \to \mathscr{X}$ of the equation $(T - \lambda I)f(\lambda) = 0$ for all $\lambda \in U$

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is the zero function on U. If T possesses SVEP at every $\lambda_0 \in \mathbb{C}$, then we call T has SVEP. It is obvious that every operator whose interior of point spectrum is empty enjoys this property. The interested reader is referred to [3] and [17] for more details.

Preserver problems aim to characterize those linear or nonlinear maps on operator algebras preserving certain properties, subsets, or relations. The study of linear or nonlinear preserver problems has a long history and has established many remarkable results in past decades. The spectrum of an operator is a very fundamental and key concept in operator theory. Many authors have studied linear or additive maps preserving the spectrum as well as certain parts of the spectrum. For example, the authors in [15] showed that additive maps on standard operator algebras preserving parts of spectrum is either an isomorphism or an anti-isomorphism. Later, Bourhim and Ransford firstly put the local spectrum as the invariants of an automorphism or an anti-isomorphism on the algebra of all bounded linear operators on a Banach space (cf. [12]). After that, Costara in [13] described surjective linear maps on $\mathscr{B}(\mathscr{X})$ which preserve operators of local spectral radius zero at points of $\mathscr{X}$. Then Bourhim and Mashreghi extended this result to the nonlinear setting (cf. [6]). For more details on the preserver problems of local spectrum, see the survey article [9]. What's more, in [2, 4, 5, 10], maps on the algebra $M_n(\mathbb{C})$ of all $n \times n$ complex matrices preserving the local spectrum of the product, triple product, Jordan product and quadratic product of matrices have been characterized. Moreover, in [7, 8], the authors obtained the forms of all maps preserving the local spectrum at some fixed vectors of products and triple products of operators.

Recently, there have been a lot of results describing the forms of mappings that preserve the generalized product of operators; see for example [1, 14, 16, 19] and the references therein. In this paper, we propose to characterize the form of maps that preserve the local spectrum at fixed point of generalized product of operators, which generalizes the results in [7, 8].

For an integer $k \geqslant 2$, let $(i_1, i_2, \ldots, i_m)$ be a finite sequence with terms chosen from $\{1, 2, \ldots, k\}$ such that $\{i_1, i_2, \ldots, i_m\} = \{1, 2, \ldots, k\}$ and at least one of the terms in $(i_1, i_2, \ldots, i_m)$ appears exactly once. The generalized product of k operators $A_1, A_2, \ldots, A_k \in \mathscr{B}(\mathscr{X})$ is defined by

$$A_1 * A_2 * \ldots * A_k = A_{i_1} A_{i_2} \ldots A_{i_m},$$

m is called the width of the generalized product. Evidently, the generalized product includes the usual product and the triple product.

The following theorem is the main result of this paper.

THEOREM 1. *Let $x_0 \in \mathscr{X}$ and $y_0 \in \mathscr{Y}$ be two nonzero vectors. Let $\varphi : \mathscr{B}(\mathscr{X}) \to \mathscr{B}(\mathscr{Y})$ be a surjective map. Then*

$$\sigma_{\varphi(T_1)*\varphi(T_2)*\ldots*\varphi(T_k)}(y_0) = \sigma_{T_1*T_2*\ldots*T_k}(x_0)$$

for any $T_1, T_2, \ldots, T_k \in \mathscr{B}(\mathscr{X})$ if and only if there exist a bijective bounded linear mapping A from $\mathscr{X}$ into $\mathscr{Y}$ and some $\lambda \in \mathbb{C}$ with $\lambda^m = 1$ such that $Ax_0 = y_0$ and $\varphi(T) = \lambda ATA^{-1}$ for any $T \in \mathscr{B}(\mathscr{X})$.

Note that the only restriction on the map φ is the surjectivity; no linearity or additivity or continuity is assumed. In fact, the surjectivity condition on φ could be relaxed if one assumes that the range of φ contains the ideal of finite rank operators of $\mathscr{B}(\mathscr{Y})$. We also would like to point out that if $\mathscr{X}$ and $\mathscr{Y}$ are isomorphic Banach spaces, then the statements of the above result can be reduced to the case when $\mathscr{X}=\mathscr{Y}$ and $x_0=y_0$. But the fact that $\mathscr{X}$ and $\mathscr{Y}$ are isomorphic is a part of the conclusion of the above theorem but not of its hypothesis.

Let $A, B\in\mathscr{B}(\mathscr{X})$. Set $A_{i_p}=B$ and $A_{i_j}=A$ for $j\neq p$ where i_p appears only once in $(i_1, i_2, \ldots, i_m)$. Then $A_1 * A_2 * \ldots * A_m = A^r B A^s$ for some nonnegative integers r and s with $r+s=m-1$. This shows that the condition in Theorem 1 is more stronger. So Theorem 1 is a consequence of the following one.

THEOREM 2. *Let $x_0\in\mathscr{X}$ and $y_0\in\mathscr{Y}$ be two nonzero vectors and let r and s be two nonnegative integers with $r+s\geqslant 1$. Let $\varphi:\mathscr{B}(\mathscr{X})\to\mathscr{B}(\mathscr{Y})$ be a surjective map. Then*

$$\sigma_{\varphi(T)^r\varphi(S)\varphi(T)^s}(y_0)=\sigma_{T^rST^s}(x_0)$$

for any $T, S\in\mathscr{B}(\mathscr{X})$ if and only if there exist a bijective bounded linear mapping A from $\mathscr{X}$ into $\mathscr{Y}$ and some $\lambda\in\mathbb{C}$ with $\lambda^{r+s+1}=1$ such that $Ax_0=y_0$ and $\varphi(T)=\lambda ATA^{-1}$ for any $T\in\mathscr{B}(\mathscr{X})$.

2. Preliminaries and auxiliary results

In this section, we collect some results that will be used in the proof of the main result. The following lemma summarizes some basic properties of the local spectrum and the proof can be found in [3, 17].

LEMMA 1. *Let $T\in\mathscr{B}(\mathscr{X})$, $x, y\in\mathscr{X}$ and $\alpha\in\mathbb{C}$. Then the following statements hold:*

(i) if T has SVEP, then $\sigma_T(x)\neq\emptyset$ provided $x\neq 0$;
(ii) $\sigma_T(\alpha x)=\sigma_T(x)$ if $\alpha\neq 0$; $\sigma_{\alpha T}(x)=\alpha\sigma_T(x)$;
(iii) $\sigma_T(x+y)\subseteq\sigma_T(x)\cup\sigma_T(y)$ and the equality holds if $\sigma_T(x)\cap\sigma_T(y)=\emptyset$;
(iv) If T satisfies SVEP, $x\neq 0$ and $Tx=\lambda x$ for some $\lambda\in\mathbb{C}$, then $\sigma_T(x)=\{\lambda\}$;
(v) If T satisfies SVEP and $Tx=\alpha y$, then $\sigma_T(y)\subseteq\sigma_T(x)\subseteq\sigma_T(y)\cup\{0\}$;
(vi) If $R\in\mathscr{B}(\mathscr{X})$ and $TR=RT$, then $\sigma_T(Rx)\subseteq\sigma_T(x)$;
(vii) $\sigma_{T^n}(x)=\{\sigma_T(x)\}^n$ for any $x\in\mathscr{X}$ and $n\geqslant 1$.

Let $T\in\mathscr{B}(\mathscr{X})$ and x_0 be a nonzero vector in $\mathscr{X}$. The following symbol, which was introduced in [7, 8], is useful in the sequel. That is,

$$\sigma_T^*(x_0)=\begin{cases}\{0\}, & \text{if } \sigma_T(x_0)=\{0\},\\ \sigma_T(x_0)\setminus\{0\}, & \text{if } \sigma_T(x_0)\neq\{0\}.\end{cases}$$

The following lemma is useful in the characterization of the map φ.

LEMMA 2. *Let $R \in \mathscr{F}_1(\mathscr{X})$, $x_0 \in \mathscr{X}$ be a nonzero vector and let r and s be two nonnegative integers with $r+s \geqslant 1$. Then for any $T, S \in \mathscr{B}(\mathscr{X})$ we have*

$$\sigma^*_{R^r(T+S)R^s}(x_0) = \sigma^*_{R^rTR^s}(x_0) + \sigma^*_{R^rSR^s}(x_0).$$

Proof. Let $R = x \otimes f$ with $x \in \mathscr{X}$ and $f \in \mathscr{X}^*$. Then

$$R^r(T+S)R^s = f(x)^{r+s-2} f(Tx+Sx)x \otimes f.$$

So,

$$\begin{aligned}
&\sigma^*_{R^r(T+S)R^s}(x_0) \\
&= f(x)^{r+s-2} f(Tx+Sx)\sigma^*_{x\otimes f}(x_0) \\
&= f(x)^{r+s-2} f(Tx)\sigma^*_{x\otimes f}(x_0) + f(x)^{r+s-2} f(Sx)\sigma^*_{x\otimes f}(x_0) \\
&= \sigma^*_{R^rTR^s}(x_0) + \sigma^*_{R^rSR^s}(x_0). \quad \square
\end{aligned}$$

The following theorems give the characterizations on the rank one operators and identity principles by using the local spectrum, which are useful in the proof of the main result and are of interest in their own right.

THEOREM 3. *Let x_0 be a nonzero vector of $\mathscr{X}$ and $R \in \mathscr{B}(\mathscr{X})$ and let r and s be two nonnegative integers with $r+s \geqslant 1$. Then the following statements are equivalent.*

(i) $R \in \mathscr{F}_1(\mathscr{X})$;

(ii) $\sigma^*_{T^rRT^s}(x_0)$ contains only one element for any $T \in \mathscr{B}(\mathscr{X})$.

Proof. $(i) \Rightarrow (ii)$ If $R = 0$, then $\sigma^*_{T^rRT^s}(x_0) = \{0\}$ for any $T \in \mathscr{B}(\mathscr{X})$. If $R \neq 0$, then T^rRT^s has rank one for any $T \in \mathscr{B}(\mathscr{X})$. So, $\sigma^*_{T^rRT^s}(x_0)$ is a singleton.

$(ii) \Rightarrow (i)$ Suppose on the contrary that R has rank at least two, and let us discuss two cases.

Case 1. There exist two vectors $x_1, x_2 \in \mathscr{X}$ such that $\{x_0, Rx_1, Rx_2\}$ is a linearly independent subset of $\mathscr{X}$.

Now, we could take $x \in \mathscr{X}$ such that $\{x, x_0, Rx_1, Rx_2\}$ forms a linearly independent set. Let $Rx = u$, $Rx_0 = v_1$, $Rx_1 = y_1$ and $Rx_2 = y_2$.

Assume that $s = 0$ and $r \geqslant 1$. If $r = 1$, then we use $\mathscr{M}$ to denote the closed subspace spanned by u and v_1. Put $\mathscr{X} = \mathscr{M} \oplus \mathscr{K}$. Define $T \in \mathscr{B}(\mathscr{X})$ as

$$\begin{cases} Tu = x, \\ Tv_1 = 2x_0 - x, \\ Tz = 0, \ \forall z \in \mathscr{K}. \end{cases}$$

Clearly, T is finite rank and satisfies $TRx = x$ and $TRx_0 = 2x_0 - x$. Then we have $\sigma_{TR}(x) = \{1\}$ and $\sigma_{TR}(x_0 - x) = \{2\}$. From Lemma 1 we get $\sigma^*_{TR}(x_0) = \{1, 2\}$, a contradiction.

If $r \geqslant 2$, then we take $\{v_2 : 2 \leqslant i \leqslant r\} \subseteq \mathscr{X}$. Put $\mathscr{M}$ be the closed linear subspace spanned by the vectors $\{u, x, v_i : 1 \leqslant i \leqslant r\}$ and $\mathscr{X} = \mathscr{M} \oplus \mathscr{K}$. Define $T \in \mathscr{B}(\mathscr{X})$ as

$$\begin{cases} Tu = x, \\ Tx = x, \\ Tv_i = v_{i+1}, \ \ 1 \leqslant i \leqslant r-1, \\ Tv_r = 2x_0 - x, \\ Tz = 0, \ \forall z \in \mathscr{K}. \end{cases}$$

Then T is finite rank and satisfies that $T^r Rx = x$ and $T^r Rx_0 = 2x_0 - x$. So, $T^r R(x_0 - x) = 2(x_0 - x)$. It follows that $\sigma_{T^r R}(x) = \{1\}$ and $\sigma_{T^r R}(x_0 - x) = \{2\}$. Then we get $\sigma^*_{T^r R}(x_0) = \{1, 2\}$, a contradiction again.

Under the case where $r = 0$ and $s \geqslant 1$, by using the technique in the proof of [8, Theorem 4.1], we could find some finite rank $T \in \mathscr{B}(\mathscr{X})$ such that $\sigma^*_{RT^s}(x_0)$ consists of two elements, which reaches a contradiction again.

Assume $r \geqslant 1$ and $s \geqslant 1$. Without loss of generality, we assume $r \geqslant 2$. Then we take two sets of vectors $\{h_i : 1 \leqslant i \leqslant r-1\}$ and $\{k_i : 1 \leqslant i \leqslant r-1\}$. Put $\mathscr{M}$ be the closed subspace spanned by the vectors $\{x, y_1, y_2, x_j, h_i, k_i : 0 \leqslant j \leqslant 2, 1 \leqslant i \leqslant r-1\}$ and $\mathscr{X} = \mathscr{M} \oplus \mathscr{K}$. Define $T \in \mathscr{B}(\mathscr{X})$ as

$$\begin{cases} Tx = x_2, \\ Tx_0 = x_1, \\ Tx_j = x_j, \ \ j = 1, 2, \\ Ty_1 = h_1, \\ Ty_2 = k_1, \\ Th_i = h_{i+1}, \ \ 1 \leqslant i \leqslant r-2, \\ Tk_i = k_{i+1}, \ \ 1 \leqslant i \leqslant r-2, \\ Th_{r-1} = 2x_0 - x, \\ Tk_{r-1} = x, \\ Tz = 0, \ \forall z \in \mathscr{K}. \end{cases}$$

Then T is finite rank and it is trivial to check that $T^r RT^s x = x$ and $T^r RT^s x_0 = 2x_0 - x$. So, $T^r RT^s(x_0 - x) = 2(x_0 - x)$. It follows that $\sigma_{T^r RT^s}(x) = \{1\}$ and $\sigma_{T^r RT^s}(x_0 - x) = \{2\}$. By Lemma 1 we obtain that $\sigma^*_{T^r RT^s}(x_0) = \{1, 2\}$, a contradiction.

Case 2. $\{x_0, Rx_1, Rx_2\}$ is a linearly dependent subset of $\mathscr{X}$ for any $x_1, x_2 \in \mathscr{X}$.

Now, we see that R is rank two and $x_0 \in ran(R)$. So, there are two linearly independent vectors $x_1, x_2 \in \mathscr{X}$ such that $x_0 = \alpha_1 x_1 + \alpha_2 x_2$, where $\alpha_1, \alpha_2 \in \mathbb{C}$ and not both zero. Also, we can find $f_1, f_2 \in \mathscr{X}^*$ such that $R = x_1 \otimes f_1 + x_2 \otimes f_2$. Put $Rx_1 = y_1$ and $Rx_2 = y_2$.

Assume that $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$. If $r \geqslant 1$ and $s = 0$, then we take vectors $\{z_i : 1 \leqslant i \leqslant r-1\}$ and $\{w_i : 1 \leqslant i \leqslant r-1\}$. Let $\mathscr{M}$ be the subspace spanned by $\{y_j, z_i, w_i :$

$j=1,2, 1\leqslant i\leqslant r-1\}$ and let $\mathscr{X}=\mathscr{M}\oplus\mathscr{K}$. Define $T\in\mathscr{B}(\mathscr{X})$ as

$$\begin{cases} Ty_1=z_1,\\ Ty_2=w_1,\\ Tz_i=z_{i+1},\ \ 1\leqslant i\leqslant r-2,\\ Tw_i=w_{i+1},\ \ 1\leqslant i\leqslant r-2,\\ Tz_{r-1}=x_1,\\ Tw_{r-1}=2x_2,\\ Tz=0,\ \forall z\in\mathscr{K}. \end{cases}$$

Then by calculation we get $T^rRx_1=x_1$ and $T^rRx_2=2x_2$. It follows that $\sigma^*_{T^rR}(x_0)=\{1,2\}$, a contradiction.

If $r=0$ and $s\geqslant 1$, then similar to the process above we could construct an operator $T\in\mathscr{B}(\mathscr{X})$ such that $\sigma^*_{RT^s}(x_0)=\{1,2\}$, which reaches a contradiction again.

If $r\geqslant 1$ and $s\geqslant 1$, then we take two sets of vectors $\{h_i: 1\leqslant i\leqslant r-1\}$ and $\{k_i: 1\leqslant i\leqslant r-1\}$. Put $\mathscr{M}$ be the closed subspace spanned by the vectors $\{x_j, y_j, h_i, k_i: j=1,2, 1\leqslant i\leqslant r-1\}$ and $\mathscr{X}=\mathscr{M}\oplus\mathscr{K}$. Define $T\in\mathscr{B}(\mathscr{X})$ as

$$\begin{cases} Tx_j=x_j, j=1,2,\\ Ty_1=h_1,\\ Ty_2=k_1,\\ Th_i=h_{i+1},\ \ 1\leqslant i\leqslant r-2,\\ Tk_i=k_{i+1},\ \ 1\leqslant i\leqslant r-2,\\ Th_{r-1}=x_1,\\ Tk_{r-1}=2x_2,\\ Tz=0,\ \forall z\in\mathscr{K}. \end{cases}$$

It is easy to check that $T^rRT^sx_1=x_1$ and $T^rRT^sx_2=2x_2$. So, $\sigma^*_{T^rRT^s}(x_0)=\{1,2\}$, a contradiction.

If $\alpha_1\neq 0$ and $\alpha_2=0$, then $x_0=\alpha_1(x_1-x_2)+\alpha_1x_2$ and $R=(x_1-x_2)\otimes f_1+x_2\otimes(f_1+f_2)$. Using the process above, we could find some $T\in\mathscr{B}(\mathscr{X})$ such that $\sigma^*_{T^rRT^s}(x_0)$ contains two elements, a contradiction again.

Similarly, we could obtain the result under the assumption that $\alpha_1=0$ and $\alpha_2\neq 0$. □

THEOREM 4. *Let $A,B\in\mathscr{B}(\mathscr{X})$ and $0\neq x_0\in\mathscr{X}$ and let r and s be two non-negative integers whose sum is at least 1. Then the following statements are equivalent.*

(i) $A=B$;

(ii) $\sigma_{T^rAT^s}(x_0)=\sigma_{T^rBT^s}(x_0)$, for any $T\in\mathscr{B}(\mathscr{X})$;

(iii) $\sigma_{T^rAT^s}(x_0)=\sigma_{T^rBT^s}(x_0)$, for any $T\in\mathscr{F}_1(\mathscr{X})$;

(iv) $\sigma_{T^rAT^s}(x_0)=\sigma_{T^rBT^s}(x_0)$, for any $T\in\mathscr{F}_1(\mathscr{X})\setminus\mathscr{N}_1(\mathscr{X})$;

(v) $\sigma^*_{T^rAT^s}(x_0)=\sigma^*_{T^rBT^s}(x_0)$, for any $T\in\mathscr{F}_1(\mathscr{X})\setminus\mathscr{N}_1(\mathscr{X})$.

Proof. It is obvious that $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (v)$. Next we show $(v) \Rightarrow (i)$. Take any $T \in \mathscr{F}_1(\mathscr{X}) \setminus \mathscr{N}_1(\mathscr{X})$. Then $T = x \otimes f$, where $x \in \mathscr{X}$, $f \in \mathscr{X}^*$ and $f(x) \neq 0$. Then $T^r A T^s = f(x)^{r+s-2} f(Ax) x \otimes f$ and $T^r B T^s = f(x)^{r+s-2} f(Bx) x \otimes f$. If $f(x_0) \neq 0$, then

$$\{f(x)^{r+s-2} f(Ax)\} = \sigma^*_{T^r A T^s}(x_0) = \sigma^*_{T^r B T^s}(x_0) = \{f(x)^{r+s-2} f(Bx)\}.$$

So we get $f(Ax) = f(Bx)$. If $f(x_0) = 0$, then we could take $g \in \mathscr{X}^*$ such that $g(x_0) \neq 0$, $g(x) \neq 0$ and $g(x) \neq -f(x)$. Now, $(f+g)(x_0) \neq 0$ and $(f+g)(x) \neq 0$. By the process as above, we obtain $g(Ax) = g(Bx)$ and $(f+g)(Ax) = (f+g)(Bx)$. So, $f(Ax) = f(Bx)$. This tells us that $f(Ax) = f(Bx)$, for any $x \in \mathscr{X}$, $f \in \mathscr{X}^*$ with $f(x) \neq 0$.

If $f(x) = 0$, then we could find $f_1, f_2 \in \mathscr{X}^*$ such that $f = f_1 + f_2$ and $f_1(x) \neq 0$, $f_2(x) \neq 0$. Thus, we get $f_1(Ax) = f_1(Bx)$ and $f_2(Ax) = f_2(Bx)$, and so $f(Ax) = f(Bx)$. Therefore, the equality $f(Ax) = f(Bx)$ holds for any $x \in \mathscr{X}$ and $f \in \mathscr{X}^*$. By the arbitrariness of x and f we obtain $A = B$. $\square$

3. Proof of Theorem 2

In this section, we will display the proof of Theorem 2, which is influenced by ideas from [7, 8].

Proof of Theorem 2. In the following, we will give the proof from several steps.

Step 1. φ is injective and $\varphi(0) = 0$.

Assume that $\varphi(A) = \varphi(B)$. Then for any $T \in \mathscr{B}(\mathscr{X})$, we have

$$\sigma_{T^r A T^s}(x_0) = \sigma_{\varphi(T)^r \varphi(A) \varphi(T)^s}(y_0) = \sigma_{\varphi(T)^r \varphi(B) \varphi(T)^s}(y_0) = \sigma_{T^r B T^s}(x_0).$$

From the surjectivity of φ and Theorem 4 we obtain $A = B$.

Take any $T \in \mathscr{B}(\mathscr{X})$. It is clear that

$$\sigma_{\varphi(T)^r 0 \varphi(T)^s}(y_0) = \{0\} = \sigma_{T^r 0 T^s}(x_0) = \sigma_{\varphi(T)^r \varphi(0) \varphi(T)^s}(y_0).$$

By using Theorem 4 again we get $\varphi(0) = 0$.

Now, φ is bijective and φ^{-1} possesses the same properties as φ.

Step 2. φ preserves rank one operators in both directions.

Take any rank one operator $R \in \mathscr{B}(\mathscr{X})$. Then from Theorem 3 we see that $\sigma_{T^r R T^s}(x_0)$ contains only one element for any $T \in \mathscr{B}(\mathscr{X})$. Note that $\sigma_{\varphi(T)^r \varphi(R) \varphi(T)^s}(y_0) = \sigma_{T^r R T^s}(x_0)$. Then $\sigma_{\varphi(T)^r \varphi(R) \varphi(T)^s}(y_0)$ contains only one element for any $T \in \mathscr{B}(\mathscr{X})$. It follows from the surjectivity of φ and Theorem 3 that $\varphi(R) \in \mathscr{F}_1(\mathscr{X})$. From step 1 we obtain that $\varphi(R)$ has rank one. Since φ^{-1} has the same properties as φ, it follows that φ preserves rank one operators in both directions.

Step 3. φ is linear.

Take any $A, B \in \mathscr{B}(\mathscr{X})$. For any $R \in \mathscr{F}_1(\mathscr{X})$, we have

$$\begin{aligned}
&\sigma^*_{\varphi(R)^r\varphi(A+B)\varphi(R)^s}(y_0)\\
&= \sigma^*_{R^r(A+B)R^s}(x_0)\\
&= \sigma^*_{R^rAR^s}(x_0) + \sigma^*_{R^rBR^s}(x_0)\\
&= \sigma^*_{\varphi(R)^r\varphi(A)\varphi(R)^s}(y_0) + \sigma^*_{\varphi(R)^r\varphi(B)\varphi(R)^s}(y_0)\\
&= \sigma^*_{\varphi(R)^r(\varphi(A)+\varphi(B))\varphi(R)^s}(y_0).
\end{aligned}$$

It follows from Theorem 4 that $\varphi(A+B) = \varphi(A) + \varphi(B)$.

Take any $A \in \mathscr{B}(\mathscr{X})$ and $\lambda \neq 0$. For any $T \in \mathscr{B}(\mathscr{X})$, we have

$$\begin{aligned}
&\sigma_{\varphi(T)^r\lambda\varphi(A)\varphi(T)^s}(y_0)\\
&= \lambda\sigma_{\varphi(T)^r\varphi(A)\varphi(T)^s}(y_0)\\
&= \lambda\sigma_{T^rAT^s}(x_0)\\
&= \sigma_{T^r(\lambda A)T^s}(x_0)\\
&= \sigma_{\varphi(T)^r\varphi(\lambda A)\varphi(T)^s}(y_0).
\end{aligned}$$

Using Theorem 4 again we obtain $\varphi(\lambda A) = \lambda\varphi(A)$.

If $\lambda = 0$, then $\varphi(\lambda A) = \varphi(0) = 0 = \lambda\varphi(A)$, $\forall A \in \mathscr{B}(\mathscr{X})$. Thus, φ is linear and so φ preserves $\mathscr{F}(\mathscr{X})$ in both directions.

Step 4. $\varphi(I) = \alpha I$ for some $\alpha \in \mathbb{C}$ with $\alpha^{r+s+1} = 1$.

Assume $\varphi(R) = I$ for some $R \in \mathscr{B}(\mathscr{X})$. It is easy to see that the equality

$$\{\sigma_T(x_0)\}^{r+s+1} = \sigma_{T^{r+s+1}}(x_0) = \sigma_{\varphi(T)^{r+s+1}}(y_0) = \{\sigma_{\varphi(T)}(y_0)\}^{r+s+1} \tag{1}$$

holds for any $T \in \mathscr{B}(\mathscr{X})$. To prove step 4, we firstly give the following two assertions.

Claim 1. For any $x \in \mathscr{X}$, Rx is either linearly dependent with x or x_0.

In fact, suppose that there is $x \in \mathscr{X}$ such that Rx is linearly independent with both x and x_0. Take $f_1 \in \mathscr{X}^*$ with $f_1(Rx) = 0$, $f_1(x) \neq 0$ and $f_1(x_0) \neq 0$. Put $T_1 = x \otimes f_1$. Then we have $T_1^r R T_1^s(x_0) = 0$. So,

$$\{0\} = \sigma_{T_1^r R T_1^s}(x_0) = \sigma_{\varphi(T_1)^r\varphi(R)\varphi(T_1)^s}(y_0) = \sigma_{\varphi(T_1)^{r+s}}(y_0) = \{\sigma_{\varphi(T_1)}(y_0)\}^{r+s}.$$

From (1) we obtain $\sigma_{T_1}(x_0) = \{0\}$. Note that $T_1x_0 = f_1(x_0)x$. From Lemma 1 we have $\sigma_{T_1}(x) = \{0\}$. Since $T_1x = f_1(x)x$, it follows that $\sigma_{T_1}(x) = \{f_1(x)\} \neq \{0\}$, a desired contradiction.

Claim 2. Rx is linearly independent with x_0, for any $x \in \mathscr{X}$.

If not, then there exist some $x \in \mathscr{X}$ and $\beta \in \mathbb{C}$ such that $Rx = \beta x_0$. If $\beta = 0$, then $Rx = 0$. Now, we take $f_2 \in \mathscr{X}^*$ such that $f_2(x) = f_2(x_0) = 1$. Set $T_2 = x \otimes f_2$. It is easy to check that $T_2^r R T_2^s(x_0) = 0$. Using the process as in the proof of Claim 1 we get $\sigma_{T_2}(x_0) = \{0\}$. Note that $T_2x_0 = x$. So, from Lemma 1 we get $\sigma_{T_2}(x) \subseteq \{0\}$. However,

from the fact $T_2x = x$ we could obtain $\sigma_{T_2}(x) = \{1\}$, a contradiction. This shows $\beta \neq 0$. Take $f_3 \in \mathscr{X}^*$ such that $f_3(x_0) = 1$ and $f_3(x) = 2\beta$. Put $T_3 = x \otimes f_3$. By calculation we get $T_3^r R T_3^s(x) = \beta(2\beta)^{r+s-1}x$, from which we obtain $\sigma_{T_3^r R T_3^s}(x) = \{\beta(2\beta)^{r+s-1}\}$. Besides, $T_3^r R T_3^s(x_0) = \beta(2\beta)^{r+s-2}x$. So, $\sigma_{T_3^r R T_3^s}(x) \subseteq \sigma_{T_3^r R T_3^s}(x_0) \subseteq \sigma_{T_3^r R T_3^s}(x) \cup \{0\}$. Combining this with (1) we get $\sigma^*_{\varphi(T_3)}(y_0) = \{\mu\}$, where $\mu^{r+s} = \beta(2\beta)^{r+s-1}$. On the other hand, since $T_3x = 2\beta x$, it follows that $\sigma_{T_3}(x) = \{2\beta\}$. Besides, from $T_3x_0 = x$ we get $\{2\beta\} \subseteq \sigma_{T_3}(x_0) \subseteq \{0, 2\beta\}$ and so $\sigma^*_{T_3}(x_0) = \{2\beta\}$. By using (1) we get $\mu = 4\beta$. So, $(4\beta)^{r+s} = 2^{r+s-1}\beta^{r+s}$ and then $r+s = -1$, a contradiction.

From Claim 1 and Claim 2, we get that Rx is linearly dependent with x, for any $x \in \mathscr{X}$. So, there is $\gamma_x \in \mathbb{C}$ such that $Rx = \gamma_x x$, $\forall x \in \mathscr{X}$. Take two linearly independent vectors x and y in $\mathscr{X}$. Then $R(x+y) = \gamma_{x+y}(x+y) = \gamma_{x+y}x + \gamma_{x+y}y$. On the other hand, $R(x+y) = Rx + Ry = \gamma_x x + \gamma_y y$, which shows that $\gamma_x = \gamma_{x+y} = \gamma_y$. Thus, the constant γ_x is independent of the vector x, that is to say, there is some $\gamma \in \mathbb{C}$ such that $Rx = \gamma x$, $\forall x \in \mathscr{X}$. Therefore, $R = \gamma I$. From equation (1) we get $\gamma^{r+s+1} = 1$. Put $\alpha = \gamma^{-1}$. From the linearity of φ we obtain that $\varphi(I) = \alpha I$ and $\alpha^{r+s+1} = 1$.

Now, it is easy to prove that

$$\sigma_T(x_0) = \alpha^{r+s}\sigma_{\varphi(T)}(y_0) \tag{2}$$

for any $T \in \mathscr{B}(\mathscr{X})$.

Step 5. φ has the desired form.

From [18, Theorem 3.3] we get that either there are bijective mappings $A : \mathscr{X} \to \mathscr{X}$ and $B : \mathscr{X}^* \to \mathscr{X}^*$ such that

$$\varphi(x \otimes f) = Ax \otimes Bf,\ \forall x \in \mathscr{X}, \forall f \in \mathscr{X}^*,$$

or there are bijective mappings $C : \mathscr{X}^* \to \mathscr{X}$ and $D : \mathscr{X} \to \mathscr{X}^*$ such that

$$\varphi(x \otimes f) = Cf \otimes Dx,\ \forall x \in \mathscr{X}, \forall f \in \mathscr{X}^*.$$

Next, we show that the second form cannot occur. Suppose that $\varphi(x \otimes f) = Cf \otimes Dx$ for any $x \in \mathscr{X}$ and $f \in \mathscr{X}^*$. Take $g \in \mathscr{Y}^*$ such that $g(y_0) = 0$. Set $x = D^{-1}g \in \mathscr{X}$. Take $f \in \mathscr{X}^*$ such that $f(x_0) \neq 0$ and $f(x) \neq 0$. Put $T = x \otimes f$. Then we have $\sigma^*_T(x_0) = \{f(x)\} \neq \{0\}$. On the other hand, it is clear that $\sigma^*_{\varphi(T)}(y_0) = \sigma^*_{Cf \otimes Dx}(y_0) = \sigma^*_{Cf \otimes g}(y_0) = \{0\}$ since $g(y_0) = 0$. This contradicts with (2).

In the following, we firstly claim that

$$f(x) = \alpha^{r+s}(Bf)(Ax),\ \forall x \in \mathscr{X}, \forall f \in \mathscr{X}^*.$$

In fact, take any $f \in \mathscr{X}^*$ and any $x \in \mathscr{X}$. If $f(x_0) \neq 0$, then

$$\{0\} \neq \{f(x_0)\} = \sigma_{x_0 \otimes f}(x_0) = \alpha^{r+s}\sigma_{\varphi(x_0 \otimes f)}(y_0) = \alpha^{r+s}\sigma_{Ax_0 \otimes Bf}(y_0),$$

from which we get $(Bf)(y_0) \neq 0$. Besides, we have

$$\{f(x)\} = \sigma^*_{x \otimes f}(x_0) = \alpha^{r+s}\sigma^*_{Ax \otimes Bf}(y_0) = \{\alpha^{r+s}(Bf)(Ax)\}.$$

So, $f(x)=\alpha^{r+s}(Bf)(Ax)$ for any $x\in\mathscr{X}$ and any $f\in\mathscr{X}^*$ with $f(x_0)\neq 0$. If $f(x_0)=0$, then we take $g\in\mathscr{X}^*$ such that $g(x_0)\neq 0$. From the process above, we could get $g(x)=\alpha^{r+s}(Bg)(Ax)$ and $(f+g)(x)=\alpha^{r+s}(B(f+g))(Ax)$. By easy calculation, we obtain $f(x)=\alpha^{r+s}(Bf)(Ax)$. The claim is proved.

Next, we show A is continuous and $B=\alpha^{-1}(A^{-1})^*$.

Suppose that $\{x_n\}\subseteq\mathscr{X}$ such that $x_n\to x$ and $Ax_n\to y$ as $n\to\infty$. For any $f\in\mathscr{X}^*$, we have

$$(Bf)y=\lim_{n\to\infty}(Bf)(Ax_n)=\lim_{n\to\infty}\alpha^{-1}f(x_n)=\alpha^{-1}f(x)=(Bf)(Ax).$$

By the arbitrariness of f we get $y=Ax$, which shows that A is continuous. Besides, by the claim above we obtain $B=\alpha^{-1}(A^{-1})^*$. Thus, $\varphi(x\otimes f)=\alpha^{-1}A(x\otimes f)A^{-1}$, for any $x\in\mathscr{X}$ and any $f\in\mathscr{X}^*$. Next, we prove that Ax_0 is linearly dependent with y_0. Otherwise, there exists some $f\in\mathscr{X}^*$ such that $f(x_0)=1$ and $f(A^{-1}y_0)=0$. Put $T=x_0\otimes f$. It is easy to check that $\sigma_T(x_0)=\{1\}$ and $\sigma_{ATA^{-1}}(y_0)=\{0\}$. On the other hand, by (2) we have

$$\sigma_T(x_0)=\alpha^{r+s}\sigma_{\varphi(T)}(y_0)=\alpha^{r+s}\sigma_{\alpha^{-1}A(x_0\otimes f)A^{-1}}(y_0)=\sigma_{A(x_0\otimes f)A^{-1}}(y_0),$$

which gives a contradiction. Therefore, $Ax_0=\eta y_0$ for some $\eta\neq 0$. Without loss of generality, we may assume $\eta=1$, i.e., $Ax_0=y_0$. For any $T\in\mathscr{B}(\mathscr{X})$ and any $R\in\mathscr{F}_1(\mathscr{X})$, we have

$$\begin{aligned}
&\sigma_{\varphi(R)^r(\alpha^{-1}ATA^{-1})\varphi(R)^s}(y_0)\\
&=\sigma_{AR^rA^{-1}(ATA^{-1})AR^sA^{-1}}(y_0)\\
&=\sigma_{AR^rTR^sA^{-1}}(y_0)\\
&=\sigma_{R^rTR^s}(A^{-1}y_0)\\
&=\sigma_{R^rTR^s}(x_0)\\
&=\sigma_{\varphi(R)^r\varphi(T)\varphi(R)^s}(y_0).
\end{aligned}$$

From Theorem 4 we obtain that $\varphi(T)=\alpha^{-1}ATA^{-1}$ for any $T\in\mathscr{B}(\mathscr{X})$. Put $\lambda=\alpha^{-1}$. Then $\lambda^{r+s+1}=1$ and $\varphi(T)=\lambda ATA^{-1}$ for any $T\in\mathscr{B}(\mathscr{X})$. □

REMARK 1. (i) Fix two nonzero vectors $x_0\in\mathscr{X}$ and $y_0\in\mathscr{Y}$. In [11], Bourhim and Lee described the forms of any two maps φ_1 and φ_2 from $\mathscr{B}(\mathscr{X})$ onto $\mathscr{B}(\mathscr{Y})$ satisfy

$$\sigma_{\varphi_1(S)\varphi_2(T)}(y_0)=\sigma_{TS}(x_0)$$

for any $T,S\in\mathscr{B}(\mathscr{X})$. Fix an integer $k\geqslant 2$, take $A_1=S$, $A_2=T$ and all other $A_i's$ are I. Assume that i_1 and i_2 appear exactly once in $(i_1,i_2,\cdots,i_m)$. Without loss of generality, let $i_1=1$ and $i_2=2$. Then $A_1*A_2*\cdots*A_k=ST$ and $\varphi(A_1)*\varphi(A_2)*\cdots*\varphi(A_k)=\varphi(S)\varphi(T)\varphi(I)^{m-2}$. The condition of Theorem 1 implies that $\sigma_{\varphi(S)\varphi(T)\varphi(I)^{m-2}}(y_0)=\sigma_{ST}(x_0)$. Thus, the result in [11] applies to $\varphi_1(\cdot)=\varphi(\cdot)$ and $\varphi_2(\cdot)=\varphi(\cdot)\varphi(I)^{m-2}$.

(ii) In finite-dimensional cases, one can get rid of the surjectivity condition. For related results, see [2] and [10].

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REFERENCES

[1] Z. ABDELALI AND A. ACHCHI AND R. MARZOUKI, *Maps preserving the local spectral radius zero of generalized product of operators*, Linear Multilinear Algebra **67** (1), (2019), 2021–2029.
[2] Z. ABDELALI AND A. BOURHIM, *Maps preserving the local spectrum of quadratic products of matrices*, Acta Sci. Math. (Szeged) **84** (1), (2018), 49–64.
[3] P. AIENA, *Fredholm and Local Spectral Theory, with Applications to Multipliers*, Kluwer Academic Publishers, Dordrecht, 2004.
[4] M. BENDAOUD, *Preservers of local spectrum of matrix Jordan triple products*, Linear Algebra Appl. **471** (1), (2015), 604–614.
[5] M. BENDAOUD AND M. JABBAR AND M. SARIH, *Preservers of local spectra of operator products*, Linear Multilinear Algebra **63** (1), (2015), 806–819.
[6] A. BOURHIM AND J. MASHREGHI, *Local spectral radius preserves*, Integral Equations Operator Theory **76** (1), (2013), 95–104.
[7] A. BOURHIM AND J. MASHREGHI, *Maps preserving the local spectrum of triple product of operators*, Linear Multilinear Algebra **64** (1), (2015), 765–773.
[8] A. BOURHIM AND J. MASHREGHI, *Maps preserving the local spectrum of product of operators*, Glasgow Math. J. **57** (2), (2015), 709–718.
[9] A. BOURHIM AND J. MASHREGHI, *A survey on preservers of spectra and local spectra*, in: Invariant Subspaces of the Shift Operators, Contemp. Math. **638** (1), (2015), 45–98.
[10] A. BOURHIM AND M. MABROUK, *Maps preserving the local spectrum of Jordan product of matrices*, Linear Algebra Appl. **484** (1), (2015), 379–395.
[11] A. BOURHIM AND J. E. LEE, *Multiplicatively local spectrum-preserving maps*, Linear Algebra Appl. **549** (1), (2018), 291–308.
[12] A. BOURHIM AND T. RANSFORD, *Additive maps preserving local spectrum*, Integral Equations Operator Theory **55** (2), (2006), 377–385.
[13] C. COSTARA, *Linear maps preserving operators of local spectral radius zero*, Integral Equations Operator Theory **73**, (2), (2012), 7–16.
[14] M. ECH-CHERIF EL KETTANI AND E. SIAR, *On the some spectral domains in the spectrum preserves*, *Complex Anal. Oper. Theory* **6** (1), (2023), Paper No. 100.
[15] J. C. HOU AND J. L. CUI, *Additive maps on standard operator algebras preserving parts of the spectrum*, J. Math. Anal. Appl. **282** (3), (2003), 266–278.
[16] J. C. HOU AND C. K. LI AND N. C. WONG, *Jordan isomorphism and maps preserving spectra of certain operator product*, Studia Math. **184** (1), (2008), 31–47.
[17] K. B. LAURSEN AND M. M. NEUMANN, *An introduction to local spectral theory*, Lodan Mathematical Society Monograph, Dordrecht, 2000.
[18] M. OMLADIČ AND P. ŠEMRL, *Additive mappings preserving operators of rank one*, Linear Algebra Appl. **182** (5), (1993), 239–256.
[19] W. ZHANG AND J. C. HOU, *Maps preserving peripheral spectrum of generalized products of operators*, Linear Algebra Appl. **468** (1), (2015), 2021–2029.

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