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SOME GENERALIZATIONS OF NUMERICAL RADII INEQUALITIES

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Abstract. This paper focuses on establishing new upper bounds for the numerical radius of operators on Hilbert spaces by utilizing the Moore-Penrose inverse and the generalized Cartesian decomposition. The obtained estimates enhance the existing body of knowledge and are systematically compared with results from the current literature. Our findings not only extend but also unify several recent contributions, offering a broader and more cohesive framework for understanding numerical radius inequalities. Through the application of the generalized Cartesian decomposition, we provide deeper insights into the behavior of numerical radii, building upon previous research and opening new directions for further investigation in this field.

1. Introduction

Let $\mathbb{H}$ be a complex Hilbert space with inner product $\langle .,. \rangle$ and $\mathbb{B}(\mathbb{H})$ be the collection of all bounded linear operator on $\mathbb{H}$. For $A \in \mathbb{B}(\mathbb{H})$, A^* denote the conjugate of A. An operator $A \in \mathbb{B}(\mathbb{H})$ is called positive, denoted $A \geqslant 0$, if it satisfies $\langle Ax, x \rangle \geqslant 0$ for all $x \in \mathbb{H}$. For an operator $A \in \mathbb{B}(\mathbb{H})$, we denote $|A| := (A^*A)^{\frac{1}{2}}$. The Cartesian decomposition of A is $A = \Re(A) + i\Im(A)$, where $\Re(A) = \frac{1}{2}(A + A^*)$ and $\Im(A) = \frac{1}{2i}(A - A^*)$ are the real part and imaginary parts of A, respectively.

The numerical radius of a bounded linear operator A on a Hilbert space $\mathbb{H}$ is an important and widely studied quantity in operator theory, playing a crucial role in understanding the behavior of operators. It is defined as

$$w(A) = \sup\{|\langle Ax, x \rangle| : x \in \mathbb{H}, \|x\| = 1\}.$$

This concept provides valuable insight into the structure of operators, offering an upper bound for the spectral radius and satisfying fundamental inequalities that establish a direct connection with the operator norm. Due to its significance, the study of numerical radius inequalities has garnered considerable attention in the literature, leading to various generalizations and refinements with applications in functional analysis, matrix theory, and quantum mechanics.

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It is well known that

$$\frac{1}{2}||A|| \leqslant w(A) \leqslant ||A||. \tag{1}$$

These inequalities in (1) are fundamental and sharp, providing essential bounds for the numerical radius. The first inequality becomes an equality if $A^2 = 0$, while the second inequality attains equality if A is a normal operator. These conditions illustrate the precise limitations of the numerical radius in different operator settings, further motivating its study in various mathematical disciplines.

Kittaneh [21] improved (1) as follows

$$\frac{1}{4}\left|\left||A|^2+|A^*|^2\right|\right| \leqslant w^2(A) \leqslant \frac{1}{2}\left|\left||A|^2+|A^*|^2\right|\right|, \tag{2}$$

where $|A| = (A^*A)^{\frac{1}{2}}$ is the absolute value of A. El-Haddad and Kittaneh [13] showed the following generalizations of the second inequality in (2),

$$w^{2r}(A) \leqslant \left|\left|t|A|^{2r}+(1-t)|A^*|^{2r}\right|\right|, \tag{3}$$

where $0 < t < 1$ and $r \geqslant 1$.

Dragomir [12] presented an important upper bound for the numerical radii of products of two operators by showing that if $A, B \in \mathbb{B}(\mathbb{H})$ and $r \geqslant 1$, then

$$w^r(B^*A) \leqslant \frac{1}{2}\left|\left||A|^{2r}+|B|^{2r}\right|\right|. \tag{4}$$

Let $A \in \mathbb{B}(\mathbb{H})$ with the Cartesian decomposition $A = B + iC$. Then El-Haddad and Kittaneh [13] showed the following generalized inequalities:

1. If $0 < r \leqslant 2$, then

$$w^r(A) \leqslant \left|\left||B|^r+|C|^r\right|\right|. \tag{5}$$

2. If $r \geqslant 2$, then

$$w^r(A) \leqslant 2^{\frac{r}{2}-1}\left|\left||B|^r+|C|^r\right|\right|. \tag{6}$$

and

$$2^{-\frac{r+2}{2}}\left|\left||B+C|^r+|B-C|^r\right|\right| \leqslant w^r(A) \leqslant \frac{1}{2}\left|\left||B+C|^r+|B-C|^r\right|\right|. \tag{7}$$

Very recently, Bhunia, Sen, and Paul [4] introduced the generalized Cartesian decomposition of $A \in \mathbb{B}(\mathbb{H})$, which is defined as

$$A = \Re_\lambda(A) + i\Im_\lambda(A),$$

where $\Re_\lambda(A) = \frac{A+\lambda A^*}{2}$ and $\Im_\lambda(A) = \frac{A-\lambda A^*}{2i}$ for $\lambda \in \mathbb{T}$, with $\mathbb{T} = \left\{z \in \mathbb{C} : |z| = 1\right\}$. Observe that both operators $\Re_\lambda(A)$ and $\Im_\lambda(A)$ are normal, which allows for a more refined analysis of operator inequalities and numerical radius bounds. This decomposition has been instrumental in obtaining new inequalities and improving existing results in the study of numerical radii.

For further inequalities involving the numerical radius, readers are encouraged to consult the following references: [1, 2, 4, 9, 11, 20, 21, 22, 23, 26, 28, 30, 31, 32].

A central notion in operator theory is the Moore-Penrose inverse, which extends the classical idea of inverses to a broader class of operators.

Given an operator $A \in \mathbb{B}(\mathbb{H})$, the Moore-Penrose inverse of A, denoted by $A^{\dagger}$, is uniquely characterized by the following set of four Penrose equations [16]:

$$AA^{\dagger}A = A, \quad A^{\dagger}AA^{\dagger} = A^{\dagger}, \quad (AA^{\dagger})^{*} = AA^{\dagger}, \quad (A^{\dagger}A)^{*} = A^{\dagger}A. \tag{8}$$

These conditions guarantee both the existence and uniqueness of $A^{\dagger}$, provided that A has a closed range.

We define the collection of all operators in $\mathbb{B}(\mathbb{H})$ admitting a Moore-Penrose inverse as follows:

$$\mathbb{CR}(\mathbb{H}) = \{A \in \mathbb{B}(\mathbb{H}) \mid \mathrm{range}(A) \text{ is closed}\}.$$

Whenever $A \in \mathbb{CR}(\mathbb{H})$, its Moore-Penrose inverse $A^{\dagger}$ belongs to $\mathbb{B}(\mathbb{H})$ as well.

The Moore-Penrose inverse enjoys several important properties when applied to elements of $\mathbb{CR}(\mathbb{H})$:

- **Self-adjointness:** If A is self-adjoint, meaning $A = A^{*}$, then $A^{\dagger}$ is also self-adjoint:
$$(A^{\dagger})^{*} = A^{\dagger}.$$
- **Involutive property:** The Moore-Penrose inverse satisfies
$$(A^{\dagger})^{\dagger} = A.$$
- **Order reversal:** If $A_1 \leqslant A_2$ in the operator ordering, then
$$A_2^{\dagger} \leqslant A_1^{\dagger}.$$
- **Product rule:** For $A_1, A_2 \in \mathbb{CR}(\mathbb{H})$ with $\mathrm{range}(A_1) \subseteq \mathrm{range}(A_2)$, it holds that
$$(A_2A_1)^{\dagger} = A_1^{\dagger}A_2^{\dagger}.$$

For further properties and developments on the Moore-Penrose inverse, we refer the reader to [10, 14].

Recently, Sababheh and collaborators [29] introduced a refined inequality for the numerical raduis involving the Moore-Penrose inverse. Specifically, for $A \in \mathbb{CR}(\mathbb{H})$,

$$w(A) \leqslant \frac{1}{2}\left\| |A|^2 + AA^{\dagger} \right\|. \tag{9}$$

A significant topic in operator theory concerns establishing sharp estimates linking the numerical radius and the operator norm. Several researchers have provided improvements in this direction. The next result offers a notable estimate:

THEOREM 1. ([29]) *Let* $A \in \mathbb{CR}(\mathbb{H})$. *Then the following inequality holds:*

$$w(A) \leqslant \frac{1}{4}\left\||A|^2 + A^{\dagger}A\right\| + \frac{1}{2}w(A^{\dagger}A^2). \tag{10}$$

An alternative bound is also valid

$$w(A) \leqslant \frac{1}{4}\left\||A^*|^2 + AA^{\dagger}\right\| + \frac{1}{2}w(A^2A^{\dagger}). \tag{11}$$

Throughout this paper, we assume that every function under consideration is continuous and satisfies the following conditions: J is a subinterval of $(0,\infty)$, and $f : J \to (0,\infty)$.

This paper is organized as follows. In the next section, we introduce several key lemmas that form the foundation for our subsequent analysis. Section 3 is devoted to deriving new upper bounds for the numerical radius of operators on Hilbert spaces, incorporating the Moore-Penrose inverse. These new estimates enrich the current body of knowledge and are carefully compared against existing results from the literature. In Section 4, we present our principal findings by establishing fresh numerical radius inequalities based on the generalized Cartesian decomposition. We also explore various generalizations and enhancements of known inequalities, highlighting their relevance and impact within the broader study of operator inequalities.

2. Preliminary

In this section, we begin by revisiting and summarizing fundamental lemmas that are crucial for our analysis.

LEMMA 1. ([25]) *Let* $A \in \mathbb{B}(\mathbb{H})$ *be positive, and let* $x \in \mathbb{H}$ *be any unit vector. Then*

$$\begin{aligned} \langle A^{v}x,x\rangle &\leqslant \langle Ax,x\rangle^{v} \ \text{ for } \ 0<v\leqslant 1;\\ \langle Ax,x\rangle^{v} &\leqslant \langle A^{v}x,x\rangle \ \text{ for } \ 1\leqslant v. \end{aligned}$$

LEMMA 2. ([8]) *Let* $A_1, A_2, \cdots A_n \geqslant 0$. *Then for every non-negative concave function* f *on* $[0,\infty)$ *and for every unitarily invariant norm* $|||\cdot|||$,

$$\left|\left|\left| f\left(\sum_{j=1}^{n} A_j\right)\right|\right|\right| \leqslant \left|\left|\left| \sum_{j=1}^{n} f(A_j)\right|\right|\right|.$$

LEMMA 3. ([17] p. 26) *For* $a,b \geqslant 0$, $0<v<1$, *and* $r \neq 0$, *let* $M_r(a,b,v) = (va^r + (1-v)b^r)^{\frac{1}{r}}$ *and let* $M_0(a,b,v) = a^v b^{1-v}$. *Then*

$$M_r(a,b,v) \leqslant M_s(a,b,v) \ \text{ for } \ r \leqslant s.$$

LEMMA 4. ([15] p. 118) (Operator Jensen inequality for convex function [27]). *Let $A \in \mathbb{B}(\mathbb{H})$ be a self-adjoint operator with $Sp(A) \subseteq [m,M]$ for some scalars $m < M$. If f is a convex function on $[m,M]$, then*

$$f\big(\langle Ax,x\rangle\big) \leqslant \big\langle f(A)x,x\big\rangle \tag{12}$$

for every unit vector $x \in \mathbb{H}$.

LEMMA 5. ([19]) *Let $a,b,x \in \mathbb{H}$ with $||x|| = 1$, let J be a set such that $(0,1) \subset J \subset \mathbb{R}$, let χ be a mapping $\chi : J \to [\frac{1}{2},1]$, and $r \geqslant 1$. Then we have*

$$\big|\langle a,x\rangle\langle x,b\rangle\big|^r \leqslant \chi(\alpha)||a||^r||b||^r + (1-\chi(\alpha))\big|\langle a,b\rangle\big|^r. \tag{13}$$

Taking $r = 2$ in (13), we obtain

$$\big|\langle a,x\rangle\langle x,b\rangle\big|^2 \leqslant \chi(\alpha)||a||^2||b||^2 + (1-\chi(\alpha))\big|\langle a,b\rangle\big|^2. \tag{14}$$

The following lemma is a direct consequence of the well-known Young's inequality.

LEMMA 6. *Let $u,v \in \mathbb{R}$ be positive, let J be a set such that $(0,1) \subset J \subset \mathbb{R}$, let ζ be a mapping $\zeta : J \to [0,1]$ be such that $\zeta(\alpha) + \zeta(1-\alpha) = 1$. Then, we have*

$$uv \leqslant (\zeta(\alpha)u + \zeta(1-\alpha)v)(\zeta(1-\alpha)u + \zeta(\alpha)v).$$

Proof. Let $\alpha = \zeta(t)$ and $\beta = \zeta(1-t)$. Then $\alpha,\beta \in [0,1]$ and $\alpha+\beta = 1$. Since $u,v > 0$, we have

$$uv = u^{\alpha+\beta}v^{\alpha+\beta} = (u^\alpha v^\beta)(u^\beta v^\alpha).$$

By the classical Young inequality $x^\lambda y^{1-\lambda} \leqslant \lambda x + (1-\lambda)y$ (applied with $x = u$, $y = v$), we obtain

$$u^\alpha v^\beta \leqslant \alpha u + \beta v \quad \text{and} \quad u^\beta v^\alpha \leqslant \beta u + \alpha v.$$

Multiplying these two inequalities yields

$$uv = (u^\alpha v^\beta)(u^\beta v^\alpha) \leqslant (\alpha u + \beta v)(\beta u + \alpha v),$$

which is the desired inequality. $\square$

The last lemma is stated as follows.

LEMMA 7. ([1]) *Let $f : [0,\infty) \to [0,\infty)$ be a convex function, and let $A,B \in \mathbb{B}(\mathbb{H})$ be positive operators. If $0 \leqslant \alpha \leqslant 1$, then*

$$\|f((1-\alpha)A + \alpha B)\| \leqslant \|(1-\alpha)f(A) + \alpha f(B)\|.$$

3. New numerical radius via the Moore-Penrose inverse

To achieve our result, we will incorporate concepts related to convexity into our discussion. Recall that if $f:[0,\infty)\to[0,\infty)$, then f is referred to as doubly convex if f is convex in the conventional sense and

$$f\left(a^{1-\theta}b^{\theta}\right)\leqslant f^{1-\theta}(a)f^{\theta}(b),\quad a,b\geqslant 0,0\leqslant\theta\leqslant 1. \tag{15}$$

The functions $f(t)=\sinh t$ and $f(t)=\cosh t$ are examples of functions that are doubly convex on $[0,\infty)$.

The first key result of this section is the following theorem, which provides a new upper bound for the numerical radius of operators on Hilbert spaces. This bound is derived by leveraging the Moore-Penrose inverse. The theorem extends existing results by offering a refined estimation technique for numerical radii. We present the detailed proof and implications in the following subsections.

THEOREM 2. *Let $A\in\mathbb{CR}(\mathbb{H})$, let J be a set such that $(0,1)\subset J\subset\mathbb{R}$ and let $\zeta:J\to[0,1]$ be a mapping such that $\zeta(\alpha)+\zeta(1-\alpha)=1$ with $0\leqslant\alpha\leqslant 1$. If $f:[0,\infty)\to[0,\infty)$ is an increasing doubly convex function, then*

$$f(\omega(A))\leqslant\sqrt{\left\|\zeta(\alpha)f\left(AA^{\dagger}\right)+\zeta(1-\alpha)f\left(|A|^2\right)\right\|\left\|\zeta(1-\alpha)f\left(AA^{\dagger}\right)+\zeta(\alpha)f\left(|A|^2\right)\right\|}. \tag{16}$$

In particular, for any $r\geqslant 1$,

$$\omega^r(A)\leqslant\sqrt{\left\|\zeta(\alpha)AA^{\dagger}+\zeta(1-\alpha)|A|^{2r}\right\|\left\|\zeta(1-\alpha)AA^{\dagger}+\zeta(\alpha)|A|^{2r}\right\|}. \tag{17}$$

Proof. Let $x\in\mathbb{H}$ be a unit vector. Applying the well-known Cauchy-Schwarz inequality together with Lemma 6, we obtain

$$\begin{aligned}
|\langle Ax,x\rangle| &= |\langle AA^{\dagger}Ax,x\rangle| = |\langle Ax,AA^{\dagger}x\rangle|\\
&\leqslant\sqrt{\|AA^{\dagger}x\|^2\|Ax\|^2}\\
&=\sqrt{\langle AA^{\dagger}x,x\rangle\langle|A|^2x,x\rangle}\\
&\leqslant\sqrt{(\zeta(\alpha)\langle AA^{\dagger}x,x\rangle+\zeta(1-\alpha)\langle|A|^2x,x\rangle)(\zeta(1-\alpha)\langle AA^{\dagger}x,x\rangle+\zeta(\alpha)\langle|A|^2x,x\rangle)}\\
&=\sqrt{\langle(\zeta(\alpha)AA^{\dagger}+\zeta(1-\alpha)|A|^2)x,x\rangle\,\langle(\zeta(1-\alpha)AA^{\dagger}+\zeta(\alpha)|A|^2)x,x\rangle}.
\end{aligned}$$

Using the fact that f is an increasing doubly convex function, we obtain

$$\begin{aligned}
f\left(|\langle Ax,x\rangle|\right) &\leqslant f\left(\sqrt{\langle(\zeta(\alpha)AA^{\dagger}+\zeta(1-\alpha)|A|^2)x,x\rangle\,\langle(\zeta(1-\alpha)AA^{\dagger}+\zeta(\alpha)|A|^2)x,x\rangle}\right)\\
&\leqslant\sqrt{f\left(\langle(\zeta(\alpha)AA^{\dagger}+\zeta(1-\alpha)|A|^2)x,x\rangle\right)f\left(\langle(\zeta(1-\alpha)AA^{\dagger}+\zeta(\alpha)|A|^2)x,x\rangle\right)}
\end{aligned}$$

$$\leqslant \sqrt{((f\langle(\zeta(\alpha)AA^{\dagger}+\zeta(1-\alpha)|A|^2)x,x\rangle)\,(\langle f(\zeta(1-\alpha)AA^{\dagger}+\zeta(\alpha)|A|^2)x,x\rangle)}$$
$$\leqslant \sqrt{\|f\,(\zeta(\alpha)AA^{\dagger}+\zeta(1-\alpha)|A|^2\|\,\|f\,(\zeta(1-\alpha)AA^{\dagger}+\zeta(\alpha)|A|^2)\,\|)\,\|}$$
$$= \sqrt{\|\zeta(\alpha)f\,(AA^{\dagger})+\zeta(1-\alpha)f\,(|A|^2)\,\|\,\|\zeta(1-\alpha)f\,(AA^{\dagger})+\zeta(\alpha)f\,(|A|^2)\,\|}.$$

Taking the supremum over all unit vectors $x \in \mathbb{H}$, we obtain the desired inequality. □

REMARK 1. Setting $r = 1$ in inequality (17) and choosing $\chi(\alpha) = \frac{1}{2}$, we recover inequality (9).

The following alternative bound is likewise valid.

THEOREM 3. *Let $A \in \mathbb{CR}(\mathbb{H})$, let J be a set such that $(0,1) \subset J \subset \mathbb{R}$ and let $\zeta : J \to [0,1]$ be a mapping such that $\zeta(\alpha)+\zeta(1-\alpha) = 1$ with $0 \leqslant \alpha \leqslant 1$. If $f : [0,\infty) \to [0,\infty)$ is an increasing doubly convex function, then*

$$f(\omega(A)) \leqslant \sqrt{\|\zeta(\alpha)f\,(A^{\dagger}A)+\zeta(1-\alpha)f\,(|A^*|^2)\,\|\,\|\zeta(1-\alpha)f\,(A^{\dagger}A)+\zeta(\alpha)f\,(|A^*|^2)\,\|}.$$

In particular, for any $r \geqslant 1$,

$$\omega^r(A) \leqslant \sqrt{\|\zeta(\alpha)A^{\dagger}A+\zeta(1-\alpha)|A^*|^{2r}\|\,\|\zeta(1-\alpha)A^{\dagger}A+\zeta(\alpha)|A^*|^{2r}\|}.$$

Proof. Let $x \in \mathbb{H}$ be a unit vector. We have

$$\begin{aligned}
&|\langle Ax,x\rangle| \\
&= |\langle AA^{\dagger}Ax,x\rangle| = |\langle A^{\dagger}Ax,A^*x\rangle| \\
&\leqslant \sqrt{\|A^{\dagger}Ax\|^2\|A^*x\|^2} \\
&= \sqrt{\langle A^{\dagger}Ax,x\rangle\langle|A^*|^2x,x\rangle} \\
&\leqslant \sqrt{(\zeta(\alpha)\langle A^{\dagger}Ax,x\rangle+\zeta(1-\alpha)\langle|A^*|^2x,x\rangle)(\zeta(1-\alpha)\langle A^{\dagger}Ax,x\rangle+\zeta(\alpha)\langle|A^*|^2x,x\rangle)} \\
&= \sqrt{\langle(\zeta(\alpha)A^{\dagger}A+\zeta(1-\alpha)|A^*|^2)x,x\rangle\,\langle(\zeta(1-\alpha)A^{\dagger}A+\zeta(\alpha)|A^*|^2)x,x\rangle}.
\end{aligned}$$

Exploiting the fact that f is increasing and doubly convex, we find

$$\begin{aligned}
&f\,(|\langle Ax,x\rangle|) \\
&\leqslant f\left(\sqrt{\langle(\zeta(\alpha)A^{\dagger}A+\zeta(1-\alpha)|A^*|^2)x,x\rangle\,\langle(\zeta(1-\alpha)A^{\dagger}A+\zeta(\alpha)|A^*|^2)x,x\rangle}\right) \\
&\leqslant \sqrt{f\,(\langle(\zeta(\alpha)A^{\dagger}A+\zeta(1-\alpha)|A^*|^2)x,x\rangle)\,f\,(\langle(\zeta(1-\alpha)A^{\dagger}A+\zeta(\alpha)|A^*|^2)x,x\rangle)} \\
&\leqslant \sqrt{((f\langle(\zeta(\alpha)A^{\dagger}A+\zeta(1-\alpha)|A^*|^2)x,x\rangle)\,(\langle f(\zeta(1-\alpha)A^{\dagger}A+\zeta(\alpha)|A^*|^2)x,x\rangle)}
\end{aligned}$$

$$\leqslant \sqrt{\|f\left(\zeta(\alpha)A^{\dagger}A+\zeta(1-\alpha)|A^{*}|^{2}\|\|f\left(\zeta(1-\alpha)A^{\dagger}A+\zeta(\alpha)|A^{*}|^{2}\right)\|\right)\|}$$
$$= \sqrt{\|\zeta(\alpha)f\left(A^{\dagger}A\right)+\zeta(1-\alpha)f\left(|A^{*}|^{2}\right)\|\|\zeta(1-\alpha)f\left(A^{\dagger}A\right)+\zeta(\alpha)f\left(|A^{*}|^{2}\right)\|}.$$

Considering the supremum over all unit vectors $x \in \mathbb{H}$, we deduce the desired inequality. $\square$

The next main result of this section is a theorem that establishes a novel upper estimate for the numerical radius of operators on Hilbert spaces, utilizing the generalized Buzano inequality introduced in Lemma 5. This estimate is obtained through the application of the Moore-Penrose inverse. The theorem advances previous findings by providing a sharper method for estimating numerical radii.

THEOREM 4. *Let $A \in \mathbb{CR}(\mathbb{H})$ is normal, let J be a set such that $(0,1) \subset J \subset \mathbb{R}$ and $\zeta : J \to [0,1]$ be a mapping such that $\zeta(\alpha)+\zeta(1-\alpha)=1$ for all $0 \leqslant \alpha \leqslant 1$, and let χ be a function defined as in Lemma 5. If $f : [0,\infty) \to [0,\infty)$ is an increasing doubly convex function, then*

$$\begin{aligned} &f(\omega^{2}(A)) \\ &\leqslant \chi(\alpha)\sqrt{\|\zeta(\alpha)f\left(AA^{\dagger}\right)+\zeta(1-\alpha)f\left(|A|^{4}\right)\|\|\zeta(1-\alpha)f\left(AA^{\dagger}\right)+\zeta(\alpha)f\left(|A|^{4}\right)\|} \\ &\quad +(1-\chi(\alpha))\|f\left(|A|^{2}\right)\|. \end{aligned} \tag{18}$$

In particular, for any $r \geqslant 1$,

$$\begin{aligned} \omega^{2r}(A) \leqslant \chi(\alpha)&\sqrt{\|\zeta(\alpha)AA^{\dagger}+\zeta(1-\alpha)|A|^{4r}\|\|\zeta(1-\alpha)AA^{\dagger}+\zeta(\alpha)|A|^{4r}\|} \\ &+(1-\chi(\alpha))\||A|^{2r}\|. \end{aligned} \tag{19}$$

Proof. Let $x \in \mathbb{H}$ be a unit vector. Using the Cauchy–Schwarz inequality together with Lemma 5, we get

$$\begin{aligned} &|\langle Ax,x\rangle|^{2} = |\langle AA^{\dagger}Ax,x\rangle|^{2} = |\langle Ax,AA^{\dagger}x\rangle|^{2} \\ &\leqslant \langle AA^{\dagger}x,x\rangle\langle |A|^{2}x,x\rangle \\ &\leqslant \chi(\alpha)\|AA^{\dagger}x\|\||A|^{2}x\|+(1-\chi(\alpha))\left|\langle AA^{\dagger}x,|A|^{2}x\rangle\right| \\ &= \chi(\alpha)\sqrt{\langle AA^{\dagger}x,x\rangle\langle |A|^{4}x,x\rangle}+(1-\chi(\alpha))\left|\langle |A|^{2}AA^{\dagger}x,x\rangle\right| \\ &\leqslant \chi(\alpha)\sqrt{(\zeta(\alpha)\langle AA^{\dagger}x,x\rangle+\zeta(1-\alpha)\langle |A|^{4}x,x\rangle)(\zeta(1-\alpha)\langle AA^{\dagger}x,x\rangle+\zeta(\alpha)\langle |A|^{4}x,x\rangle)} \\ &\quad +(1-\chi(\alpha))\left|\langle |A|^{2}x,x\rangle\right| \text{ since } (|A|^{2}AA^{\dagger}=|A|^{2}) \\ &= \sqrt{\langle(\zeta(\alpha)AA^{\dagger}+\zeta(1-\alpha)|A|^{4})x,x\rangle\,\langle(\zeta(1-\alpha)AA^{\dagger}+\zeta(\alpha)|A|^{4})x,x\rangle} \\ &\quad +(1-\chi(\alpha))\left|\langle |A|^{2}x,x\rangle\right|. \end{aligned}$$

Using the fact that f is an increasing doubly convex function, we obtain

$$\begin{aligned}
&f\left(|\langle Ax,x\rangle|^2\right)\\
&\leqslant \chi(\alpha) f\left(\sqrt{\langle(\zeta(\alpha)AA^{\dagger}+\zeta(1-\alpha)|A|^4)x,x\rangle\,\langle(\zeta(1-\alpha)AA^{\dagger}+\zeta(\alpha)|A|^4)x,x\rangle}\right)\\
&\quad+(1-\chi(\alpha))f\left(\left|\langle|A|^2x,x\rangle\right|\right)\\
&\leqslant \chi(\alpha)\sqrt{f\left(\langle(\zeta(\alpha)AA^{\dagger}+\zeta(1-\alpha)|A|^4)x,x\rangle\right)f\left(\langle(\zeta(1-\alpha)AA^{\dagger}+\zeta(\alpha)|A|^4)x,x\rangle\right)}\\
&\quad+(1-\chi(\alpha))\left|\langle f\left(|A|^2\right)x,x\rangle\right|\\
&\leqslant \chi(\alpha)\sqrt{((f\langle(\zeta(\alpha)AA^{\dagger}+\zeta(1-\alpha)|A|^4)x,x\rangle)\,(\langle f(\zeta(1-\alpha)AA^{\dagger}+\zeta(\alpha)|A|^4)x,x\rangle)}\\
&\quad+(1-\chi(\alpha))\left|\langle f\left(|A|^2\right)x,x\rangle\right|\\
&\leqslant \chi(\alpha)\sqrt{\|f\left(\zeta(\alpha)AA^{\dagger}+\zeta(1-\alpha)|A|^4\right\|\|f\left(\zeta(1-\alpha)AA^{\dagger}+\zeta(\alpha)|A|^4\right)\|)\|}\\
&\quad+(1-\chi(\alpha))\left|\langle f\left(|A|^2\right)x,x\rangle\right|\\
&=\chi(\alpha)\sqrt{\|\zeta(\alpha)f\left(AA^{\dagger}\right)+\zeta(1-\alpha)f\left(|A|^4\right)\|\|\zeta(1-\alpha)f\left(AA^{\dagger}\right)+\zeta(\alpha)f\left(|A|^4\right)\|}\\
&\quad+(1-\chi(\alpha))\|f\left(|A|^2\right)\|.
\end{aligned}$$

Considering the supremum over all unit vectors $x\in\mathbb{H}$, we deduce the desired inequality. □

4. New numerical radius via the generalized Cartesian decomposition

In this section, we present further theorems on numerical radius inequalities, with the first one extending a recently established result from the literature. Our findings build upon existing work, offering a broader perspective on numerical radius properties and their implications.

THEOREM 5. *Let $A\in\mathbb{B}(\mathbb{H})$ and f be an increasing convex function. Let J be a set such that $(0,1)\subset J\subset\mathbb{R}$, and let χ be a mapping $\chi: J\to[\frac{1}{2},1]$. Then we have Then*

$$f\left(w^4(A)\right)\leqslant\frac{\chi(\alpha)}{2}\left|\left|f\left(|A|^4+|A^*|^4\right)\right|\right|+(1-\chi(\alpha))f\left(w^2(A^2)\right).$$

Proof. For every unit vector $x\in\mathbb{H}$, replace a,b by Ax,A^*x in (14), respectively, we have

$$\begin{aligned}
f\left(|\langle Ax,x\rangle|^4\right)&\leqslant f\left(\chi(\alpha)||Ax||^2||A^*x||^2+(1-\chi(\alpha))\left|\langle Ax,A^*x\rangle\right|^2\right)\\
&=f\left(\chi(\alpha)\left\langle|A|^2x,x\right\rangle\left\langle|A^*|^2x,x\right\rangle+(1-\chi(\alpha))\left|\langle A^2x,x\rangle\right|^2\right)\\
&\leqslant\chi(\alpha)f\left(\left\langle|A|^2x,x\right\rangle\left\langle|A^*|^2x,x\right\rangle\right)+(1-\chi(\alpha))f\left(\left|\langle A^2x,x\rangle\right|^2\right)
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \chi(\alpha) f\left(\frac{\langle |A|^2x,x\rangle^2+\langle |A^*|^2x,x\rangle^2}{2}\right)+(1-\chi(\alpha))f\big(w^2(A^2)\big)\\
&\leqslant \chi(\alpha) f\left(\frac{\langle |A|^4x,x\rangle+\langle |A^*|^4x,x\rangle}{2}\right)+(1-\chi(\alpha))f\big(w^2(A^2)\big)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\text{(by Lemma 1)}\\
&\leqslant \chi(\alpha)\frac{f\Big(\langle |A|^4x,x\rangle\Big)+f\Big(\langle |A^*|^4x,x\rangle\Big)}{2}+(1-\chi(\alpha))f\big(w^2(A^2)\big)\\
&\leqslant \frac{\chi(\alpha)}{2}\left(\Big\langle f\big(|A|^4\big)x,x\Big\rangle+\Big\langle f\big(|A^*|^4\big)x,x\Big\rangle\right)+(1-\chi(\alpha))f\big(w^2(A^2)\big)\\
&\leqslant \frac{\chi(\alpha)}{2}\Big\langle f\big(|A|^4+|A^*|^4\big)x,x\Big\rangle+(1-\chi(\alpha))f\big(w^2(A^2)\big),
\end{aligned}
$$

then we can get the desired results by taking the supremum over unit vector x. $\square$

REMARK 2. Let $f(x)=x$ in Theorem 5. We obtain

$$
w^4(A)\leqslant \frac{\chi(\alpha)}{2}\big|\big||A|^4+|A^*|^4\big|\big|+(1-\chi(\alpha))w^2(A^2). \tag{20}
$$

Next, we explain inequality (20) improved (3) when $r=2$ and $t=\frac{1}{2}$. In fact,

$$
\begin{aligned}
w^4(A)&\leqslant \frac{\chi(\alpha)}{2}\big|\big||A|^4+|A^*|^4\big|\big|+(1-\chi(\alpha))w^2(A^2)\\
&\leqslant \frac{\chi(\alpha)}{2}\big|\big||A|^4+|A^*|^4\big|\big|+(1-\chi(\alpha))\frac{1}{2}\big|\big||A|^4+|A^*|^4\big|\big| \quad \big(\text{by (4)}\big)\\
&=\frac{1}{2}\big|\big||A|^4+|A^*|^4\big|\big|. \quad \square
\end{aligned}
$$

Our second theorem extends inequality (6) using the generalized Cartesian decomposition, providing a broader framework for numerical radius inequalities.

THEOREM 6. *Let* $A\in\mathbb{B}(\mathbb{H})$ *with the generalized cartesian decomposition* $A=\Re_\lambda(A)+i\Im_\lambda(A)$ *and* f *be an increasing concave function. Then*

$$
f\big(w^2(A)\big)\leqslant \big|\big|f\big(|\Re_\lambda(A)|^2\big)+f\big(|\Im_\lambda(A)|^2\big)\big|\big|.
$$

Proof. For every unit vector $x\in\mathbb{H}$, we have

$$
\begin{aligned}
f\big(w^2(A)\big)&=f\left(\sup_{||x||=1}|\langle Ax,x\rangle|^2\right)\\
&=\sup_{||x||=1} f\big(|\langle Ax,x\rangle|^2\big)\\
&=\sup_{||x||=1} f\big(\langle \Re_\lambda(A)x,x\rangle^2+\langle \Im_\lambda(A)x,x\rangle^2\big)
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant \sup_{||x||=1} f\big(\langle|\Re_\lambda(A)|x,x\rangle^2+\langle|\Im_\lambda(A)|x,x\rangle^2\big)\\
&\leqslant \sup_{||x||=1} f\big(\langle|\Re_\lambda(A)|^2x,x\rangle+\langle|\Im_\lambda(A)|^2x,x\rangle\big) \quad \text{(by Lemma 1)}\\
&= \sup_{||x||=1} f\Big(\big\langle\big(|\Re_\lambda(A)|^2+|\Im_\lambda(A)|^2\big)x,x\big\rangle\Big)\\
&= f\Big(\sup_{||x||=1}\big\langle\big(|\Re_\lambda(A)|^2+|\Im_\lambda(A)|^2\big)x,x\big\rangle\Big)\\
&= f\big(\big|\big|\Re_\lambda(A)|^2+|\Im_\lambda(A)|^2\big|\big|\big)\\
&= \big|\big|f\big(\Re_\lambda(A)|^2+|\Im_\lambda(A)|^2\big)\big|\big|\\
&\leqslant \big|\big|f\big(\Re_\lambda(A)|^2\big)+f\big(|\Im_\lambda(A)|^2\big)\big|\big| \quad \text{(by Lemma 2)}. \qquad \square
\end{aligned}
$$

REMARK 3. The inequality (5) comes from Theorem 6 when $\lambda=1$ and $f(t)=t^r$ with $0<r\leqslant 1$.

In the following theorem, we broaden the applicability of inequality (6).

THEOREM 7. *Let $A\in\mathbb{B}(\mathbb{H})$ with the generalized cartesian decomposition $A=\Re_\lambda(A)+i\Im_\lambda(A)$ and f be an increasing double convex function and $f(1)=1$. Then*

$$
f^r\left(\frac{w(A)}{\sqrt{2}}\right)\leqslant\left|\left|\frac{f(|\Re_\lambda(A)|^r)+f(|\Im_\lambda(A)|^r)}{2}\right|\right|,
$$

where $r\geqslant 2$.

Proof. For every unit vector $x\in\mathbb{H}$, we have

$$
\begin{aligned}
f\left(\frac{|\langle Ax,x\rangle|}{\sqrt{2}}\right)&=f\left(\left(\frac{\langle\Re_\lambda(A)x,x\rangle^2+\langle\Im_\lambda(A)x,x\rangle^2}{2}\right)^{\frac12}\right)\\
&\leqslant f\left(\left(\frac{\langle|\Re_\lambda(A)|x,x\rangle^2+\langle|\Im_\lambda(A)|x,x\rangle^2}{2}\right)^{\frac12}\right)\\
&\leqslant f\left(\left(\frac{\langle|\Re_\lambda(A)|x,x\rangle^r+\langle|\Im_\lambda(A)|x,x\rangle^r}{2}\right)^{\frac1r}\right) \quad \text{(by Lemma 3)}\\
&\leqslant \left(f\left(\frac{\langle|\Re_\lambda(A)|x,x\rangle^r+\langle|\Im_\lambda(A)|x,x\rangle^r}{2}\right)\right)^{\frac1r}\big(f(1)\big)^{1-\frac1r}\\
&\leqslant \left(f\left(\frac{\langle|\Re_\lambda(A)|^rx,x\rangle+\langle|\Im_\lambda(A)|^rx,x\rangle}{2}\right)\right)^{\frac1r} \quad \text{(by Lemma 1)}\\
&\leqslant \left(\frac{f(\langle|\Re_\lambda(A)|^rx,x\rangle)+f(\langle|\Im_\lambda(A)|^rx,x\rangle)}{2}\right)^{\frac1r}
\end{aligned}
$$

$$\leqslant \left(\frac{\langle f(|\Re_\lambda(A)|^r)x,x\rangle + \langle f(|\Im_\lambda(A)|^r)x,x\rangle}{2} \right)^{\frac{1}{r}} \quad \text{(by Lemma 4)}$$
$$= \left(\frac{\langle \big(f(|\Re_\lambda(A)|^r) + f(|\Im_\lambda(A)|^r)\big)x,x\rangle}{2} \right)^{\frac{1}{r}}.$$

Since f is continuous and increasing, we have

$$\begin{aligned} f^r\left(\frac{w(A)}{\sqrt{2}}\right) &= f^r\left(\sup_{||x||=1} \frac{|\langle Ax,x\rangle|}{\sqrt{2}}\right) \\ &= \sup_{||x||=1} f^r\left(\frac{|\langle Ax,x\rangle|}{\sqrt{2}}\right) \\ &\leqslant \sup_{||x||=1} \frac{\big\langle \big(f(|\Re_\lambda(A)|^r) + f(|\Im_\lambda(A)|^r)\big)x,x\big\rangle}{2} \\ &= \left|\left| \frac{f(|\Re_\lambda(A)|^r) + f(|\Im_\lambda(A)|^r)}{2} \right|\right|. \quad \square \end{aligned}$$

REMARK 4. The inequality (6) comes from Theorem 7 when $\lambda = 1$ and $f(t) = t$.

Another theorem, which presents the lower bound of $f\big(w(A)\big)$ using the generalized Cartesian decomposition, is given by the following result. This theorem establishes a connection between the function f and the numerical radius. The generalized Cartesian decomposition plays a crucial role in deriving this bound. We now state the theorem that provides this lower bound.

THEOREM 8. *Let $A \in \mathbb{B}(\mathbb{H})$ with the generalized cartesian decomposition $A = \Re_\lambda(A) + i\Im_\lambda(A)$ and f be an increasing concave function. Then*

$$\frac{1}{2}\left|\left| f\big(|\Re_\lambda(A) + \Im_\lambda(A)|\big) + f\big(|\Re_\lambda(A) - \Im_\lambda(A)|\big) \right|\right| \leqslant f\big(w(A)\big).$$

Proof. For every unit vector $x \in \mathbb{H}$, we have

$$\begin{aligned} f\big(w(A)\big) &= f\left(\sup_{||x||=1} |\langle Ax,x\rangle|\right) \\ &= \sup_{||x||=1} f\big(|\langle Ax,x\rangle|\big) \\ &= \sup_{||x||=1} f\left(\big(\langle \Re_\lambda(A)x,x\rangle^2 + \langle \Im_\lambda(A)x,x\rangle^2\big)^{\frac{1}{2}}\right) \\ &= \sup_{||x||=1} f\left(\left(\frac{\big(\langle(\Re_\lambda(A)+\Im_\lambda(A))x,x\rangle\big)^2 + \big(\langle(\Re_\lambda(A)-\Im_\lambda(A))x,x\rangle\big)^2}{2}\right)^{\frac{1}{2}}\right) \\ &\geqslant \sup_{||x||=1} f\left(\frac{\big|\langle(\Re_\lambda(A)+\Im_\lambda(A))x,x\rangle\big| + \big|\langle(\Re_\lambda(A)-\Im_\lambda(A))x,x\rangle\big|}{2}\right) \end{aligned}$$

$$
\begin{aligned}
&\geqslant \sup_{||x||=1} \frac{f\left(\left|\langle(\Re_\lambda(A)+\Im_\lambda(A))x,x\rangle\right|\right)+f\left(\left|\langle(\Re_\lambda(A)-\Im_\lambda(A))x,x\rangle\right|\right)}{2}\\
&=\frac{1}{2}\left(f\left(\sup_{||x||=1}\left|\langle(\Re_\lambda(A)+\Im_\lambda(A))x,x\rangle\right|\right)\right.\\
&\quad\left.+f\left(\sup_{||x||=1}\left|\langle(\Re_\lambda(A)-\Im_\lambda(A))x,x\rangle\right|\right)\right)\\
&=\frac{1}{2}\left(f\left(||\Re_\lambda(A)+\Im_\lambda(A)||\right)+f\left(||\Re_\lambda(A)-\Im_\lambda(A)||\right)\right)\\
&=\frac{1}{2}\left(||f\left(|\Re_\lambda(A)+\Im_\lambda(A)|\right)||+||f\left(|\Re_\lambda(A)-\Im_\lambda(A)|\right)||\right)\\
&\geqslant\frac{1}{2}||f\left(|\Re_\lambda(A)+\Im_\lambda(A)|\right)+f\left(|\Re_\lambda(A)-\Im_\lambda(A)|\right)||. \quad \square
\end{aligned}
$$

In the next remark, we provide an example to illustrate our theorem, offering a practical demonstration to clarify its application and the concepts discussed.

REMARK 5. Putting $\lambda=1$ and $f(t)=t^r$ when $0<r\leqslant 1$ in Theorem 8, we get

$$\frac{1}{2}\left|\left||\Re(A)+\Im(A)|^r+|\Re(A)-\Im(A)|^r\right|\right|\leqslant w^r(A),$$

which can be regard as a complement of the first inequality (7). Taking $A=\begin{pmatrix}1&0\\0&1\end{pmatrix}$, then $\frac{1}{2}\left|\left||\Re(A)+\Im(A)|+|\Re(A)-\Im(A)|\right|\right|=1=w(A)$, which implies the obtained result is sharp.

Motivated by the ideas in Theorem 7 and Theorem 8, we now present a generalization of the inequality in (7), which extends the previous result and provides a broader perspective on the topic.

THEOREM 9. *Let $A\in\mathbb{B}(\mathbb{H})$ with the generalized cartesian decomposition $A=\Re_\lambda(A)+i\Im_\lambda(A)$ and f be an increasing double convex function with $f(1)=1$. Then*

$$f^r\left(w(A)\right)\leqslant\frac{1}{2}||f\left(|\Re_\lambda(A)+\Im_\lambda(A)|^r\right)+f\left(|\Re_\lambda(A)-\Im_\lambda(A)|^r\right)||,$$

where $r\geqslant 2$.

Proof. For every unit vector $x\in\mathbb{H}$, we have

$$
\begin{aligned}
f\left(|\langle Ax,x\rangle|\right)&=f\left(\left(\langle\Re_\lambda(A)x,x\rangle^2+\langle\Im_\lambda(A)x,x\rangle^2\right)^{\frac{1}{2}}\right)\\
&=f\left(\left(\frac{\left(\langle(\Re_\lambda(A)+\Im_\lambda(A))x,x\rangle\right)^2+\left(\langle(\Re_\lambda(A)-\Im_\lambda(A))x,x\rangle\right)^2}{2}\right)^{\frac{1}{2}}\right)\\
&\leqslant\left(\frac{\langle\left(f(|\Re_\lambda(A)+\Im_\lambda(A)|^r)+f(|\Re_\lambda(A)-\Im_\lambda(A)|^r)\right)x,x\rangle}{2}\right)^{\frac{1}{r}}.
\end{aligned}
$$

Since f is continuous and increasing, we have

$$\begin{aligned} f^r\big(w(A)\big) &= f^r\left(\sup_{||x||=1}\left|\langle Ax,x\rangle\right|\right) \\ &= \sup_{||x||=1} f^r\big(|\langle Ax,x\rangle|\big) \\ &\leqslant \sup_{||x||=1} \frac{\big\langle \big(f(|\Re_\lambda(A)+\Im_\lambda(A)|^r)+f(|\Re_\lambda(A)-\Im_\lambda(A)|^r)\big)x,x\big\rangle}{2} \\ &= \frac{1}{2}\left|\left|f\big(|\Re_\lambda(A)+\Im_\lambda(A)|^r\big)+f\big(|\Re_\lambda(A)-\Im_\lambda(A)|^r\big)\right|\right|. \quad \square \end{aligned}$$

REMARK 6. The second inequality in (7) follows from Theorem 9 when $\lambda = 1$ and $f(t) = t$. This specific case simplifies the general result of the theorem. Thus, we obtain the desired inequality for these parameter values.

Concluding remark

In this paper, we derived new upper bounds for the numerical radius of bounded linear operators on Hilbert spaces by employing the Moore-Penrose inverse and the generalized Cartesian decomposition. Our approach, based on increasing doubly convex functions together with refined inequalities of Young and Buzano type, provides a unified framework that extends and improves several known results in the literature.

In particular, our inequalities extend some of Bhunia, Kittaneh, and Sahoo [6], where numerical radius estimates involving the Moore-Penrose inverse were first explored. Our results generalize their bounds by introducing a flexible weight function ζ and a functional parameter f, which lead to strictly stronger estimates in many cases. Furthermore, our results complement those of Bhunia, Jana, and Paul [7], who refined lower bounds of the numerical radius using the classical Cartesian decomposition; by contrast, we established new upper bounds through the λ-generalized Cartesian decomposition, showing that their inequalities are recovered when $\lambda = 1$. Moreover, our work is closely related to that of Barik, Bhunia, and Paul [3], who recently introduced the P-Cartesian decomposition and the P-numerical radius. Our results correspond to the special case $P = I$, and thus naturally suggest a possible extension of our methods to the semi-Hilbertian setting.

The results presented here enhance the existing theory of numerical radius inequalities and open potential directions for future research. In particular, it would be interesting to investigate extensions of our inequalities to P-numerical radii, reverse-type bounds, operator matrices, and applications to perturbation theory and spectral approximation problems.

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Declarations

Author contributions. All authors contributed almost the same amount of work to manuscript. All authors have read and approved the final version of the manuscript for publication.

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