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UPPER BOUNDS OF NUMERICAL RADIUS AND a–NUMERICAL RADIUS IN C^*–ALGEBRA SETTING USING ORLICZ FUNCTIONS

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Abstract. In this paper, several significant upper bounds for the numerical radius and a-numerical radius of an element in a C^*-algebra are obtained using Orlicz functions. Many well-known results are obtained from our findings, depending on specific choices of Orlicz functions.

1. Introduction and preliminaries

Let $\mathcal{A}$ be a unital C^*-algebra with the unit denoted by $\mathbf{1}$ and $\mathcal{A}^*$ represents the dual space of $\mathcal{A}$. A linear functional $g \in \mathcal{A}^*$ is said to be positive if $g(x^*x) \geqslant 0$ for all $x \in \mathcal{A}$ and denoted by $g \geqslant 0$. By $\mathcal{S}(\mathcal{A})$ we denote the set of all states on $\mathcal{A}$, i.e., $\mathcal{S}(\mathcal{A}) = \{g \in \mathcal{A}^* : g \geqslant 0 \text{ and } g(\mathbf{1}) = 1\}$. The algebraic numerical range $V(x)$ and the algebraic numerical radius $v(x)$ of an element $x \in \mathcal{A}$ are defined respectively by

$$V(x) = \{g(x) : g \in \mathcal{S}(\mathcal{A})\}, \ \ v(x) = \sup\{|z| : z \in V(x)\}.$$

Let $\mathcal{B}(H)$ be the unital C^*-algebra of all bounded linear operators on a complex separable Hilbert space $\big(H, \langle\cdot\rangle\big)$. In particular, if $\mathcal{A} = \mathcal{B}(H)$ and $T \in \mathcal{B}(H)$, the spatial numerical range and the spatial numerical radius of T are denoted by $W(T)$ and $\omega(T)$, respectively. It is well known that $V(T)$ is the closure of $W(T) = \big\{\big\langle Tx, x\big\rangle : x \in H, \|x\| = 1\big\}$.

For any $x \in \mathcal{A}$, the fundamental inequality between the numerical radius $v(x)$ and the C^*-norm is as follows:

$$\frac{1}{2}\|x\| \leqslant v(x) \leqslant \|x\|. \tag{1.1}$$

A positive element $x \in \mathcal{A}$ is denoted by $x \geqslant 0$. To indicate that $x - y \geqslant 0$, we use $x \geqslant y$. By convention, the value of x^0 is taken as $\mathbf{1}$ for any element $x \in \mathcal{A}$. Let $\mathcal{A}^+$ denote the set of all non-zero positive elements in $\mathcal{A}$. Now, for a non-zero $a \in \mathcal{A}^+$, we

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define $\mathcal{S}_a(\mathcal{A}) = \{f \in \mathcal{A}^* : f \geqslant 0,\ f(a) = 1\}$ and it is nothing but $\mathcal{S}_a(\mathcal{A}) = \left\{\frac{g}{g(a)} : g \in \mathcal{S}(\mathcal{A}),\ g(a) \neq 0\right\}$. Throughout this paper, a stands for a non-zero positive element of $\mathcal{A}$ unless otherwise specified. Recently, a-numerical range and a-numerical radius of elements in unital $\mathcal{C}^*$-algebras were introduced and studied by Bourhim and Mabrouk in [9]. The a-numerical range and a-numerical radius of an element $x \in \mathcal{A}$ are defined, respectively, by

$$V_a(x) = \{f(ax) : f \in \mathcal{S}_a(\mathcal{A})\},\ \ v_a(x) = \sup\{|z| : z \in V_a(x)\}.$$

In particular, when $a = \mathbf{1}$, then a-numerical range and a-numerical radius of x become algebraic numerical range and algebraic numerical radius of x respectively, i.e., $V_a(x) = V(x)$ and $v_a(x) = v(x)$. These ideas were presented in [9] as an extensions of the spatial A-numerical range and spatial A-numerical radius of a bounded linear operator $T \in \mathcal{B}(H)$ defined respectively by $W_A(T) = \left\{\left\langle Tx, x\right\rangle_A : x \in H, \|x\|_A = 1\right\}$ and $w_A(T) = \sup\{|z| : z \in W_A(T)\}$. Here A be a positive bounded operator on a Hilbert space $(H, \langle\cdot\rangle)$ and $\|x\|_A^2 = \langle x, x\rangle_A = \langle Ax, x\rangle$, $x \in H$.

For an element $x \in \mathcal{A}$, let $\|x\|_a = \sup\{\sqrt{f(x^*ax)} : f \in \mathcal{S}_a(\mathcal{A})\}$. It is clear that $\|x\|_a = 0$ if and only if $ax = 0$. Note that, it may happen that $\|x\|_a = \infty$ for some $x \in \mathcal{A}$. Let $\mathcal{A}^a$ denote the set of all such elements $x \in \mathcal{A}$ such that $\|x\|_a < \infty$. From [9, Proposition 3.3] it is known that $\|\cdot\|_a$ is a semi-norm on $\mathcal{A}^a$ and satisfies $\|xy\|_a \leqslant \|x\|_a\|y\|_a$ for all $x, y \in \mathcal{A}^a$. Thus $\mathcal{A}^a$ is a subalgebra of $\mathcal{A}$. For $x \in \mathcal{A}$, an element $x^{\#_a} \in \mathcal{A}$ is said to be an a-adjoint of x if $ax^{\#_a} = x^*a$. The existence and uniqueness of a-adjoint elements for $x \in \mathcal{A}$ are not assured. The collection of all elements in $\mathcal{A}$ that have a-adjoints is denoted by $\mathcal{A}_a$. That is

$$\mathcal{A}_a = \{x \in \mathcal{A} : \text{there is } x^{\#_a} \in \mathcal{A} \text{ such that } ax^{\#_a} = x^*a\}.$$

Additionally, $\mathcal{A}_a$ is a subalgebra of $\mathcal{A}$ and $\mathcal{A}_a$ is a subset of $\mathcal{A}^a$ (see [9]). If $x \in \mathcal{A}_a$ and $x^{\#_a}$ is an a-adjoint of x, then by [9, Corollary 4.9]

$$\|x\|_a^2 = \|xx^{\#_a}\|_a = \|x^{\#_a}x\|_a = \|x^{\#_a}\|_a^2.$$

An element $x \in \mathcal{A}$ is considered a-self adjoint if ax is self-adjoint, i.e., $ax = x^*a$. It is noted that every element x in $\mathcal{A}_a$ can be expressed as $x = y + iz$, where y and z are a-self-adjoint but this decomposition is not unique in general. In fact if $x^{\#_a}$ is an a-adjoint of x, then $x = \Re_a(x) + i\Im_a(x)$, where $\Re_a(x) = \frac{x+x^{\#_a}}{2}$ and $\Im_a(x) = \frac{x-x^{\#_a}}{2i}$. An element $x \in \mathcal{A}$ is said to be a-positive if ax is positive, i.e., $ax \geqslant 0$ and it is denoted by $x \geqslant_a 0$.

In [9], Bourhim and Mabrouk have shown that for any $x \in \mathcal{A}_a$, the inequality between the a-numerical radius $v_a(x)$ and the semi-norm $\|\cdot\|_a$ holds as below:

$$\frac{1}{2}\|x\|_a \leqslant v_a(x) \leqslant \|x\|_a. \tag{1.2}$$

If x is a-self-adjoint, the second part of the above inequality becomes equality and the first part of the above inequality becomes equality if $ax^2 = 0$. In [19], Mabrouk and Zamani further have given some upper bounds of a-numerical radius, which are as follow:

$$v_a^2(x) \leqslant \frac{1}{2}\|xx^{\#_a}+x^{\#_a}x\|_a \tag{1.3}$$

$$v_a^2(x) \leqslant \frac{1}{2}v_a(x^2)+\frac{1}{4}\|xx^{\#_a}+x^{\#_a}x\|_a. \tag{1.4}$$

Clearly, inequality in (1.3) refines the right side inequality in (1.2) and the inequality in (1.4) is sharper than the inequality in (1.3) for the power inequality $v_a(x^2) \leqslant v_a^2(x)$. It is worth mentioning that several research articles exist in the literature that focus on the estimation of quantities $w(T)$, $w_A(T)$, $v(x)$, and $v_a(x)$. Here, we refer to the articles [5, 7, 9, 11, 13, 14, 15, 17, 19, 23] and the books [4, 12].

In this article, by employing the concept of Orlicz functions, we obtain several upper bounds of a-numerical radius of an element in unital $\mathcal{C}^*$-algebras. In addition, several well-known results are established from our findings. Some examples are provided to supplement our results.

2. Main result

Throughout this article, let $\mathcal{A}$ denote a complex unital $\mathcal{C}^*$-algebra with unit $\mathbf{1}$. We start this section with the following lemmas, which will be useful for our study.

LEMMA 2.1. [8] *Let g be a non-zero positive linear functional on a $\mathcal{C}^*$-algebra $\mathcal{A}$. Then $|g(x^*y)|^2 \leqslant g(x^*x)g(y^*y)$ for all $x,y \in \mathcal{A}$.*

As a consequence of Lemma 2.1 we have the following result.

LEMMA 2.2. *Let $f \in \mathcal{S}_a(\mathcal{A})$ and $x,y \in \mathcal{A}$, then $|f(x^*ay)|^2 \leqslant f(x^*ax)f(y^*ay)$.*

LEMMA 2.3. [1] *Let $\mathcal{A}$ be a unital $\mathcal{C}^*$ -algebra and $g \in \mathcal{S}(\mathcal{A})$. If $x \in \mathcal{A}^+$, then for every continuous, convex function ϕ on $[0,\infty)$, the inequality $\phi\big(g(x)\big) \leqslant g\big(\phi(x)\big)$ holds.*

COROLLARY 2.4. *Let $\mathcal{A}$ be a unital C^*-algebra. If $x \in \mathcal{A}^+$, $g \in \mathcal{S}(\mathcal{A})$ and $0 < r \leqslant 1$, then*

$$\big(g(x)\big)^r \geqslant g\big((x)^r\big).$$

Proof. Taking $\phi(t)=t^s$, $t \geqslant 0$ and $s \geqslant 1$ in Lemma 2.3, we get $\big(g(x)\big)^s \leqslant g\big((x)^s\big)$. Since $x \in \mathcal{A}^+$ so, $\big(g(x)\big)^r = \left(g\big((x^r)^{\frac{1}{r}}\big)\right)^r$. Since $\frac{1}{r} \geqslant 1$ and $g \geqslant 0$, we get $g\big((x^r)^{\frac{1}{r}}\big) \geqslant \big(g(x^r)\big)^{\frac{1}{r}}$. Hence, the result follows. □

If ϕ is a continuous, convex function on $[0,\infty)$ with $\phi(0)=0$, then using Lemma 2.3 we get the following useful results.

LEMMA 2.5. *Let x be an a-self-adjoint element in a unital $\mathcal{C}^*$-algebra $\mathcal{A}$ such that the spectrum of ax is contained in $[0,\infty)$ and $a \geqslant \mathbf{1}$. If $\phi(t)$ is a continuous, convex function on $[0,\infty)$ with $\phi(0)=0$, then for all $f \in \mathcal{S}_a(\mathcal{A})$*

$$f\big(\phi(ax)\big) \geqslant \phi\big(f(ax)\big).$$

Proof. Let $f \in \mathcal{S}_a(\mathcal{A})$. Then there exists $g \in \mathcal{S}(\mathcal{A})$ such that $f = \frac{g}{g(a)}$, where $g(a) \neq 0$. Since $a \geqslant \mathbf{1}$, it implies $\frac{1}{g(a)} \leqslant 1$. Again $\phi(0)=0$, it gives $\phi(\lambda t) \leqslant \lambda\phi(t)$ for $0 \leqslant \lambda \leqslant 1$. Now, using Lemma 2.3, we get

$$f\big(\phi(ax)\big) = \frac{g\big(\phi(ax)\big)}{g(a)} \geqslant \frac{\phi\big(g(ax)\big)}{g(a)} \geqslant \phi\left(\frac{g(ax)}{g(a)}\right) = \phi\big(f(ax)\big). \quad \square$$

COROLLARY 2.6. *Let $\mathcal{A}$ be a unital $\mathcal{C}^*$-algebra. If x is an a-self adjoint element in $\mathcal{A}$ such that the spectrum of ax is contained in $[0,\infty)$, $a \geqslant \mathbf{1}$ and $f \in \mathcal{S}_a(\mathcal{A})$, then for any $r \geqslant 1$*

$$\big(f(ax)\big)^r \leqslant f\big((ax)^r\big).$$

Proof. Taking $\phi(t) = t^r$, $t \geqslant 0$ and $r \geqslant 1$ in Lemma 2.5, we get the above inequality. $\square$

LEMMA 2.7. *Let $x,y \in \mathcal{A}_a$ and $f \in \mathcal{S}_a(\mathcal{A})$. Then*

$$f(y^*x^*axy) \leqslant \|x\|_a^2 f(y^*ay).$$

Proof. See the Proposition 3.3 of [9]. $\square$

Next result follows from [2, Lemma 2.12].

LEMMA 2.8. *Let $\big(H,\langle\cdot\rangle\big)$ be a semi inner product space. If $x,y,e \in H$ with $\langle e,e\rangle = 1$, then*

$$\big|\langle x,e\rangle\langle e,y\rangle\big| \leqslant \frac{1}{2}\Big(\sqrt{\langle x,x\rangle}\sqrt{\langle y,y\rangle} + \big|\langle x,y\rangle\big|\Big).$$

Let $\mathcal{A}$ be a complex unital $\mathcal{C}^*$-algebra. Now, take $\mathcal{A}^n = \{(x_1,x_2,\cdots,x_n) : x_i \in \mathcal{A}\}$. Let $(x_1,x_2,\cdots,x_n),(y_1,y_2,\cdots,y_n) \in \mathcal{A}^n$ and $\lambda \in \mathbb{C}$. Define

$$\begin{aligned}(x_1,x_2,\cdots,x_n)+(y_1,y_2,\cdots,y_n) &= (x_1+y_1,x_2+y_2,\cdots,x_n+y_n)\\ \lambda(x_1,x_2,\cdots,x_n) &= (\lambda x_1,\lambda x_2,\cdots,\lambda x_n).\end{aligned}$$

Under these operations, $\mathcal{A}^n$ forms a complex vector space.

A vector $\mathbf{P} = (p_1,p_2,\cdots,p_n)$ is called probability distribution over n elements if $p_i \geqslant 0$ for every $i \in \{1,2,\cdots,n\}$ and $\sum_{i=1}^n p_i = 1$. The set of all such distributions over n elements is denoted by $\mathcal{P}_n$. The following two lemmas are generalizations of Lemma 2.8 in a $\mathcal{C}^*$-algebra setting.

LEMMA 2.9. *Let $\mathcal{A}$ be a unital C^*-algebra and $\boldsymbol{P}=(p_1,p_2,\cdots,p_n)\in\mathcal{P}_n$ be a probability distribution. If $X=(x_1,x_2,\cdots,x_n)$, $Y=(y_1,y_2,\cdots,y_n)\in\mathcal{A}^n$, and $f\in\mathcal{S}_a(\mathcal{A})$, then*

$$\begin{aligned}&\Big|\sum_{i=1}^{n}p_i f(ax_i)\Big|\Big|\sum_{i=1}^{n}p_i f(y_i^*a)\Big|\\ &\leqslant \frac{\Big|\sum_{i=1}^{n}p_i f(y_i^*ax_i)\Big|+\sqrt{\sum_{i=1}^{n}p_i f(x_i^*ax_i)}\sqrt{\sum_{i=1}^{n}p_i f(y_i^*ay_i)}}{2}.\end{aligned}$$

Proof. Let $\mathcal{A}$ be a unital C^*-algebra with unit $\mathbf{1}$ and $X=(x_1,x_2,\cdots,x_n)$, $Y=(y_1,y_2,\cdots,y_n)$, $I=(\mathbf{1},\mathbf{1},\cdots,\mathbf{1})\in\mathcal{A}^n$. Define a semi-inner product on $\mathcal{A}^n$ by

$$\langle X,Y\rangle=\sum_{i=1}^{n}p_i f(y_i^*ax_i),$$

which satisfies $\langle I,I\rangle=1$. Now, from Lemma 2.8 for the vectors X,Y and I, we get the required inequality. $\square$

As a consequence of Lemma 2.9 we have the following result.

LEMMA 2.10. *Let $x_i,y_i\in\mathcal{A}_a$, $i=1,2,\cdots,n$. If $\boldsymbol{P}=(p_1,p_2,\cdots,p_n)\in\mathcal{P}_n$ is a probability distribution and $f\in\mathcal{S}_a(\mathcal{A})$, then*

$$\begin{aligned}&\Big|\sum_{i=1}^{n}p_i f(ax_i)\Big|\Big|\sum_{i=1}^{n}p_i f(ay_i)\Big|\\ &\leqslant \frac{1}{2}\sqrt{\sum_{i=1}^{n}p_i f(ax_i^{\#_a}x_i)}\sqrt{\sum_{i=1}^{n}p_i f(ay_iy_i^{\#_a})}+\frac{1}{2}\Big|\sum_{i=1}^{n}p_i f(ay_ix_i)\Big|.\end{aligned}$$

Proof. Since $y_i\in\mathcal{A}_a$ so there is $y_i^{\#_a}\in\mathcal{A}$ such that $ay_i^{\#_a}=y_i^*a$ for all $1\leqslant i\leqslant n$. Hence $Y^{\#_a}=(y_1^{\#_a},y_2^{\#_a},\cdots,y_n^{\#_a})\in\mathcal{A}^n$. Now, if we apply Lemma 2.9 for the elements $X=(x_1,x_2,\cdots,x_n)$ and $Y^{\#_a}=(y_1^{\#_a},y_2^{\#_a},\cdots,y_n^{\#_a})$, then we get the following inequality

$$\begin{aligned}&\Big|\sum_{i=1}^{n}p_i f(ax_i)\Big|\Big|\sum_{i=1}^{n}p_i f\big((y_i^{\#_a})^*a\big)\Big|\\ &\leqslant \frac{1}{2}\sqrt{\sum_{i=1}^{n}p_i f(x_i^*ax_i)}\sqrt{\sum_{i=1}^{n}p_i f\big((y_i^{\#_a})^*ay_i^{\#_a}\big)}+\frac{1}{2}\Big|\sum_{i=1}^{n}p_i f\big((y_i^{\#_a})^*ax_i\big)\Big|.\end{aligned}$$

Again, we know that if $f \in \mathcal{S}_a(\mathcal{A})$, then $f(x^*) = \overline{f(x)}$. Now, from the above inequality, we get

$$\begin{aligned}
&\Big|\sum_{i=1}^{n} p_i f(ax_i)\Big|\Big|\sum_{i=1}^{n} p_i f(ay_i)\Big| \\
&\leqslant \frac{1}{2}\sqrt{\sum_{i=1}^{n} p_i f(ax_i^{\#_a}x_i)}\sqrt{\sum_{i=1}^{n} p_i f(ay_i y_i^{\#_a})} + \frac{1}{2}\Big|\sum_{i=1}^{n} p_i f(x_i^* a y_i^{\#_a})\Big| \\
&= \frac{1}{2}\sqrt{\sum_{i=1}^{n} p_i f(ax_i^{\#_a}x_i)}\sqrt{\sum_{i=1}^{n} p_i f(ay_i y_i^{\#_a})} + \frac{1}{2}\Big|\sum_{i=1}^{n} p_i f(x_i^* y_i^* a)\Big| \\
&= \frac{1}{2}\sqrt{\sum_{i=1}^{n} p_i f(ax_i^{\#_a}x_i)}\sqrt{\sum_{i=1}^{n} p_i f(ay_i y_i^{\#_a})} + \frac{1}{2}\Big|\sum_{i=1}^{n} p_i f(ay_i x_i)\Big|. \qquad \square
\end{aligned}$$

Now, we derive several upper bounds of a-numerical radius of an element in a $\mathcal{C}^*$-algebra using Orlicz function.

A map $\phi : [0,\infty) \to [0,\infty)$ is said to be an Orlicz function if it is continuous, convex, non-decreasing, $\phi(0) = 0$ and $\phi(u) \to \infty$ as $u \to \infty$ (see [18]). An Orlicz function is said to be non-degenerate if $\phi(u) > 0$ when $u > 0$ and it is said to be degenerate if $\phi(u) = 0$ for some $u > 0$. Throughout this paper, whenever the Orlicz function appears, we interpret it as non-degenerate. An Orlicz function can be expressed by $\phi(u) = \int_0^u p(t)dt$, where p is a non decreasing function such that $p(0) = 0$, $p(t) > 0$ for $t > 0$, $\lim_{t\to\infty} p(t) = \infty$. When $\phi(u)$ is equivalent to the function u then those restrictions on p are excluded. Let q be the right inverse of p and it is defined as $q(s) = \sup\{t : p(t) \leqslant s\}$, $s \geqslant 0$. The Orlicz function ψ, defined as $\psi(v) = \int_0^v q(s)ds$, is called the complementary Orlicz function of ϕ. If $\phi(t) = \frac{t^P}{p}$, $t \geqslant 0$ and $p > 1$ then the corresponding Orlicz function is $\psi(t) = \frac{t^q}{q}$, where $\frac{1}{p} + \frac{1}{q} = 1$. Recently, Manna and Majhi [20] derived several numerical radius inequalities using Orlicz functions.

LEMMA 2.11. [18] *Let ϕ and ψ be two complementary Orlicz functions. Then for any $x, y \geqslant 0$, the following inequality holds*

$$xy \leqslant \phi(x) + \psi(y).$$

The following lemma is used to prove the Theorem 2.13.

LEMMA 2.12. *Let $x \in \mathcal{A}_a$ and ϕ be a Orlicz function. If $a \geqslant \mathbf{1}$, $f \in \mathcal{S}_a(\mathcal{A})$ and $0 \leqslant \alpha \leqslant 1$, then*

$$\phi\left(\left|f(ax)\right|^2\right) \leqslant \alpha f\big(\phi(ax^{\#_a}x)\big) + (1-\alpha) f\big(\phi(axx^{\#_a})\big).$$

Proof. Let $x \in \mathcal{A}_a$ and $f \in \mathcal{S}_a(\mathcal{A})$. Then we have

$$\begin{aligned}
\big|f(ax)\big|^2 &= \alpha\big|f(ax)\big|^2 + (1-\alpha)\big|f(ax)\big|^2 \\
&= \alpha\big|f(ax)\big|^2 + (1-\alpha)\big|\overline{f(ax)}\big|^2 \\
&= \alpha\big|f(ax)\big|^2 + (1-\alpha)\big|f(x^*a)\big|^2 \\
&= \alpha\big|f(ax)\big|^2 + (1-\alpha)\big|f(ax^{\#_a})\big|^2 \\
&\leqslant \alpha f(x^*ax) + (1-\alpha) f\big((x^{\#_a})^*ax^{\#_a}\big) \\
&= \alpha f(x^*ax) + (1-\alpha) f\big((x^{\#_a})^*x^*a\big) \\
&= \alpha f(ax^{\#_a}x) + (1-\alpha) f(axx^{\#_a}),
\end{aligned}$$

where the third and last equality come from the fact that $f(y^*) = \overline{f(y)}$ for all $y \in \mathcal{A}$. Applying Lemma 2.2, it follows that $|f(ay)|^2 = |f(\mathbf{1}^*ay)|^2 \leqslant f(a)f(y^*ay) = f(y^*ay)$ holds for all $y \in \mathcal{A}$. Therefore, the above inequality holds.

Now, using the non-decreasing and convexity property of ϕ and from Lemma 2.5, we get

$$\begin{aligned}
\phi\Big(\big|f(ax)\big|^2\Big) &\leqslant \phi\Big(\alpha f(ax^{\#_a}x) + (1-\alpha) f(axx^{\#_a})\Big) \\
&\leqslant \alpha\ \phi\big(f(ax^{\#_a}x)\big) + (1-\alpha)\phi\big(f(axx^{\#_a})\big) \qquad (2.1) \\
&\leqslant \alpha\ f\big(\phi(ax^{\#_a}x)\big) + (1-\alpha) f\big(\phi(axx^{\#_a})\big). \quad \square
\end{aligned}$$

THEOREM 2.13. *Let $\mathcal{A}$ be a unital $\mathcal{C}^*$-algebra and $0 \leqslant \alpha \leqslant 1$, then the following results hold:*

(a) *If $x \in \mathcal{A}_a$, then*

$$v_a^2(x) \leqslant \big\|\alpha x^{\#_a}x + (1-\alpha)xx^{\#_a}\big\|_a.$$

(b) *If $x \in \mathcal{A}$ and ϕ be an Orlicz function then*

$$\phi\big(v^2(x)\big) \leqslant \big\|\alpha\phi(x^*x) + (1-\alpha)\phi(xx^*)\big\|.$$

Proof.

(a) Considering $\phi(t) = t$, $t \geqslant 0$ in the inequality (2.1), we get

$$\begin{aligned}
\big|f(ax)\big|^2 &\leqslant \alpha f(ax^{\#_a}x) + (1-\alpha) f(axx^{\#_a}) \\
\big|f(ax)\big|^2 &\leqslant f\big(a(\alpha x^{\#_a}x + (1-\alpha)xx^{\#_a})\big).
\end{aligned}$$

Now, taking supremum over $f \in \mathcal{S}_a(\mathcal{A})$ on both sides of the above inequality, we get

$$v_a^2(x) \leqslant v_a\big(\alpha x^{\#_a}x + (1-\alpha)xx^{\#_a}\big).$$

Again, we know that for any a-self-adjoint element y, $v_a(y) = \|y\|_a$ holds. Hence, the result follows.

(b) Let $x \in \mathcal{A}$ and $f \in \mathcal{S}(\mathcal{A})$. Now, if we take $a = \mathbf{1}$ in Lemma 2.12, we get

$$\begin{aligned}\phi(|f(x)|^2) &\leqslant \alpha f\big(\phi(x^*x)\big) + (1-\alpha) f\big(\phi(xx^*)\big) \\ &= f\big(\alpha\, \phi(x^*x) + (1-\alpha)\phi(xx^*)\big).\end{aligned}$$

Taking supremum over $f \in \mathcal{S}(\mathcal{A})$ on both sides of the above inequality, we get

$$\phi(v^2(x)) \leqslant v\big(\alpha\phi\big(x^*x\big) + (1-\alpha)\phi\big(xx^*\big)\big).$$

Since, for a self-adjoint element $y \in \mathcal{A}$, we have $v(y) = \|y\|$, the desired inequality follows. □

The following corollary is an immediate consequence of Theorem 2.13.

COROLLARY 2.14. *Let $\mathcal{A}$ be a unital $\mathcal{C}^*$-algebra and $x \in \mathcal{A}$. If $0 \leqslant \alpha \leqslant 1$ and $r \geqslant 1$, then*

$$v^{2r}(x) \leqslant \big\|\alpha(x^*x)^r + (1-\alpha)(xx^*)^r\big\|.$$

Proof. Taking $\phi(t) = t^r$, for $t \geqslant 0$ and $r \geqslant 1$ in Theorem 2.13 (b), we get the required result. □

The above result is also derived in [3, Theorem 3.5] for the case where r is a natural number.

REMARK 2.15. Let $\mathcal{A}$ be a unital $\mathcal{C}^*$-algebra and $0 \leqslant \alpha \leqslant 1$, then we get the following known results from Theorem 2.13 and Corollary 2.14.

(i) If we take $\alpha = \frac{1}{2}$ in Theorem 2.13 (a), we get the right hand inequality of [19, Corollary 3.5].

(ii) If we consider the $\mathcal{C}^*$-algebra $\mathcal{A} = \mathcal{B}(H)$, the Theorem 2.13 (a) becomes [6, Theorem 1].

(iii) If $\alpha = \frac{1}{2}$ and $\phi(t) = t$, $t \geqslant 0$ in Theorem 2.13 (b), we get [23, Corollary 2.8].

(iv) If we take $\mathcal{A} = \mathcal{B}(H)$, in the Corollary 2.14 becomes [13, Theorem 2].

Next result is a generalization of the inequality (1.4).

THEOREM 2.16. *Let $x \in \mathcal{A}_a$ and ϕ be an Orlicz function. Then*

$$\phi\left(v_a^2(x)\right) \leqslant \frac{1}{2}\phi\left(v_a(x^2)\right) + \frac{1}{2}\phi\left(\frac{\|xx^{\#_a} + x^{\#_a}x\|_a}{2}\right).$$

Proof. Let $x \in \mathcal{A}_a$ and $f \in \mathcal{S}_a(\mathcal{A})$. Now, taking $n = 1$ in Lemma 2.10, we get

$$\left|f(ax)\right|^2 \leqslant \frac{1}{2}\left|f(ax^2)\right| + \frac{1}{2}\sqrt{f(ax^{\#_a}x)f(axx^{\#_a})}.$$

Again, using the non-decreasing and convexity property of ϕ we obtain

$$\begin{aligned}
\phi\left(\left|f(ax)\right|^2\right) &\leqslant \frac{1}{2}\phi\left(\left|f(ax^2)\right|\right) + \frac{1}{2}\phi\left(\sqrt{f(ax^{\#_a}x)f(axx^{\#_a})}\right) \\
&\leqslant \frac{1}{2}\phi\left(v_a(x^2)\right) + \frac{1}{2}\phi\left(\frac{f(ax^{\#_a}x) + f(axx^{\#_a})}{2}\right) \\
&= \frac{1}{2}\phi\left(v_a(x^2)\right) + \frac{1}{2}\phi\left(\frac{f\left(a(x^{\#_a}x + xx^{\#_a})\right)}{2}\right) \\
&\leqslant \frac{1}{2}\phi\left(v_a(x^2)\right) + \frac{1}{2}\phi\left(\frac{\|xx^{\#_a} + x^{\#_a}x\|_a}{2}\right).
\end{aligned}$$

Since, ϕ is continuous, non-decreasing and the above inequality holds for all $f \in \mathcal{S}_a(\mathcal{A})$, we get the desired result. □

An immediate consequence of Theorem 2.16 is the following corollary.

COROLLARY 2.17. *Let $x \in \mathcal{A}_a$. If $r \geqslant 1$, then*

$$v_a^{2r}(x) \leqslant \frac{1}{2}v_a^r(x^2) + \frac{1}{2^{r+1}}\left\|xx^{\#_a} + x^{\#_a}x\right\|_a^r.$$

Proof. If we take $\phi(t) = t^r$, $t \geqslant 0$ and $r \geqslant 1$ in Theorem 2.16, we get the required result. □

REMARK 2.18. Let $\mathcal{A}$ be a unital C^*-algebra. Then we get the following known results:

(i) If we take $\phi(t) = t$, $t \geqslant 0$ in Theorem 2.16, then we get [19, Theorem 3.3].

(ii) If we take $\mathcal{A} = \mathcal{B}(H)$, the Corollary 2.17 becomes [6, Theorem 2].

The following lemma is required to obtain the next theorem.

LEMMA 2.19. *Let $x \in \mathcal{A}_a$ and ϕ be an Orlicz function. If $a \geqslant \mathbf{1}$, $f \in \mathcal{S}_a(\mathcal{A})$ and $0 \leqslant \alpha \leqslant 1$, then*

(a) $\phi\left(\left|f(ax)\right|^2\right) \leqslant \frac{\alpha}{2}\phi\left(\left|f(ax^2)\right|\right) + \frac{\alpha}{4}f\left(\phi(axx^{\#_a})\right) + (1 - \frac{3\alpha}{4})f\left(\phi(ax^{\#_a}x)\right)$.

(b) $\phi\left(\left|f(ax)\right|^2\right) \leqslant \frac{\alpha}{2}\phi\left(\left|f(ax^2)\right|\right) + \frac{\alpha}{4}f\left(\phi(ax^{\#_a}x)\right) + (1 - \frac{3\alpha}{4})f\left(\phi(axx^{\#_a})\right)$.

Proof.

(a) Let $x \in \mathcal{A}_a$ and $f \in \mathcal{S}_a(\mathcal{A})$. Then we have,

$$\begin{aligned}
&\phi(|f(ax)|^2) \\
&= \alpha\phi\left(|f(ax)|^2\right) + (1-\alpha)\phi\left(|f(ax)|^2\right) \qquad (2.2)\\
&\leqslant \alpha\phi(|f(ax)|^2) + (1-\alpha)\phi(f(x^*ax)) \\
&\leqslant \alpha\ \phi\left(\frac{|f(axx)| + \sqrt{f(ax^{\#_a}x)}\sqrt{f(axx^{\#_a})}}{2}\right) + (1-\alpha)\phi(f(ax^{\#_a}x)) \\
&\leqslant \frac{\alpha}{2}\phi(|f(ax^2)|) + \frac{\alpha}{2}\phi\left(\sqrt{f(ax^{\#_a}x)f(axx^{\#_a})}\right) + (1-\alpha)\phi(f(ax^{\#_a}x)) \\
&\leqslant \frac{\alpha}{2}\phi(|f(ax^2)|) + \frac{\alpha}{2}\phi\left(\frac{f(ax^{\#_a}x)+f(axx^{\#_a})}{2}\right) + (1-\alpha)\phi(f(ax^{\#_a}x)) \\
&\leqslant \frac{\alpha}{2}\phi(|f(ax^2)|) + \frac{\alpha}{4}\phi(f(axx^{\#_a})) + \left(1-\frac{3\alpha}{4}\right)\phi(f(ax^{\#_a}x)) \qquad (2.3)\\
&\leqslant \frac{\alpha}{2}\phi(|f(ax^2)|) + \frac{\alpha}{4}f(\phi(axx^{\#_a})) + \left(1-\frac{3\alpha}{4}\right)f(\phi(ax^{\#_a}x)),
\end{aligned}$$

where the third and fifth inequality hold for the convexity property of ϕ and the first inequality comes from Lemma 2.2. The second inequality comes from Lemma 2.10. Again, from the inequality $2pq \leqslant p^2+q^2$ for all $p,q \in \mathbb{R}$, we get the fourth inequality. The last inequality follows from Lemma 2.5.

(b) Let $x \in \mathcal{A}_a$. Then for any $f \in \mathcal{S}_a(\mathcal{A})$, $|f(ax)| = |f(ax^{\#_a})|$ holds. Now, if we replace $|f(ax)|$ by $|f(ax^{\#_a})|$ in the inequality (2.2) and proceed similarly as in part (a) of this lemma, then we get

$$\begin{aligned}
&\phi\left(|f(ax)|^2\right) \\
&\leqslant \frac{\alpha}{2}\phi\left(\left|f(ax^2)\right|\right) + \frac{\alpha}{4}\phi\left(f(ax^{\#_a}x)\right) + \left(1-\frac{3\alpha}{4}\right)\phi\left(f(axx^{\#_a})\right) \qquad (2.4)\\
&\leqslant \frac{\alpha}{2}\phi\left(\left|f(ax^2)\right|\right) + \frac{\alpha}{4}f\left(\phi(ax^{\#_a}x)\right) + \left(1-\frac{3\alpha}{4}\right)f\left(\phi(axx^{\#_a})\right). \quad \square
\end{aligned}$$

THEOREM 2.20. *Let $\mathcal{A}$ be a unital $\mathcal{C}^*$-algebra and $0 \leqslant \alpha \leqslant 1$, then we get the following results*

(a) *If $x \in \mathcal{A}_a$, then*

  (i) $v_a^2(x) \leqslant \frac{\alpha}{2}v_a(x^2) + \left\|\frac{\alpha}{4}xx^{\#_a} + (1-\frac{3\alpha}{4})x^{\#_a}x\right\|_a$

  (ii) $v_a^2(x) \leqslant \frac{\alpha}{2}v_a(x^2) + \left\|\frac{\alpha}{4}x^{\#_a}x + (1-\frac{3\alpha}{4})xx^{\#_a}\right\|_a.$

(b) *If $x \in \mathcal{A}$ and ϕ be an Orlicz function, then*

  (i) $\phi(v^2(x)) \leqslant \frac{\alpha}{2}\phi(v(x^2)) + \left\|\frac{\alpha}{4}\phi(xx^*) + (1-\frac{3\alpha}{4})\phi(x^*x)\right\|$

(ii) $\phi(v^2(x)) \leqslant \frac{\alpha}{2}\phi(v(x^2)) + \left\|\frac{\alpha}{4}\phi(x^*x) + (1-\frac{3\alpha}{4})\phi(xx^*)\right\|$.

Proof.

(a) Let $x \in \mathcal{A}_a$ and $f \in \mathcal{S}_a(\mathcal{A})$.

(i) Taking $\phi(t) = t$, $t \geqslant 0$ in inequality (2.3) and using the fact that $|f(ay)| \leqslant \|y\|_a \ \forall \ y \in \mathcal{A}$, we get

$$\begin{aligned} \left|f(ax)\right|^2 &\leqslant \frac{\alpha}{2}\left|f(ax^2)\right| + \frac{\alpha}{4}f(axx^{\#_a}) + \left(1-\frac{3\alpha}{4}\right)f(ax^{\#_a}x) \\ &\leqslant \frac{\alpha}{2}v_a(x^2) + f\left(a\left(\frac{\alpha}{4}xx^{\#_a} + \left(1-\frac{3\alpha}{4}\right)x^{\#_a}x\right)\right) \\ &\leqslant \frac{\alpha}{2}v_a(x^2) + \left\|\frac{\alpha}{4}xx^{\#_a} + \left(1-\frac{3\alpha}{4}\right)x^{\#_a}x\right\|_a. \end{aligned}$$

Since, the above inequality holds for all $f \in \mathcal{S}_a(\mathcal{A})$, we get our required result.

(ii) Taking $\phi(t) = t$, $t \geqslant 0$ in inequality (2.4) and proceeding similarly as in the first result of part (a) of this theorem, we obtain the required inequality.

(b) Let $x \in \mathcal{A}$ and $f \in \mathcal{S}(\mathcal{A})$.

(i) Taking $a = \mathbf{1}$ in Lemma 2.19 (a), we get

$$\begin{aligned} \phi\left(\left|f(x)\right|^2\right) &\leqslant \frac{\alpha}{2}\phi\left(\left|f(x^2)\right|\right) + \frac{\alpha}{4}f\left(\phi(xx^*)\right) + \left(1-\frac{3\alpha}{4}\right)f\left(\phi(x^*x)\right) \\ &= \frac{\alpha}{2}\phi\left(\left|f(x^2)\right|\right) + f\left(\frac{\alpha}{4}\phi(xx^*) + \left(1-\frac{3\alpha}{4}\right)\phi(x^*x)\right) \\ &\leqslant \frac{\alpha}{2}\phi\left(v(x^2)\right) + v\left(\frac{\alpha}{4}\phi(xx^*) + \left(1-\frac{3\alpha}{4}\right)\phi(x^*x)\right) \\ &= \frac{\alpha}{2}\phi\left(v(x^2)\right) + \left\|\frac{\alpha}{4}\phi(xx^*) + \left(1-\frac{3\alpha}{4}\right)\phi(x^*x)\right\|. \end{aligned}$$

Now, taking supremum over $f \in \mathcal{S}(\mathcal{A})$, we get

$$\phi\left(v^2(x)\right) \leqslant \frac{\alpha}{2}\phi\left(v(x^2)\right) + \left\|\frac{\alpha}{4}\phi(xx^*) + \left(1-\frac{3\alpha}{4}\right)\phi(x^*x)\right\|.$$

(ii) Taking $a = \mathbf{1}$ in Lemma 2.19 (b) and proceeding Similarly as in the first result of part (b) of this theorem, we obtain that required inequality. $\square$

COROLLARY 2.21. *Let $\mathcal{A}$ be a unital $\mathcal{C}^*$-algebra and $x \in \mathcal{A}$. If $0 \leqslant \alpha \leqslant 1$ and $r \geqslant 1$, then*

(i) $v^{2r}(x) \leqslant \frac{\alpha}{2}v^r(x^2) + \left\|\frac{\alpha}{4}(xx^*)^r + (1-\frac{3\alpha}{4})(x^*x)^r\right\|$

(ii) $v^{2r}(x) \leqslant \frac{\alpha}{2} v^r(x^2) + \left\| \frac{\alpha}{4}(x^*x)^r + (1-\frac{3\alpha}{4})(xx^*)^r \right\|$.

Proof. If we take $\phi(t) = t^r$, $t \geqslant 0$ and $r \geqslant 1$ in Theorem 2.20 (b), then we get the required results. □

REMARK 2.22. Let $\mathcal{A}$ be a unital $\mathcal{C}^*$-algebra and $0 \leqslant \alpha \leqslant 1$, then we get the following results:

(i) If we we put $\alpha = 1$ in Theorem 2.20 (a), it becomes [19, Theorem 3.3].

(ii) If we consider $\mathcal{A} = \mathcal{B}(H)$ in Theorem 2.20 (a), we get [6, Theorem 3].

(iii) If we consider $\mathcal{A} = \mathcal{B}(H)$, the Corollary 2.21 becomes [7, Theorem 2.11].

(iv) If we take $\phi(t) = t$, $t \geqslant 0$ and $\alpha = 1$ in Theorem 2.20 (b), we get the upper bound given in [23, Theorem 2.4].

In order to obtain our next inequality that gives an upper bound for the a-numerical radius of the sum of two elements of $\mathcal{A}_a$, we need the following lemma.

LEMMA 2.23. *Let $x, y \in \mathcal{A}_a$. If ψ_1 be the complementary Orlicz function of ψ_2 and $f \in \mathcal{S}_a(\mathcal{A})$, then*

$$\begin{aligned} &\left|f\big(a(x+y)\big)\right|^2 \\ &\leqslant |f(ax)|^2 + |f(ay)|^2 + |f(ayx)| + \psi_1\Big(\sqrt{f(ax^{\#_a}x)}\Big) + \psi_2\Big(\sqrt{f(ayy^{\#_a})}\Big). \end{aligned}$$

Proof. Let $x, y \in \mathcal{A}_a$ and $f \in \mathcal{S}_a(\mathcal{A})$. Then we have

$$\begin{aligned} &|f\big(a(x+y)\big)|^2 \\ &\leqslant |f(ax)|^2 + |f(ay)|^2 + 2|f(ax)||f(ay)| \\ &\leqslant |f(ax)|^2 + |f(ay)|^2 + |f(ayx)| + \sqrt{f(ax^{\#_a}x)}\sqrt{f(ayy^{\#_a})} \\ &\leqslant |f(ax)|^2 + |f(ay)|^2 + |f(ayx)| + \psi_1\Big(\sqrt{f(ax^{\#_a}x)}\Big) + \psi_2\Big(\sqrt{f(ayy^{\#_a})}\Big), \end{aligned}$$

where the second inequality comes from Lemma 2.10 and third inequality holds from Lemma 2.11. □

REMARK 2.24. Taking $\psi_1(t) = \frac{t^2}{2} = \psi_2(t)$ in Lemma 2.23, we get

$$\begin{aligned} &\left|f\big(a(x+y)\big)\right|^2 \\ &\leqslant |f(ax)|^2 + |f(ay)|^2 + |f(ayx)| + \frac{f(ax^{\#_a}x) + f(ayy^{\#_a})}{2} \\ &\leqslant v_a^2(x) + v_a^2(y) + v_a(yx) + \frac{f\big(a(x^{\#_a}x + yy^{\#_a})\big)}{2} \\ &\leqslant v_a^2(x) + v_a^2(y) + v_a(yx) + \frac{\left\|x^{\#_a}x + yy^{\#_a}\right\|_a}{2}. \end{aligned}$$

Taking supremum over $f \in \mathcal{S}_a(\mathcal{A})$, we get

$$v_a^2(x+y) \leqslant v_a^2(x) + v_a^2(y) + v_a(yx) + \frac{1}{2}\left\|x^{\#_a}x + yy^{\#_a}\right\|_a. \tag{2.5}$$

Now, if we take $\mathcal{A} = \mathcal{B}(H)$, then the inequality (2.5) becomes that in [5, Theorem 3.6].

THEOREM 2.25. *Let $x, y \in \mathcal{A}_a$. Then for any $n \geqslant 1$*

$$v_a^2(x+y) \leqslant v_a^2(x) + v_a^2(y) + v_a(yx) + \frac{1}{n}\|x\|_a^n + \frac{n-1}{n}\|y\|_a^{\frac{n}{n-1}}.$$

Proof. Let $x, y \in \mathcal{A}_a$. Taking $\psi_1(t) = \frac{t^n}{n}$ and $\psi_2(t) = \frac{n-1}{n}t^{\frac{n}{n-1}}$, $t \geqslant 0$ and $n \geqslant 1$ in Lemma 2.23, we get

$$\begin{aligned}
&\left|f\left(a(x+y)\right)\right|^2 \\
&\leqslant |f(ax)|^2 + |f(ay)|^2 + |f(ayx)| + \frac{1}{n}\left(f(ax^{\#_a}x)\right)^{\frac{n}{2}} + \frac{n-1}{n}\left(f(ayy^{\#_a})\right)^{\frac{n}{2(n-1)}} \\
&\leqslant v_a^2(x) + v_a^2(y) + v_a(yx) + \frac{1}{n}\|x\|_a^n + \frac{n-1}{n}\|y^{\#_a}\|_a^{\frac{n}{n-1}} \\
&= v_a^2(x) + v_a^2(y) + v_a(yx) + \frac{1}{n}\|x\|_a^n + \frac{n-1}{n}\|y\|_a^{\frac{n}{n-1}}.
\end{aligned}$$

Now, taking supremum over $f \in \mathcal{S}_a(\mathcal{A})$, we get the desired inequality. $\square$

Next, we consider an example and show that the bound obtained in Theorem 2.25 is better than that given in [5, Theorem 3.6].

EXAMPLE 2.26. Consider $\mathcal{A} = \mathcal{B}(H)$, $T = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}$, $S = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{5} \end{pmatrix}$, and $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then Theorem 3.6 in [5] gives $w^2(T+S) \leqslant 0.59375$, whereas for $n = 3$ Theorem 2.25 gives $w^2(T+S) \leqslant 0.5625$.

Next, we obtain the following inequality for the sum of n elements of $\mathcal{A}_a$.

THEOREM 2.27. *Let ϕ be an Orlicz function and $x_i \in \mathcal{A}_a$, $i = 1, 2, \cdots, n$. If $\boldsymbol{P} = (p_1, p_2, \cdots, p_n) \in \mathcal{P}_n$ is a probability distribution, then*

$$\phi\left(v_a^2\left(\sum_{i=1}^n p_i x_i\right)\right) \leqslant \frac{1}{2}\phi\left(v_a\left(\sum_{i=1}^n p_i x_i^2\right)\right) + \frac{1}{2}\phi\left(\left\|\sum_{i=1}^n p_i x_i^{\#_a} x_i\right\|_a^{\frac{1}{2}}\left\|\sum_{i=1}^n p_i x_i x_i^{\#_a}\right\|_a^{\frac{1}{2}}\right).$$

Proof. Let $x_i \in \mathcal{A}_a$ and $f \in \mathcal{S}_a(\mathcal{A})$. Then we get

$$\begin{aligned}
&\phi\Big(\big|f\big(a\sum_{i=1}^{n} p_i x_i\big)\big|^2\Big) = \phi\Big(\big|f\big(\sum_{i=1}^{n} p_i a x_i\big)\big|^2\Big)\\
&\leqslant \phi\left(\frac{\big|\sum_{i=1}^{n} p_i f(a x_i^2)\big| + \sqrt{\sum_{i=1}^{n} p_i f(a x_i^{\#_a} x_i)}\sqrt{\sum_{i=1}^{n} p_i f(a x_i x_i^{\#_a})}}{2}\right)\\
&\leqslant \frac{1}{2}\phi\Big(\big|\sum_{i=1}^{n} p_i f(a x_i^2)\big|\Big) + \frac{1}{2}\phi\left(\sqrt{\sum_{i=1}^{n} p_i f(a x_i^{\#_a} x_i)}\sqrt{\sum_{i=1}^{n} p_i f(a x_i x_i^{\#_a})}\right)\\
&= \frac{1}{2}\phi\Big(\big|f\big(a\sum_{i=1}^{n} p_i x_i^2\big)\big|\Big) + \frac{1}{2}\phi\left(\sqrt{f\big(a\sum_{i=1}^{n} p_i x_i^{\#_a} x_i\big)}\sqrt{f\big(a\sum_{i=1}^{n} p_i x_i x_i^{\#_a}\big)}\right)\\
&\leqslant \frac{1}{2}\phi\Big(\big|f\big(a\sum_{i=1}^{n} p_i x_i^2\big)\big|\Big) + \frac{1}{2}\phi\Big(\big\|\sum_{i=1}^{n} p_i x_i^{\#_a} x_i\big\|_a^{\frac{1}{2}}\big\|\sum_{i=1}^{n} p_i x_i x_i^{\#_a}\big\|_a^{\frac{1}{2}}\Big),
\end{aligned}$$

where Lemma 2.10 and the non-decreasing property of ϕ give us the first inequality and the second inequality holds for the convexity property of ϕ. Now, using the fact that $|f(ay)| \leqslant \|y\|_a$ for all $y \in \mathcal{A}$, we obtain the last inequality. The desired inequality is obtained by taking supremum over $f \in \mathcal{S}_a(\mathcal{A})$ on both sides of the above inequality. $\square$

REMARK 2.28. Let $\mathcal{A} = \mathcal{B}(H)$ and a be the identity operator. Now, if we take $\phi(t) = t$, $t \geqslant 0$ in Theorem 2.27, we get [11, Theorem 2.6].

To obtain the next theorem, we need the following lemma.

LEMMA 2.29. *Let $x,y,z \in \mathcal{A}_a$, $\|y\|_a \leqslant 1$ and ϕ be an Orlicz function. If $a \geqslant \mathbf{1}$ and $f \in \mathcal{S}_a(\mathcal{A})$, then*

$$\phi\Big(\big|f(axyz^{\#_a})\big|\Big) \leqslant \frac{\|y\|_a}{2}\Big(f\big(\phi(axx^{\#_a}) + \phi(azz^{\#_a})\big)\Big).$$

Proof. Let $x,y,z \in \mathcal{A}_a$ and $f \in \mathcal{S}_a(\mathcal{A})$. Then we have,

$$\begin{aligned}
\big|f\big(axyz^{\#_a}\big)\big| &= \big|f\big((yz^{\#_a})^* x^* a\big)\big|\\
&= \big|f\big((yz^{\#_a})^* a x^{\#_a}\big)\big|\\
&\leqslant \big|f\big((x^{\#_a})^* a x^{\#_a}\big)\big|^{\frac{1}{2}}\big|f\big((yz^{\#_a})^* a y z^{\#_a}\big)\big|^{\frac{1}{2}}\\
&= \big|f\big(axx^{\#_a}\big)\big|^{\frac{1}{2}}\big|f\big((z^{\#_a})^* y^* a y z^{\#_a}\big)\big|^{\frac{1}{2}}\\
&\leqslant \|y\|_a\big|f\big(axx^{\#_a}\big)\big|^{\frac{1}{2}}\big|f\big((z^{\#_a})^* a z^{\#_a}\big)\big|^{\frac{1}{2}}\\
&= \|y\|_a\big|f\big(axx^{\#_a}\big)\big|^{\frac{1}{2}}\big|f\big(azz^{\#_a}\big)\big|^{\frac{1}{2}}\\
&\leqslant \|y\|_a\frac{f\big(axx^{\#_a}\big) + f\big(azz^{\#_a}\big)}{2},
\end{aligned}$$

where the first equality comes from the fact that $f(y^*)=\overline{f(y)}$ for all $y\in\mathcal{A}$ and the first inequality holds from Lemma 2.2. Now, the second inequality follows from Lemma 2.7 and the last inequality holds because $2pq\leqslant p^2+q^2$ for any $p,q\in\mathbb{R}$. Now, using the convexity, non-decreasing property of ϕ and $\phi(\lambda t)\leqslant\lambda\phi(t)$ for $0\leqslant\lambda\leqslant 1$, $t\geqslant 0$, we get

$$\begin{aligned}\phi\left(\left|f(axyz^{\#_a})\right|\right) &\leqslant \phi\left(\|y\|_a\frac{f\left(axx^{\#_a}\right)+f\left(azz^{\#_a}\right)}{2}\right)\\ &\leqslant \frac{\|y\|_a}{2}\left(\phi\left(f(axx^{\#_a})\right)+\phi\left(f(azz^{\#_a})\right)\right).\end{aligned}$$

Again, from Lemma 2.5, we get

$$\begin{aligned}\phi\left(\left|f(axyz^{\#_a})\right|\right) &\leqslant \frac{\|y\|_a}{2}\left(f\left(\phi(axx^{\#_a})\right)+f\left(\phi(azz^{\#_a})\right)\right)\\ &= \frac{\|y\|_a}{2}\left(f\left(\phi(axx^{\#_a})+\phi(azz^{\#_a})\right)\right). \quad\square\end{aligned}$$

Next, we prove the following theorem.

THEOREM 2.30. *Let $\mathcal{A}$ be a unital C^*-algebra, then the following results hold:*

(a) *If $x,y,z\in\mathcal{A}_a$ and $\|y\|_a\leqslant 1$, then*

$$v_a\left(xyz^{\#_a}\right)\leqslant\frac{\|y\|_a}{2}\left\|xx^{\#_a}+zz^{\#_a}\right\|_a.$$

(b) *If $x,y,z\in\mathcal{A}$, $\|y\|\leqslant 1$ and ϕ be an Orlicz function, then*

$$\phi\left(v\left(xyz^*\right)\right)\leqslant\frac{\|y\|}{2}\left\|\phi(xx^*)+\phi(zz^*)\right\|.$$

Proof.

(a) Taking $\phi(t)=t$, $t\geqslant 0$ in Lemma 2.29, we get

$$\begin{aligned}\left|f\left(axyz^{\#_a}\right)\right| &\leqslant \frac{\|y\|_a}{2}\left(f\left(axx^{\#_a}\right)+f\left(azz^{\#_a}\right)\right)\\ &= \frac{\|y\|_a}{2}f\left(a(xx^{\#_a}+zz^{\#_a})\right)\\ &\leqslant \frac{\|y\|_a}{2}\left\|xx^{\#_a}+zz^{\#_a}\right\|_a,\end{aligned}$$

where the last inequality comes from the fact that $|f(ay)|\leqslant\|y\|_a$ for all $y\in\mathcal{A}$. Now, taking supremum over $f\in\mathcal{S}_a(\mathcal{A})$, we get the desired inequality.

(b) Putting $a = \mathbf{1}$ in Lemma 2.29 and using the fact that $|f(y)| \leqslant \|y\|$ for all $y \in \mathcal{A}$, we get

$$\begin{aligned}\phi\left(\left|f(xyz^*)\right|\right) &\leqslant \frac{\|y\|}{2}\Big(f\big(\phi(xx^*)+\phi(zz^*)\big)\Big)\\ &\leqslant \frac{\|y\|}{2}\Big\|\phi(xx^*)+\phi(zz^*)\Big\|.\end{aligned}$$

Since the above inequality holds for all $f \in \mathcal{S}(\mathcal{A})$, so the desired result follows. □

COROLLARY 2.31. *Let $\mathcal{A}$ be a unital $\mathcal{C}^*$-algebra and $x,y,z \in \mathcal{A}$. If $r \geqslant 1$ and $\|y\| \leqslant 1$, then*

$$v^r\left(xyz^*\right) \leqslant \frac{\|y\|}{2}\left\|(xx^*)^r + (zz^*)^r\right\|. \tag{2.6}$$

Proof. If we put $\phi(t) = t^r$, where $t \geqslant 0$ and $r \geqslant 1$ in Theorem 2.30 (b), we get the required inequality. □

REMARK 2.32.

(i) It is shown in [22, Lemma 4.4] that if $T,S,R \in \mathcal{B}_A(H)$, then

$$w_A(SRT^{\#_A}) \leqslant \frac{1}{2}\|TT^{\#_A} + SS^{\#_A}\|_A\|R\|_A.$$

Now, if we take $\mathcal{A} = \mathcal{B}(H)$ and put $x = S$, $y = R$ and $z = T$ in Theorem 2.30 (a), we get the above upper bound of $w_A(SRT^{\#_A})$ for $\|R\|_A \leqslant 1$.

(ii) If we take $\mathcal{A} = \mathcal{B}(H)$ and y as the identity operator I in Corollary 2.31, we get the existing inequality given in [10, Theorem 1].

A Orlicz function ϕ is said to be sub-multiplicative if for every $u,v \geqslant 0$, $\phi(uv) \leqslant \phi(u)\phi(v)$ holds. In the following result, we use the sub-multiplicative property of ϕ to find an upper bound of the numerical radius.

THEOREM 2.33. *Let $x,y,z \in \mathcal{A}$ and ϕ be a sub-multiplicative Orlicz function. If x,z are positive, $r \geqslant 2$ and $0 \leqslant \alpha \leqslant 1$, then*

$$\phi\left(v^r\left(x^\alpha yz^{(1-\alpha)}\right)\right) \leqslant \phi\left(\|y\|^r\right)\Big\|\alpha\phi(x^r) + (1-\alpha)\phi(z^r)\Big\|.$$

Proof. Let x,z be positive, $y \in \mathcal{A}$ and $f \in \mathcal{S}(\mathcal{A})$. Then we have,

$$\begin{aligned}\left|f\left(x^\alpha yz^{(1-\alpha)}\right)\right|^r &= \left|f\left((yz^{(1-\alpha)})^*x^\alpha\right)\right|^r\\ &\leqslant \left(f(x^{2\alpha})\right)^{\frac{r}{2}}\left(f(z^{(1-\alpha)}y^*yz^{1-\alpha})\right)^{\frac{r}{2}}\\ &\leqslant \|y\|^r\left(f(x^{2\alpha})\right)^{\frac{r}{2}}\left(f(z^{2(1-\alpha)})\right)^{\frac{r}{2}}\\ &\leqslant \|y\|^r\left(f(x^{r\alpha})\right)\left(f(z^{r(1-\alpha)})\right)\\ &\leqslant \|y\|^r\left(f(x^r)\right)^\alpha\left(f(z^r)\right)^{(1-\alpha)}\\ &\leqslant \|y\|^r\Big(\alpha f(x^r) + (1-\alpha)f(z^r)\Big),\end{aligned}$$

where the first inequality comes from Lemma 2.1 and the second inequality comes from Lemma 2.7. Third and fourth inequalities follow from Corollary 2.6 and Corollary 2.4, respectively. The last inequality holds for the fact that $a^{\lambda}b^{1-\lambda} \leqslant \lambda a + (1-\lambda)b$ where $a,b \geqslant 0$ and $0 \leqslant \lambda \leqslant 1$.

Now, using non-decreasing, sub-multiplicative and convexity property of ϕ, we get

$$\phi\left(\left|f\left(x^{\alpha}yz^{(1-\alpha)}\right)\right|^{r}\right) \leqslant \phi\left(\|y\|^{r}\right)\left(\alpha\phi\left(f(x^{r})\right) + (1-\alpha)\phi\left(f(z^{r})\right)\right).$$

Again, from Lemma 2.3, we get

$$\begin{aligned}\phi\left(\left|f\left(x^{\alpha}yz^{(1-\alpha)}\right)\right|^{r}\right) &\leqslant \phi\left(\|y\|^{r}\right)\left(f\left(\alpha\phi\left(x^{r}\right) + (1-\alpha)\phi\left(z^{r}\right)\right)\right)\\ &\leqslant \phi\left(\|y\|^{r}\right)\left\|\alpha\phi\left(x^{r}\right) + (1-\alpha)\phi\left(z^{r}\right)\right\|.\end{aligned}$$

Taking supremum over $f \in \mathcal{S}(\mathcal{A})$ of the above inequality, we get the desired result. $\square$

REMARK 2.34. If we put $\phi(t) = t$, $t \geqslant 0$ and consider $\mathcal{A} = \mathcal{B}(H)$ in Theorem 2.33, we get [21, Theorem 3.3].

THEOREM 2.35. *Let ϕ be a multiplicative Orlicz function and $r,s \geqslant 1$. If $w,x,y,z \in \mathcal{A}$, then*

$$\begin{aligned}&\phi\left(\frac{v^{2}(x^{*}w + z^{*}y)}{2}\right)\\ &\leqslant \left\|\frac{\left(\phi(w^{*}w)\right)^{r} + \left(\phi(y^{*}y)\right)^{r}}{2}\right\|^{\frac{1}{r}}\left\|\frac{\left(\left(\phi(x^{*}x)\right)^{s} + \left(\phi(z^{*}z)\right)^{s}}{2}\right\|^{\frac{1}{s}}.\end{aligned}$$

Proof. Let $w,x,y,z \in \mathcal{A}$ and $f \in \mathcal{S}(\mathcal{A})$. Then we get

$$\begin{aligned}\left|f\left(\frac{x^{*}w + z^{*}y}{2}\right)\right|^{2} &= \left|\frac{f(x^{*}w)}{2} + \frac{f(z^{*}y)}{2}\right|^{2}\\ &\leqslant \left(\left|\frac{f(x^{*}w)}{2}\right| + \left|\frac{f(z^{*}y)}{2}\right|\right)^{2}\\ &\leqslant \left(\frac{1}{2}\sqrt{f(x^{*}x)}\sqrt{f(w^{*}w)} + \frac{1}{2}\sqrt{f(z^{*}z)}\sqrt{f(y^{*}y)}\right)^{2}\\ &\leqslant \frac{1}{4}\left(f(w^{*}w) + f(y^{*}y)\right)\left(f(x^{*}x) + f(z^{*}z)\right)\\ &= \left(\frac{1}{2}f(w^{*}w) + \frac{1}{2}f(y^{*}y)\right)\left(\frac{1}{2}f(x^{*}x) + \frac{1}{2}f(z^{*}z)\right),\end{aligned}$$

where second inequality comes from Lemma 2.1 and the third inequality comes from the fact that $(ab+cd)^{2} \leqslant (a^{2}+c^{2})(b^{2}+d^{2})$ for all $a,b,c,d \geqslant 0$.

Again,

$$
\begin{aligned}
&\phi\left(\left|f\left(\frac{x^*w+z^*y}{2}\right)\right|^2\right)\\
&\leqslant \phi\left(\left(\frac{1}{2}f(w^*w)+\frac{1}{2}f(y^*y)\right)\left(\frac{1}{2}f(x^*x)+\frac{1}{2}f(z^*z)\right)\right)\\
&= \phi\left(\frac{1}{2}f(w^*w)+\frac{1}{2}f(y^*y)\right)\phi\left(\frac{1}{2}f(x^*x)+\frac{1}{2}f(z^*z)\right)\\
&\leqslant \left(\frac{1}{2}\phi(f(w^*w))+\frac{1}{2}\phi(f(y^*y))\right)\left(\frac{1}{2}\phi(f(x^*x))+\frac{1}{2}\phi(f(z^*z))\right)\\
&\leqslant \left(\frac{(\phi(f(w^*w)))^r+(\phi(f(y^*y)))^r}{2}\right)^{\frac{1}{r}}\left(\frac{(\phi(f(x^*x)))^s+(\phi(f(z^*z)))^s}{2}\right)^{\frac{1}{s}}\\
&\leqslant \left(\frac{(f(\phi(w^*w)))^r+(f(\phi(y^*y)))^r}{2}\right)^{\frac{1}{r}}\left(\frac{(f(\phi(x^*x)))^s+(f(\phi(z^*z)))^s}{2}\right)^{\frac{1}{s}}\\
&\leqslant \left(\frac{f((\phi(w^*w))^r)+f((\phi(y^*y))^r)}{2}\right)^{\frac{1}{r}}\left(\frac{f((\phi(x^*x))^s)+f((\phi(z^*z))^s)}{2}\right)^{\frac{1}{s}}\\
&= \left(\frac{f((\phi(w^*w))^r+(\phi(y^*y))^r)}{2}\right)^{\frac{1}{r}}\left(\frac{f((\phi(x^*x))^s+(\phi(z^*z))^s)}{2}\right)^{\frac{1}{s}}\\
&\leqslant \left\|\frac{(\phi(w^*w))^r+(\phi(y^*y))^r}{2}\right\|^{\frac{1}{r}}\left\|\frac{((\phi(x^*x))^s+(\phi(z^*z))^s}{2}\right\|^{\frac{1}{s}}.
\end{aligned}
$$

Where the first and the second inequalities hold for non-decreasing and convexity property of ϕ, respectively. The third inequality comes from the fact that $(\frac{a+b}{2})^r \leqslant (\frac{a^r}{2}+\frac{b^r}{2})$ for all $a,b \geqslant 0$ and $r \geqslant 1$. Next, the fourth inequality follows from Lemma 2.5 and the fifth inequality comes from Corollary 2.6. Now, taking supremum over $f \in \mathcal{S}(\mathcal{A})$, we get the desired result. $\square$

COROLLARY 2.36. *Let $p,q \in \mathcal{A}$ and $r,s \geqslant 1$, then*

$$
\phi\left(v^2\left(\frac{pq-qp}{2}\right)\right) \leqslant \left\|\frac{\left(\phi(q^*q)\right)^r+\left(\phi(p^*p)\right)^r}{2}\right\|^{\frac{1}{r}}\left\|\frac{\left(\phi(qq^*)\right)^s+\left(\phi(pp^*)\right)^s}{2}\right\|^{\frac{1}{s}}.
$$

Proof. If we take $x^*=p$, $w=q$, $z^*=q$ and $y=-p$ in Theorem 2.35, then we get the required result. $\square$

REMARK 2.37. If we take $\mathcal{A}=\mathcal{B}(H)$ and $\phi(t)=t$, $t \geqslant 0$ in Theorem 2.35, we get the inequality in [10, Theorem 3].

LEMMA 2.38. [16] *Let $T=U|T|$ be the polar decomposition of an operator T on a Hilbert space H. Then for any positive number α the following holds*

$$
|T^*|^\alpha = U|T|^\alpha U^*.
$$

Let x be an invertible element in a unital $\mathcal{C}^*$-algebra $\mathcal{A}$. Then there exists a unique unitary u such that $x = u|x|$, where $|x| = (x^*x)^{\frac{1}{2}}$. Now, using the same technique as used in Lemma 2.38, it can be proven that if x is an invertible element in a unital $\mathcal{C}^*$-algebra $\mathcal{A}$, then $|x^*|^\alpha = u|x|^\alpha u^*$ for any positive number α.

LEMMA 2.39. *Let $x,y,z \in \mathcal{A}$ and y be an invertible element in $\mathcal{A}$. If $0 \leqslant \alpha \leqslant 1$ and $f \in \mathcal{S}(\mathcal{A})$, then*

$$\left|f(xyz)\right|^2 \leqslant f\left(x|y^*|^{2(1-\alpha)}x^*\right) f\left(z^*|y|^{2\alpha}z\right).$$

Proof. Let y be an invertible in $\mathcal{A}$. Then by polar decomposition there exists a unique unitary element u in $\mathcal{A}$ such that $y = u|y|$. Now,

$$\begin{aligned}
\left|f(xyz)\right|^2 &= \left|f\left(xu|y|z\right)\right|^2 \\
&= \left|f\left(xu|y|^{(1-\alpha)}|y|^\alpha z\right)\right|^2 \\
&\leqslant f\left(xu|y|^{(1-\alpha)}|y|^{(1-\alpha)}u^*x^*\right) f\left(z^*|y|^\alpha|y|^\alpha z\right) \\
&= f\left(xu|y|^{2(1-\alpha)}u^*x^*\right) f\left(z^*|y|^{2\alpha}z\right) \\
&= f\left(x|y^*|^{2(1-\alpha)}x^*\right) f\left(z^*|y|^{2\alpha}z\right) \ ,
\end{aligned}$$

where the above inequality comes from the Lemma 2.1 and the last equality comes from the fact

$$u|y|^\alpha u^* = |y^*|^\alpha. \quad \square \tag{2.7}$$

LEMMA 2.40. *Let $x,y,p,q,r,s \in \mathcal{A}$. Let ψ_1 be the complementary Orlicz function of ψ_2 and ψ_3 be the complementary Orlicz function of ψ_4. If x,y are invertible, $f \in \mathcal{S}(\mathcal{A})$ and $0 \leqslant \alpha \leqslant 1$, then*

$$\begin{aligned}
&\left|f(pxq+rys)\right| \\
&\leqslant \psi_1\left(\left(f\left(p|x^*|^{2(1-\alpha)}p^*\right)\right)^{\frac{1}{2}}\right) + \psi_2\left(\left(f\left(q^*|x|^{2\alpha}q\right)\right)^{\frac{1}{2}}\right) \\
&\quad + \psi_3\left(\left(f\left(r|y^*|^{2(1-\alpha)}r^*\right)\right)^{\frac{1}{2}}\right) + \psi_4\left(\left(f\left(s^*|y|^{2\alpha}s\right)\right)^{\frac{1}{2}}\right) .
\end{aligned}$$

Proof. Let $x,y,p,q,r,s \in \mathcal{A}$. Now, $f \in \mathcal{S}(\mathcal{A})$ and x,y are invertible. Then we have

$$\begin{aligned}
&|f(pxq+rys)| \\
&\leqslant |f(pxq)| + \left|f(rys)\right| \\
&\leqslant (f(p|x^*|^{2(1-\alpha)}p^*))^{\frac{1}{2}}(f(q^*|x|^{2\alpha}q))^{\frac{1}{2}} + (f(r|y^*|^{2(1-\alpha)}r^*))^{\frac{1}{2}}\left(f(s^*|y|^{2\alpha}s)\right)^{\frac{1}{2}} \\
&\leqslant \psi_1\left((f(p|x^*|^{2(1-\alpha)}p^*))^{\frac{1}{2}}\right) + \psi_2\left((f(q^*|x|^{2\alpha}q))^{\frac{1}{2}}\right) \\
&\quad + \psi_3\left((f(r|y^*|^{2(1-\alpha)}r^*))^{\frac{1}{2}}\right) + \psi_4\left((f(s^*|y|^{2\alpha}s))^{\frac{1}{2}}\right) ,
\end{aligned}$$

where the second inequality comes from Lemma 2.39 and last inequality follows from Lemma 2.11. $\square$

THEOREM 2.41. *Let $x,y,p,q,r,s \in \mathcal{A}$. If x,y are invertible and $0 \leqslant \alpha \leqslant 1$, then for any $n \geqslant 2$,*

$$v(pxq+rys) \leqslant \frac{1}{n}\left\|\left(p|x^*|^{2(1-\alpha)}p^*\right)^{\frac{n}{2}}+\left(r|y^*|^{2(1-\alpha)}r^*\right)^{\frac{n}{2}}\right\| + \frac{n-1}{n}\left(\left\|q^*|x|^{2\alpha}q\right\|^{\frac{n}{2(n-1)}}+\left\|s^*|y|^{2\alpha}s\right\|^{\frac{n}{2(n-1)}}\right).$$

Proof. Putting $\psi_1(t)=\frac{t^n}{n}=\psi_3(t)$, $\psi_2(t)=\frac{n-1}{n}t^{\frac{n}{n-1}}=\psi_4(t)$ for $t \geqslant 0$ and $n \geqslant 2$ in Lemma 2.40, we get

$$\begin{aligned}
&\left|f((pxq+rys))\right| \\
&\leqslant \frac{1}{n}\left(\left(f\left(p|x^*|^{2(1-\alpha)}p^*\right)\right)^{\frac{n}{2}}+\left(f\left(r|y^*|^{2(1-\alpha)}r^*\right)\right)^{\frac{n}{2}}\right) \\
&\quad+\frac{n-1}{n}\left(\left(f\left(q^*|x|^{2\alpha}q\right)\right)^{\frac{n}{2(n-1)}}+\left(f\left(s^*|y|^{2\alpha}s\right)\right)^{\frac{n}{2(n-1)}}\right) \\
&\leqslant \frac{1}{n}\left(f\left(\left(p|x^*|^{2(1-\alpha)}p^*\right)^{\frac{n}{2}}\right)+f\left(\left(r|y^*|^{2(1-\alpha)}r^*\right)^{\frac{n}{2}}\right)\right) \\
&\quad+\frac{n-1}{n}\left(\left\|q^*|x|^{2\alpha}q\right\|^{\frac{n}{2(n-1)}}+\left\|s^*|y|^{2\alpha}s\right\|^{\frac{n}{2(n-1)}}\right) \\
&= \frac{1}{n}\left(f\left(\left(p|x^*|^{2(1-\alpha)}p^*\right)^{\frac{n}{2}}+\left(r|y^*|^{2(1-\alpha)}r^*\right)^{\frac{n}{2}}\right)\right) \\
&\quad+\frac{n-1}{n}\left(\left\|q^*|x|^{2\alpha}q\right\|^{\frac{n}{2(n-1)}}+\left\|s^*|y|^{2\alpha}s\right\|^{\frac{n}{2(n-1)}}\right) \\
&\leqslant \frac{1}{n}\left\|\left(p|x^*|^{2(1-\alpha)}p^*\right)^{\frac{n}{2}}+\left(r|y^*|^{2(1-\alpha)}r^*\right)^{\frac{n}{2}}\right\| \\
&\quad+\frac{n-1}{n}\left(\left\|q^*|x|^{2\alpha}q\right\|^{\frac{n}{2(n-1)}}+\left\|s^*|y|^{2\alpha}s\right\|^{\frac{n}{2(n-1)}}\right).
\end{aligned}$$

Where Corollary 2.6 and the fact that $|f(y)| \leqslant \|y\|$ for all $y \in \mathcal{A}$ are used to get the second inequality. Now, taking supremum over $f \in \mathcal{S}(\mathcal{A})$, we get the desired result. $\square$

In the following example, we show that the bound obtained in Theorem 2.41 is better than that given in [17, Theorem 2].

EXAMPLE 2.42. Consider $\mathcal{A} = \mathcal{B}(H)$, $A = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}$, $B = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$, $C = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$, $D = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}$ and $T = S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then Theorem 2 in [17] gives $w(AB+CD) \leqslant \frac{5929}{288}$, whereas for $n = 3$ Theorem 2.41 gives $w(AB+CD) \leqslant \frac{65+40\sqrt{5}}{12}$.

Declarations

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REFERENCES

[1] J. S. AUJLA, SEVER S. DRAGOMIR, M. KHOSRAVI AND M. S, MOSLEHIAN, *Refinements of Choi–Davis–Jensen's inequality*, Bull. Math. Anal. Appl., **3**, 2011.

[2] ANIKET BHANJA, PINTU BHUNIA AND KALLOL PAUL, *On generalized Davis-Wielandt radius inequalities of semi-Hilbertian space operators*, Oper. Matrices, **15** (4): 1201–1225, 2021.

[3] PINTU BHUNIA, *Inequalities for linear functionals and numerical radii on C^*-algebras*, Acta Math. Hungar., **176** (1), 111–138, 2025.

[4] PINTU BHUNIA, SILVESTRU SEVER DRAGOMIR, MOHAMMAD SAL MOSLEHIAN AND KALLOL PAUL, *Lectures on numerical radius inequalities*, Springer Nature, 2022.

[5] PINTU BHUNIA, KAIS FEKI AND KALLOL PAUL, *Numerical radius inequalities for products and sums of semi-Hilbertian space operators*, Filomat, **36** (4): 1415–1431, 2022.

[6] PINTU BHUNIA, RAJ KUMAR NAYAK AND KALLOL PAUL, *Improvement of A-numerical radius inequalities of semi-Hilbertian space operators*, Results Math., **76** (3): 120, 2021.

[7] PINTU BHUNIA AND KALLOL PAUL, *Proper improvement of well-known numerical radius inequalities and their applications*, Results Math., **76**: 1–12, 2021.

[8] BRUCE BLACKADAR, *Operator algebras: theory of C^*-algebras and von Neumann algebras*, vol. **122**, Springer Science & Business Media, 2006.

[9] ABDELLATIF BOURHIM AND MOHAMED MABROUK, *a-Numerical range on C^*-algebras*, Positivity **25**: 1489–1510, 2021.

[10] SEVER S. DRAGOMIR, *Power inequalities for the numerical radius of a product of two operators in Hilbert spaces*, Research report collection, **11** (4), 2008.

[11] SEVER S. DRAGOMIR, *Generalizations of Buzano inequality for n-tuples of vectors in inner product spaces with applications*, Tbilisi Math. J., **10** (2): 29–42, 2017.

[12] SEVER S. DRAGOMIR, *Inequalities for the numerical radius of linear operators in Hilbert spaces*, Springer, 2013.

[13] MOHAMMAD EL-HADDAD AND FUAD KITTANEH, *Numerical radius inequalities for Hilbert space operators, II*, Studia Math., **182**: 133–140, 2007.

[14] KAIS FEKI, *A note on the A-numerical radius of operators in semi-Hilbert spaces*, Archiv der Mathematik, **115** (5): 535–544, 2020.

[15] KAIS FEKI AND FUAD KITTANEH, *Some new refinements of generalized numerical radius inequalities for Hilbert space operators*, Mediterranean Journal of Mathematics, **19** (1): 17, 2022.

[16] TAKAYUKI FURUTA, *Invitation to linear operators: From matrices to bounded linear operators on a Hilbert space*, CRC Press, 2001.

[17] FUAD KITTANEH, *Numerical radius inequalities for Hilbert space operators*, Studia Math., **168** (1): 73–80, 2005.

[18] J. LINDENSTRAUSS AND L. TZAFRIRI, *Classical Banach Spaces I and II, Sequence spaces, Function spaces*, Printed in Germany, 1996.

[19] MOHAMED MABROUK AND ALI ZAMANI, *An extension of the a-numerical radius on C^*-algebras*, Banach J. Math. Anal., **17** (3): 42, 2023.

[20] AMIT MAJI, ATANU MANNA AND RAM MOHAPATRA, *Orlicz extension of numerical radius inequalities*, arXiv:2207.01915v2[math.FA], 2024.

[21] MOSTAFA SATTARI, MOHAMMAD SAL MOSLEHIAN AND TAKEAKI YAMAZAKI, *Some generalized numerical radius inequalities for Hilbert space operators*, Linear Algebra Appl., **470**: 216–227, 2015.

[22] ALI ZAMANI, *A-numerical radius inequalities for semi-Hilbertian space operators*, Linear Algebra Appl., **578**: 159–183, 2019.

[23] ALI ZAMANI, *Characterization of numerical radius parallelism in C^*-algebras*, Positivity, **23** (2): 397–411, 2019.

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