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A CHARACTERIZATION OF AN ADDITIVE IDENTITY ON MATRIX ALGEBRAS

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Abstract. Let $M_n(\mathcal{R})$ be the algebra of $n\times n$ matrices over $\mathcal{R}$, where $\mathcal{R}$ is a commutative two-torsion free ring. In this manuscript, we obtain a structure theorem for the map ψ on $M_n(\mathcal{R})$ satisfying $\psi(\mathscr{A}\mathscr{B}^*\mathscr{A}) = \psi(\mathscr{A})\mathscr{B}^*\mathscr{A} - \mathscr{A}\psi(\mathscr{B})^*\mathscr{A} + \mathscr{A}\mathscr{B}^*\psi(\mathscr{A})$. Moreover, a complete characterization of ψ on $B(\mathcal{H})$, algebra of all bounded linear operators on $\mathcal{H}$, infinite dimensional complex Hilbert space, is given.

1. Introduction

Let $\mathcal{A}$ be an algebra. Following [1], a mapping $\psi : \mathcal{A} \to \mathcal{A}$, for any $\mathscr{A}, \mathscr{B} \in \mathcal{A}$, satisfying

$$\psi(\mathscr{A}\mathscr{B}\mathscr{A}) = \psi(\mathscr{A})\psi(\mathscr{B})\psi(\mathscr{A})$$

is called a Jordan semi-triple mapping. Molnar showed in [11] that in the case of standard operator algebras acting on infinite dimensional Banach spaces every bijective semi-triple mapping is additive. Lu [10], demonstrated Jordan semi-triple map on standard operator algebra $\mathcal{A}$ in different way. He proved that ψ is additive on $\mathcal{A}$ if

$$\psi(k\mathscr{A}\mathscr{B}\mathscr{A}) = k\psi(\mathscr{A})\psi(\mathscr{B})\psi(\mathscr{A})$$

for all $\mathscr{A}, \mathscr{B} \in \mathcal{A}$ and $k \in \mathbb{Q}$. Later, Gorazd Lešnjak and Nung-Sing Sze [4] gave a characterization of injective Jordan semi-triple mapping on matrix algebra $M_n(F)$ with entries from a field F. A map $\psi : \mathcal{A} \to \mathcal{A}$ is called Jordan semi-triple derivable mapping if

$$\psi(\mathscr{A}\mathscr{B}\mathscr{A}) = \psi(\mathscr{A})\mathscr{B}\mathscr{A} + \mathscr{A}\psi(\mathscr{B})\mathscr{A} + \mathscr{A}\mathscr{B}\psi(\mathscr{A})$$

for all $\mathscr{A}, \mathscr{B} \in \mathcal{A}$. In [3], Du and Zhang characterized the Jordan semi-triple derivable mapping on matrix algebra over 2-torsion free commutative ring with unity. Gao in [6] introduced and characterized $*$-Jordan semi-triple mapping ψ on $B(\mathcal{H})$, the algebra of all bounded linear operators on $\mathcal{H}$, where $\mathcal{H}$ is a Hilbert space over real or complex field, as

$$\psi(\mathscr{A}\mathscr{B}^*\mathscr{A}) = \psi(\mathscr{A})\psi(\mathscr{B})^*\psi(\mathscr{A})$$

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for all $\mathscr{A},\mathscr{B} \in B(\mathcal{H})$. In the context of Gao's result, Chen and Zhang [2] characterized a semi-triple Jordan derivable mapping using matrix algebra $M_n(\mathcal{R})$ as

$$\psi(\mathscr{A}\mathscr{B}^*\mathscr{A}) = \psi(\mathscr{A})\mathscr{B}^*\mathscr{A} + \mathscr{A}\psi(\mathscr{B})^*\mathscr{A} + \mathscr{A}\mathscr{B}^*\psi(\mathscr{A})$$

for all $\mathscr{A},\mathscr{B} \in \mathcal{A}$. In [13], Vukman and his team define a map $\psi : \mathcal{R} \to \mathcal{R}$, where $\mathcal{R}$ is a 2-torsion free prime ring, as

$$\psi(\mathscr{A}\mathscr{B}\mathscr{A}) = \psi(\mathscr{A})\mathscr{B}\mathscr{A} - \mathscr{A}\psi(\mathscr{B})\mathscr{A} + \mathscr{A}\mathscr{B}\psi(\mathscr{A})$$

for all $\mathscr{A},\mathscr{B} \in \mathcal{R}$, proved that it is of the form $2\psi(\mathscr{A}) = q\mathscr{A} + \mathscr{A}q$, where q is a fixed element in the symmetric Martindale ring of quotients of $\mathcal{R}$. Further extensions of aforementioned results can be seen in [4, 5].

In this paper, motivated by the above discussed maps and results, we follow this line of investigation and consider the map $\psi : \mathcal{A} \to \mathcal{A}$ satisfying

$$\psi(\mathscr{A}\mathscr{B}^*\mathscr{A}) = \psi(\mathscr{A})\mathscr{B}^*\mathscr{A} - \mathscr{A}\psi(\mathscr{B})^*\mathscr{A} + \mathscr{A}\mathscr{B}^*\psi(\mathscr{A}) \tag{1.1}$$

for all $\mathscr{A},\mathscr{B} \in \mathcal{A}$, and prove the following:

THEOREM 1.1. *Let $M_n(\mathcal{R})$ be the algebra of $n \times n$ matrices over $\mathcal{R}$, where $\mathcal{R}$ is a commutative two-torsion free ring. If $\psi : M_n(\mathcal{R}) \to M_n(\mathcal{R})$ is satisfying* (1.1)*, then there exists $\mathcal{T} \in M_n(\mathcal{R})$ such that $\mathcal{T}^* = \mathcal{T}$ ($*$ is transpose) and an additive endomorphism ϕ of $\mathcal{R}$ such that*

$$\psi(\mathscr{A}) = \frac{3}{2}\psi(I)\mathscr{A} - \frac{1}{2}\mathscr{A}\psi(I) + \mathscr{A}\mathcal{T} - \mathcal{T}\mathscr{A} + \mathscr{A}_\phi$$

for all $\mathscr{A} \in M_n(\mathcal{R})$, where $\mathscr{A}_\phi$ is the image of $\mathscr{A}$ under ϕ applied entrywise.

On the basis of this, as an application, we give a full characterization of (1.1) on $B(\mathcal{H})$, algebra of all bounded linear operators on $\mathcal{H}$, infinite dimensional complex Hilbert space.

2. Preliminaries and Proof of Main Theorem

Prior to beginning the proof, a few notations and preliminary steps must be fixed. Throughout this article, $\mathcal{R}$ refers to the commutative two-torsion free ring, and $M_n(\mathcal{R})$ represents the algebra of all $n \times n$ matrices over $\mathcal{R}$. For any $1 \leqslant j,k \leqslant n$, we write $\mathscr{E}_{jk}$ for the matrix having 1 as its (j,k)-th entry and zeros elsewhere. For a matrix $\mathscr{A} \in M_n(\mathcal{R})$ and an endomorphism ϕ of $\mathcal{R}$ let $\mathscr{A}_\phi$ be the matrix obtained by applying ϕ entrywise. The following result is a special case of Theorem 1.1 for $n = 2$, and is essential to complete the proof.

THEOREM 2.1. *Let ψ be a map satisfying* (1.1)*. Then there exists $\mathcal{T} \in M_2(\mathcal{R})$ such that $\mathcal{T}^* = \mathcal{T}$ ($*$ is transpose) and an additive endomorphism ϕ of $\mathcal{R}$ such that*

$$\psi(\mathscr{A}) = \frac{3}{2}\psi(I)\mathscr{A} - \frac{1}{2}\mathscr{A}\psi(I) + \mathscr{A}\mathcal{T} - \mathcal{T}\mathscr{A} + \mathscr{A}_\phi$$

for all $\mathscr{A} \in M_2(\mathcal{R})$, where $\mathscr{A}_\phi$ is the image of $\mathscr{A}$ under ϕ applied entrywise.

The proof is based on the following lemmas.

LEMMA 2.2. $\psi(0)=0$.

Proof. For $\mathscr{A}=0$, we have

$$\psi(0)=\psi(0)0^*0-0\psi(0)^*0+00^*\psi(0)=0. \quad \square$$

LEMMA 2.3. $\psi(I)^*=\psi(I)$.

Proof. It follows from (1.1) that

$$\psi(I)=\psi(II^*I)=\psi(I)I^*I-I\psi(I)^*I+II^*\psi(I)=2\psi(I)-\psi(I)^*.$$

This gives $\psi(I)^*=\psi(I)$. $\square$

Assume that $\psi(\mathscr{E}_{11})=(w_{ij})$, $\psi(\mathscr{E}_{12})=(x_{ij})$, $\psi(\mathscr{E}_{21})=(y_{ij})$ and $\psi(\mathscr{E}_{22})=(z_{ij})$ for some $w_{ij},x_{ij},y_{ij},z_{ij}\in\mathcal{R}$ and $1\leqslant i,j\leqslant 2$.

LEMMA 2.4. $\psi(\mathscr{E}_{11})=\begin{pmatrix} w_{11} & w_{12}\\ w_{21} & 0\end{pmatrix}$, $\psi(\mathscr{E}_{12})=\begin{pmatrix} w_{12} & x_{12}\\ 0 & x_{22}\end{pmatrix}$, $\psi(\mathscr{E}_{21})=\begin{pmatrix} w_{21} & 0\\ y_{21} & y_{22}\end{pmatrix}$, *and* $\psi(\mathscr{E}_{22})=\begin{pmatrix} 0 & x_{22}\\ y_{22} & z_{22}\end{pmatrix}$.

Proof. Observe that

$$\begin{aligned}\psi(\mathscr{E}_{11}) &= \psi(\mathscr{E}_{11}\mathscr{E}_{11}^*\mathscr{E}_{11})\\ &= \psi(\mathscr{E}_{11})\mathscr{E}_{11}^*\mathscr{E}_{11}-\mathscr{E}_{11}\psi(\mathscr{E}_{11})^*\mathscr{E}_{11}+\mathscr{E}_{11}\mathscr{E}_{11}^*\psi(\mathscr{E}_{11})\\ &= \psi(\mathscr{E}_{11})\mathscr{E}_{11}-\mathscr{E}_{11}\psi(\mathscr{E}_{11})^*\mathscr{E}_{11}+\mathscr{E}_{11}\psi(\mathscr{E}_{11}).\end{aligned}$$

This gives $w_{22}=0$. So we have $\psi(\mathscr{E}_{11})=\begin{pmatrix} w_{11} & w_{12}\\ w_{21} & 0\end{pmatrix}$. Also, we have

$$\begin{aligned}\psi(\mathscr{E}_{12}) &= \psi(\mathscr{E}_{12}\mathscr{E}_{12}^*\mathscr{E}_{12})\\ &= \psi(\mathscr{E}_{12})\mathscr{E}_{12}^*\mathscr{E}_{12}-\mathscr{E}_{12}\psi(\mathscr{E}_{12})^*\mathscr{E}_{12}+\mathscr{E}_{12}\mathscr{E}_{12}^*\psi(\mathscr{E}_{12})\\ &= \psi(\mathscr{E}_{12})\mathscr{E}_{22}-\mathscr{E}_{12}\psi(\mathscr{E}_{12})^*\mathscr{E}_{12}+\mathscr{E}_{11}\psi(\mathscr{E}_{12}).\end{aligned}$$

This yields $x_{21}=0$ and $\psi(\mathscr{E}_{12})=\begin{pmatrix} x_{11} & x_{12}\\ 0 & x_{22}\end{pmatrix}$. Similarly, we can easily get $\psi(\mathscr{E}_{21})=\begin{pmatrix} y_{11} & 0\\ y_{21} & y_{22}\end{pmatrix}$ and $\psi(\mathscr{E}_{22})=\begin{pmatrix} 0 & z_{12}\\ z_{21} & z_{22}\end{pmatrix}$. Next, we have

$$0=\psi(\mathscr{E}_{11}\mathscr{E}_{12}^*\mathscr{E}_{11})=\psi(\mathscr{E}_{11})\mathscr{E}_{12}^*\mathscr{E}_{11}-\mathscr{E}_{11}\psi(\mathscr{E}_{12})^*\mathscr{E}_{11}+\mathscr{E}_{11}\mathscr{E}_{12}^*\psi(\mathscr{E}_{11}).$$

This yields $x_{11}=w_{12}$. Note that $\mathscr{E}_{11}\mathscr{E}_{21}^*\mathscr{E}_{11}=\mathscr{E}_{22}\mathscr{E}_{12}^*\mathscr{E}_{22}=\mathscr{E}_{22}\mathscr{E}_{21}^*\mathscr{E}_{22}=0$. Reasoning as above yields $y_{11}=w_{21}$ and $z_{12}=x_{22}$ and $z_{21}=y_{22}$. This completes the proof. $\square$

Now define a map $\Psi : M_2(\mathcal{R}) \to M_2(\mathcal{R})$ such that

$$\Psi(\mathscr{A}) = \psi(\mathscr{A}) - \frac{3}{2}\mathscr{A}\psi(I) + \frac{1}{2}\psi(I)\mathscr{A}. \tag{2.1}$$

One can check that Ψ is a $*$-Jordan semi-triple like derivation and $\Psi(I) = 0$. Also

$$\Psi(\mathscr{A}^2) = \Psi(\mathscr{A})\mathscr{A} + \mathscr{A}\Psi(\mathscr{A}) \tag{2.2}$$

for all $\mathscr{A} \in M_2(\mathcal{R})$.

LEMMA 2.5. *$\Psi(\mathscr{A}^*) = -\Psi(\mathscr{A})^*$ for all $\mathscr{A} \in M_2(\mathcal{R})$.*

Proof. It follows from hypothesis and Lemma 2.3 that

$$\begin{aligned}
\Psi(\mathscr{A}^*) &= \psi(\mathscr{A}^*) - \frac{3}{2}\mathscr{A}^*\psi(I) + \frac{1}{2}\psi(I)\mathscr{A}^* \\
&= \psi(I)\mathscr{A}^* - \psi(\mathscr{A})^* + \mathscr{A}^*\psi(I) - \frac{3}{2}\mathscr{A}^*\psi(I) + \frac{1}{2}\psi(I)\mathscr{A}^* \\
&= -\psi(\mathscr{A})^* + \frac{3}{2}\psi(I)\mathscr{A}^* - \frac{1}{2}\mathscr{A}^*\psi(I) \\
&= -\left(\psi(\mathscr{A}) - \frac{3}{2}\mathscr{A}\psi(I) + \frac{1}{2}\psi(I)\mathscr{A}\right)^* \\
&= -\Psi(\mathscr{A})^*. \quad \square
\end{aligned}$$

LEMMA 2.6. *$\Psi(\mathscr{E}_{11}) = \begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix}$, $\Psi(\mathscr{E}_{12}) = \begin{pmatrix} \alpha & -\beta \\ 0 & -\alpha \end{pmatrix}$, $\Psi(\mathscr{E}_{21}) = \begin{pmatrix} -\alpha & 0 \\ \beta & \alpha \end{pmatrix}$ and $\Psi(\mathscr{E}_{22}) = \begin{pmatrix} 0 & -\alpha \\ \alpha & 0 \end{pmatrix}$ for some $\alpha, \beta \in \mathcal{R}$.*

Proof. Without loss of generality, assume that $\psi(I) = \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}$ for some $b \in \mathcal{R}$ as per Lemma 2.3. Now, it follows from hypothesis and Lemma 2.3 that

$$\begin{aligned}
\Psi(\mathscr{E}_{11}) &= \psi(\mathscr{E}_{11}) - \frac{3}{2}\mathscr{E}_{11}\psi(I) + \frac{1}{2}\psi(I)\mathscr{E}_{11} \\
&= \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & 0 \end{pmatrix} - \frac{3}{2}\begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} w_{11} & w_{12} - \frac{3}{2}b \\ w_{21} + \frac{1}{2}b & 0 \end{pmatrix}.
\end{aligned} \tag{2.3}$$

Similarly, we can get

$$\Psi(\mathscr{E}_{12}) = \begin{pmatrix} w_{12} - \frac{3}{2}b & x_{12} \\ 0 & x_{22} + \frac{1}{2}b \end{pmatrix}, \Psi(\mathscr{E}_{21}) = \begin{pmatrix} w_{21} + \frac{1}{2}b & 0 \\ y_{21} & y_{22} - \frac{3}{2}b \end{pmatrix},$$

$$\Psi(\mathscr{E}_{22}) = \begin{pmatrix} 0 & x_{22}+\frac{1}{2}b \\ y_{22}-\frac{3}{2}b & z_{22} \end{pmatrix}.$$

Since we know that $\mathscr{E}_{12}^* = \mathscr{E}_{21}$, so it follows from Lemma 2.5 that $\Psi(\mathscr{E}_{12})^* = -\Psi(\mathscr{E}_{21})$ i.e.,

$$\begin{pmatrix} w_{12}-\frac{3}{2}b & x_{12} \\ 0 & x_{22}+\frac{1}{2}b \end{pmatrix}^* = \begin{pmatrix} w_{12}-\frac{3}{2}b & 0 \\ x_{12} & x_{22}+\frac{1}{2}b \end{pmatrix} = -\begin{pmatrix} w_{21}+\frac{1}{2}b & 0 \\ y_{21} & y_{22}-\frac{3}{2}b \end{pmatrix}.$$

Above expression gives

$$w_{12}+w_{21} = b, x_{22}+y_{22} = b \quad \text{and} \quad x_{12} = -y_{21}. \tag{2.4}$$

Also we know from above that Ψ is a Jordan derivation. Thus, we have

$$0 = \Psi(\mathscr{E}_{12}^2) = \Psi(\mathscr{E}_{12})\mathscr{E}_{12} + \mathscr{E}_{12}\Psi(\mathscr{E}_{12}).$$

This yields $w_{12}+x_{22} = b$ and it follows from (2.4) that $w_{12} = y_{22}$ and $w_{21} = x_{22}$. Next, apply the similar arguments for $\mathscr{E}_{11}$ and $\mathscr{E}_{22}$, we get $w_{11} = z_{22} = 0$. Let $2\alpha = w_{12} + 3w_{21}$ and $\beta = y_{21}$. Then $w_{21}+\frac{1}{2}b = -w_{12}+\frac{3}{2}b = \alpha$ and $x_{12} = -\beta$. Thus, we have $\Psi(\mathscr{E}_{11}) = \begin{pmatrix} 0 & -\alpha \\ \alpha & 0 \end{pmatrix}$, $\Psi(\mathscr{E}_{12}) = \begin{pmatrix} -\alpha & -\beta \\ 0 & \alpha \end{pmatrix}$, $\Psi(\mathscr{E}_{21}) = \begin{pmatrix} \alpha & 0 \\ \beta & -\alpha \end{pmatrix}$ and $\Psi(\mathscr{E}_{22}) = \begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix}$ for some $\alpha, \beta \in \mathcal{R}$. Hence the lemma. $\square$

Observe from Lemma 2.6 that $\Psi(\mathscr{E}_{ij}) = \mathscr{E}_{ij}\mathscr{X} - \mathscr{X}\mathscr{E}_{ij}$, where $\mathscr{X} = \begin{pmatrix} \beta & \alpha \\ \alpha & 0 \end{pmatrix}$ and $1 \leqslant i, j \leqslant 2$. Now define

$$\delta(\mathscr{A}) = \Psi(\mathscr{A}) - (\mathscr{A}\mathscr{X} - \mathscr{X}\mathscr{A}) \tag{2.5}$$

for all $\mathscr{A} \in M_2(\mathcal{R})$. It is clear that δ satifies (1.1) and $\delta(\mathscr{E}_{ij}) = 0$.

LEMMA 2.7. *$\delta(\mathscr{A}) = \mathscr{A}_\phi$, where ϕ is any map on $\mathcal{R}$ satisfying $\phi(ab) = \phi(a)\phi(b)$ and $\phi(a+b) = -(\phi(a)+\phi(b))$.*

Proof. Let $\mathscr{A} = (a_{ij})$ and $\phi(\mathscr{A}) = (b_{ij})$. Then it follows from (2.5) and $\delta(\mathscr{E}_{ij}) = 0$ that

$$-b_{ij}\mathscr{E}_{ji} = -\mathscr{E}_{ji}\delta(\mathscr{A})\mathscr{E}_{ji} = \delta(\mathscr{E}_{ij}\mathscr{A}\mathscr{E}_{ij}) = \delta(a_{ji}\mathscr{E}_{ij}).$$

This implies that (i,j)-th entry of $\delta(\mathscr{A})$ depends on (j,i)-th entry of $\mathscr{A}$. Therefore, we may write $\delta(\mathscr{A}) = -\begin{pmatrix} \rho_{11}(a_{11}) & \rho_{12}(a_{21}) \\ \rho_{21}(a_{12}) & \rho_{22}(a_{22}) \end{pmatrix}$ for some maps ρ_{ij} on $\mathcal{R}$. Next, let $\mathcal{T} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Then for any $a \in \mathcal{R}$, we have

$$\begin{aligned} -\rho(a)\mathcal{T} &= -\mathcal{T}(\rho(a)\mathscr{E}_{11})\mathcal{T} = -\mathcal{T}\delta(a\mathscr{E}_{11})\mathcal{T} = \delta(a\mathcal{T}\mathscr{E}_{11}\mathcal{T}) \\ &= \delta(a\mathcal{T}) = \begin{pmatrix} \rho_{11}(a) & \rho_{12}(a) \\ \rho_{21}(a) & \rho_{22}(a) \end{pmatrix}. \end{aligned}$$

This yields $\rho_{11} = \rho_{12} = \rho_{21} = \rho_{22} = -\rho = \phi\ (say)$. Therefore, we can write $\delta(\mathscr{A}) = \mathscr{A}_\phi$. Furthermore, from $\delta(\mathscr{E}_{ij}) = 0$, we conclude that $\phi(0) = \phi(1) = 0$. Next, for any $a, b \in \mathcal{R}$, let $\mathscr{A} = a\mathscr{E}_{11} + b\mathscr{E}_{12}$. Then $\delta(\mathscr{A}) = \phi(a)\mathscr{E}_{11} + \phi(b)\mathscr{E}_{12}$. Since $\delta(\mathscr{A}^2) = \delta(\mathscr{A})^2$, so we have

$$\phi(a^2)\mathscr{E}_{11} + \phi(ab)\mathscr{E}_{12} = \phi(a)^2\mathscr{E}_{11} + \phi(a)\phi(b)\mathscr{E}_{12}.$$

Also, we have

$$-(\phi(a) + \phi(b))\mathcal{T} = -\mathcal{T}\delta(\mathscr{A})\mathcal{T} = \delta(\mathcal{T}\mathscr{A}\mathcal{T}) = \delta((a+b)\mathcal{T}) = \phi(a+b)\mathcal{T}.$$

From above, we get $\phi(ab) = \phi(a)\phi(b)$ and $\phi(a+b) = -(\phi(a) + \phi(b))$. $\square$

The main result of this section can now be proven:

Proof of Theorem 1.1. For any $M_n(\mathcal{R})$, we define

$$\Psi(\mathscr{A}) = \psi(\mathscr{A}) - \frac{3}{2}\mathscr{A}\psi(I) + \frac{1}{2}\psi(I)\mathscr{A}. \tag{2.6}$$

In view of Lemma 2.2 and Lemma 2.5, we know that $\Psi(0) = 0$ and $\Psi(\mathscr{A}^*) = -\Psi(\mathscr{A})^*$. The Theorem is proven by induction of n for Ψ. The result holds for $n = 2$ according to Theorem 2.1. For $n = m$, we assume it is true. The proof is made for $n = m+1$. Let $\mathcal{P} = I_m \oplus [0]$ and $\mathcal{P}' = I - \mathcal{P} = [0]_m \oplus [1]$, where $[0]_m$ and I_m are zero and unit matrix in $M_n(\mathcal{R})$, respectively. Since $\Psi(\mathcal{P}^2) = \Psi(\mathcal{P})\mathcal{P} + \mathcal{P}\Psi(\mathcal{P})$, so it follows that

$$\Psi(\mathcal{P}) = \mathcal{P}\Psi(\mathcal{P})\mathcal{P}' + \mathcal{P}'\Psi(\mathcal{P})\mathcal{P} = \mathcal{P}\mathscr{U} - \mathscr{U}\mathcal{P},$$

where $\mathscr{U} = \mathcal{P}\Psi(\mathcal{P})\mathcal{P}' - \mathcal{P}'\Psi(\mathcal{P})\mathcal{P} \in M_{n+1}(\mathcal{R})$ and $\mathscr{U}^* = \mathscr{U}$. For any $\mathscr{A} \in M_{n+1}(\mathcal{R})$, replacing Ψ by the mapping

$$\mathscr{A} \mapsto \Psi(\mathscr{A}) - (\mathscr{A}\mathscr{U} - \mathscr{U}\mathscr{A}). \tag{2.7}$$

We may assume $\Psi(\mathcal{P}) = 0$. Thus, for any $\mathscr{A}_m \in M_n(\mathcal{R})$, let $\mathscr{A} = \mathscr{A}_m \oplus [0]$. Then it is simple to demonstrate that $\mathscr{A} = \mathcal{P}\mathscr{A}\mathcal{P} \in M_{n+1}(\mathcal{R})$ and $\Psi(\mathcal{P}) = 0$ implies that

$$\Psi(\mathscr{A}) = \Psi(\mathcal{P}\mathscr{A}\mathcal{P}) = -\mathcal{P}\Psi(\mathscr{A})\mathcal{P} = \mathscr{B}_m \oplus [0]$$

for some matrix $\mathscr{B}_m \in M_n(\mathcal{R})$. Define a mapping $\hat{\Psi}(\mathscr{A}) = \mathscr{B}_m$. Clearly, $\hat{\Psi}$ is a $*$-Jordan semi triple derivation on $M_n(\mathcal{R})$. By induction there is a $\mathcal{S} \in M_n(\mathcal{R})$ with $\mathcal{S}^* = \mathcal{S}$ and a multiplicative homomorphism ϕ on $\mathcal{R}$ such that $\hat{\Psi}(\mathscr{A}_m) = \mathscr{A}_m\mathcal{S} - \mathcal{S}\mathscr{A}_m + \mathscr{A}_\phi$ for all $\mathscr{A}_m \in M_n(\mathcal{R})$. Let $\mathscr{V} = \mathcal{S} \oplus [0]$. For any $\mathcal{X} \in M_{n+1}(\mathcal{R})$, define

$$\Gamma(\mathcal{X}) = \Psi(\mathcal{X}) - (\mathcal{X}\mathscr{V} - \mathscr{V}\mathcal{X}). \tag{2.8}$$

Thus we can get a $*$-Jordan semi triple derivable like mapping on $M_n(\mathcal{R})$ such that $\Gamma(\mathcal{X}_m \oplus [0]) = \hat{\Gamma}(\mathcal{X}_m) \oplus [0]$. This is equivalent to

$$\Gamma(\mathcal{X}_m \oplus [0]) = \mathcal{X}_\phi \oplus 0. \tag{2.9}$$

Note that any $\mathscr{A} \in M_{n+1}(\mathcal{R})$, we can write $\mathcal{X} = \begin{pmatrix} \mathcal{X}_{11} & \mathcal{X}_{12} \\ \mathcal{X}_{21} & \mathcal{X}_{22} \end{pmatrix}$ with $\mathcal{X}_{11} \in M_n(\mathcal{R})$. One can observe that $\mathcal{P}\mathcal{X}\mathcal{P} = \mathcal{X}_{11} \oplus [0]$. Therefore

$$-\mathcal{P}\Gamma(\mathcal{X})\mathcal{P} = \Gamma(\mathcal{P}\mathcal{X}\mathcal{P}) = \hat{\Gamma}(\mathcal{X}_{11} \oplus [0]) = (\mathcal{X}_{11})_\phi \oplus [0]. \tag{2.10}$$

Let us define matrices Δ_i for each fixed $i = \{1,2,3,\dots,m\}$ by

$$\Delta_i = I_{m+1} - \mathscr{E}_{ii} - \mathscr{E}_{(m+1)(m+1)} + \mathscr{E}_{i(m+1)} + \mathscr{E}_{(m+1)i}.$$

It follows from (2.10) that $\mathcal{P}\Gamma(\Delta_i)\mathcal{P} = 0$. Then there exist $x_i = (x_{i1}, x_{i2}, \dots, x_{im})$, $y_i = (y_{i1}, y_{i2}, \dots, y_{im}) \in \mathcal{R}^m$ and $z_i \in \mathcal{R}$ such that

$$\Gamma(\Delta_i) = \begin{pmatrix} 0_m & x_i \\ y_i & z_i \end{pmatrix}.$$

Since $\Delta_i^2 = I_{m+1}$ for each fixed i, so we have

$$\Gamma(\Delta_i^2) = \Delta_i\Gamma(\Delta_i) + \Gamma(\Delta_i)\Delta_i = \Gamma(I_{m+1}) = 0.$$

This yields $x_{ii} = -y_{ii}$ and $x_{ik} = y_{ik} = 0$ $(i \neq k)$, $z_i = 0$. So, we have $\Gamma(\Delta_i) = x_{ii}\mathscr{E}_{i(m+1)} - x_{ii}\mathscr{E}_{(m+1)i}$. Let $j \in \{1,2,\dots,m\}$ and $j \neq i$. Then one can observe that $\Delta_j\Delta_i\Delta_j = I_{m+1} - \mathscr{E}_{ii} - \mathscr{E}_{jj} + \mathscr{E}_{ij} + \mathscr{E}_{ji}$. It follows that

$$\begin{aligned} 0 &= \mathcal{P}(I_{m+1} - \mathscr{E}_{ii} - \mathscr{E}_{jj} + \mathscr{E}_{ij} + \mathscr{E}_{ji})\mathcal{P} \\ &= \mathcal{P}\Gamma(\Delta_j\Delta_i\Delta_j)\mathcal{P} \\ &= \mathcal{P}[\Gamma(\Delta_j)\Delta_i\Delta_j - \Delta_j\Gamma(\Delta_i)\Delta_j + \Delta_j\Delta_i\Gamma(\Delta_j)]\mathcal{P} \\ &= \mathcal{P}[(x_{ii} - x_{jj})\mathscr{E}_{ij} + (x_{jj} - x_{ii})\mathscr{E}_{ji}]\mathcal{P} \\ &= (x_{ii} - x_{jj})\mathscr{E}_{ij} + (x_{jj} - x_{ii})\mathscr{E}_{ji}. \end{aligned}$$

This infer us that $x_{ii} = x_{jj}$ with $i \neq j$. Therefore,

$$\Gamma(\Delta_i) = x_{11}\mathscr{E}_{i(m+1)} - x_{11}\mathscr{E}_{(m+1)i}$$

for each $i \in \{1,2,\dots,m\}$. Let $\mathscr{Q} = [0]_m \oplus x_{11}$. For any $\mathcal{Y} \in M_{n+1}(\mathcal{R})$, replacing Γ by the map

$$\mathcal{Y} \mapsto \Psi(\mathcal{Y}) - (\mathcal{Y}\mathscr{Q} - \mathscr{Q}\mathcal{Y}). \tag{2.11}$$

We may assume that $\Gamma(\Delta_i) = 0$ for all $i \in \{1,2,\dots,m\}$. As $m \geqslant 2$, there is another $j \in \{1,2,\dots,m\}$ with $i \neq j$ such that $\mathscr{E}_{i(m+1)} = \Delta_j\mathscr{E}_{ij}\Delta_j$, $\mathscr{E}_{ji} = \Delta_j\mathscr{E}_{(m+1)i}\Delta_j$ and $\mathscr{E}_{(m+1)(m+1)} = \Delta_1\mathscr{E}_{11}\Delta_1$. Then for any $a \in \mathcal{R}$, we obtain

$$\Gamma(a\mathscr{E}_{i(m+1)}) = \Gamma(\Delta_j(a\mathscr{E}_{ij})\Delta_j) = -\Delta_j\phi(a)\mathscr{E}_{ij}\Delta_j = -\phi(a)\mathscr{E}_{i(m+1)}. \tag{2.12}$$

Similarly, we can get

$$\Gamma(a\mathscr{E}_{(m+1)i}) = -\phi(a)\mathscr{E}_{(m+1)i}. \tag{2.13}$$

and

$$\Gamma(a\mathscr{E}_{(m+1)(m+1)}) = -\phi(a)\mathscr{E}_{(m+1)(m+1)}. \tag{2.14}$$

It follows from (2.9), (2.12), (2.13) and (2.14) that $\Gamma(a\mathscr{E}_{ij}) = -\phi(a)\mathscr{E}_{ij} = \phi'(a)\mathscr{E}_{ij}$ for all $i, j \in \{1,2,\ldots,m+1\}$ and $a \in \mathcal{R}$. Finally, for any $\mathscr{A} \in M_{n+1}(\mathcal{R})$, let $\Gamma(\mathscr{A}) = (b_{ij})$. Then

$$-b_{ij}\mathscr{E}_{ji} = -\mathscr{E}_{ji}\Gamma(\mathscr{A})\mathscr{E}_{ji} = \Gamma(\mathscr{E}_{ji}\mathscr{A}\mathscr{E}_{ji}) = \Gamma(a\mathscr{E}_{ji}).$$

This yields that $\Gamma(\mathscr{A}) = (\phi'(a_{ij})) = \mathscr{A}_{\phi'} = \mathscr{A}_{\phi}$ (for brevity, we assume $\phi' = \phi$) for all $\mathscr{A} \in M_{n+1}(\mathcal{R})$. Thus, in view of (2.7), (2.8) and (2.11), we get $\Psi(\mathscr{A}) = \mathscr{A}\mathcal{T} - \mathcal{T}\mathscr{A} + \mathscr{A}_{\phi}$ for all $\mathscr{A} \in M_{n+1}(\mathcal{R})$, here $\mathcal{T} = \mathscr{U} + \mathscr{V} + \mathscr{Q}$ such that $\mathcal{T}^* = \mathcal{T}$. Hence by (2.6), we have

$$\psi(\mathscr{A}) = \frac{3}{2}\mathscr{A}\psi(I) - \frac{1}{2}\psi(I) + \mathscr{A}\mathcal{T} - \mathcal{T}\mathscr{A} + \mathscr{A}_{\phi}$$

for all $\mathscr{A} \in M_{n+1}(\mathcal{R})$. Hereby the proof is completed. □

3. Applications on $B(\mathcal{H})$

In this section, we use Theorem 2.1 in the context of $B(\mathcal{H})$, the algebra of all bounded linear operators on $\mathcal{H}$, infinite dimensional complex Hilbert space. Here, for $\mathscr{A} \in B(\mathcal{H})$, $\mathscr{A}^* = \mathscr{A}$ denotes self adjoint of $\mathscr{A}$. Since $dim\mathcal{H} = \infty$, so there exists a projection $\mathcal{P} \in B(\mathcal{H})$ such that $dim(\mathcal{P}_1\mathcal{H}) = dim(\mathcal{P}_2\mathcal{H}) = \infty$. Let $\mathcal{A}_{ij} = \mathcal{P}_iB(\mathcal{H})\mathcal{P}_j$, $1 \leqslant i, j \leqslant 2$. Then $B(\mathcal{H}) = \mathcal{A}_{11} \oplus \mathcal{A}_{12} \oplus \mathcal{A}_{21} \oplus \mathcal{A}_{22}$.

THEOREM 3.1. *If a map* $\psi : B(\mathcal{H}) \to B(\mathcal{H})$ *with* $\psi(-\mathscr{A}) = -\psi(\mathscr{A})$ *for all* $\mathscr{A} \in B(\mathcal{H})$ *satisfying* (1.1), *then there exists* $\mathcal{T} \in B(\mathcal{H})$ *with* $\mathcal{T}^* = \mathcal{T}$ *such that*

$$\psi(\mathscr{A}) = \frac{3}{2}\mathscr{A}\psi(I) - \frac{1}{2}\psi(I)\mathscr{A} + \mathscr{A}\mathcal{T} - \mathcal{T}\mathscr{A}$$

for all $\mathscr{A} \in B(\mathcal{H})$.

Let

$$\Psi(\mathscr{A}) = \psi(\mathscr{A}) - \frac{3}{2}\mathscr{A}\psi(I) + \frac{1}{2}\psi(I)\mathscr{A}$$

for all $\mathscr{A} \in B(\mathcal{H})$. Then it follows from Lemma 2.3, (2.2) and Lemma 2.5 that $\Psi(I) = 0$, Ψ is a Jordan derivation and $\Psi(\mathscr{A}^*) = -\Psi(\mathscr{A})^*$. If Ψ is additive, then Ψ is an additve Jordan derivation. In view of [7, Theorem 3.1], Ψ is a derivation. By Kadison-Sakai theorem [8, 12], Ψ is an inner derivation i.e., $\Psi(\mathscr{A}) = \mathscr{A}\mathscr{X} - \mathscr{X}\mathscr{A}$ for all $\mathscr{A} \in B(\mathcal{H})$. We only need to show ψ is additive. We shall proof Theorem 3.1 in a series of lemmas.

LEMMA 3.2. *$\Psi(\mathcal{P}_i)=\mathcal{P}_i\mathscr{Y}-\mathscr{Y}\mathcal{P}_i$, where $\mathscr{Y}^*=\mathscr{Y}$ for some $\mathscr{Y}\in B(\mathcal{H})$.*

Proof. In view of (2.2), we have

$$\Psi(\mathcal{P}_i)=\Psi(\mathcal{P}_i)\mathcal{P}_i+\mathcal{P}_i\Psi(\mathcal{P}_i). \tag{3.1}$$

For $1\leqslant i\neq j\leqslant 2$, we have $\mathcal{P}_j\Psi(\mathcal{P}_i)\mathcal{P}_j=0$, and hence

$$\Psi(\mathcal{P}_i)=\mathcal{P}_i\Psi(\mathcal{P}_i)\mathcal{P}_j+\mathcal{P}_j\Psi(\mathcal{P}_i)\mathcal{P}_i. \tag{3.2}$$

From hypothesis, observe that

$$\Psi(\mathcal{P}_1)=\Psi(\mathcal{P}_1)\mathcal{P}_1-\mathcal{P}_1\Psi(\mathcal{P}_1)\mathcal{P}_1+\mathcal{P}_1\Psi(\mathcal{P}_1). \tag{3.3}$$

Multiply above expression by $\mathcal{P}_2$ from left (right), we get

$$\mathcal{P}_2\Psi(\mathcal{P}_1)=\mathcal{P}_2\Psi(\mathcal{P}_1)\mathcal{P}_1 \quad\text{and}\quad \Psi(\mathcal{P}_1)\mathcal{P}_2=\mathcal{P}_1\Psi(\mathcal{P}_1)\mathcal{P}_2. \tag{3.4}$$

Similarly, we can get

$$\mathcal{P}_1\Psi(\mathcal{P}_2)=\mathcal{P}_1\Psi(\mathcal{P}_2)\mathcal{P}_2 \quad\text{and}\quad \Psi(\mathcal{P}_2)\mathcal{P}_1=\mathcal{P}_2\Psi(\mathcal{P}_2)\mathcal{P}_1. \tag{3.5}$$

Above two equations together with (3.2) gives

$$\Psi(\mathcal{P}_i)=\mathcal{P}_j\Psi(\mathcal{P}_i)+\Psi(\mathcal{P}_i)\mathcal{P}_j. \tag{3.6}$$

Let $\mathscr{Y}=\mathcal{P}_2\Psi(\mathcal{P}_1)-\Psi(\mathcal{P}_1)\mathcal{P}_2$. Then from Lemma 2.5, we have $\mathscr{Y}^*=\mathscr{Y}$. Thus,

$$\Psi(\mathcal{P}_i)=\mathcal{P}_i\mathscr{Y}-\mathscr{Y}\mathcal{P}_i. \tag{3.7}$$

Hence the lemma. □

Now, define $\Theta(\mathscr{A})=\Psi(\mathscr{A})-[\mathscr{A},\mathscr{Y}]$ for all $\mathscr{A}\in B(\mathcal{H})$. Easily verifiable that Θ is a $*$-Jordan semi-triple like derivation on $B(\mathcal{H})$ and $\Theta(\mathcal{P}_i)=0$ for $i=1,2$.

LEMMA 3.3. *$\Theta(0)=0$.*

Proof. It is straight forward. □

LEMMA 3.4. *For $i,j=\{1,2\}$, $\Theta(\mathcal{P}_i\mathscr{A}\mathcal{P}_j)=\mathcal{P}_i\Theta(\mathscr{A})\mathcal{P}_j$ for all $\mathscr{A}\in B(\mathcal{H})$.*

Proof. According to hypothesis and $\Theta(\mathcal{P}_i)=0$, we have

$$\Theta(\mathcal{P}_i\mathscr{A}\mathcal{P}_i)=-\mathcal{P}_i\Theta(\mathscr{A})\mathcal{P}_i \tag{3.8}$$

for all $\mathscr{A}\in B(\mathcal{H})$. By polar decomposition theorem, there exists a partial isometry $\mathscr{M}\in\mathcal{A}_{12}$ such that $\mathscr{M}\mathscr{M}^*=\mathcal{P}_1$ and $\mathscr{M}^*\mathscr{M}=\mathcal{P}_2$. Thus, we have

$$0=\Theta(\mathscr{M}\mathcal{P}_1\mathscr{A}\mathcal{P}_2\mathscr{M})=-\mathscr{M}\Theta(\mathcal{P}_1\mathscr{A}\mathcal{P}_2)\mathscr{M} \tag{3.9}$$

for all $\mathscr{A} \in B(\mathcal{H})$. Multiply (3.9) by $\mathscr{M}^*$ on both sides, we get

$$\mathcal{P}_2\Theta(\mathcal{P}_1\mathscr{A}\mathcal{P}_2)\mathcal{P}_1 = 0 \tag{3.10}$$

for all $\mathscr{A} \in B(\mathcal{H})$. In view of (3.8) and (3.10), we have

$$\Theta(\mathcal{P}_1\mathscr{A}\mathcal{P}_2) = \mathcal{P}_1\Theta(\mathcal{P}_1\mathscr{A}\mathcal{P}_2)\mathcal{P}_2 \tag{3.11}$$

for all $\mathscr{A} \in B(\mathcal{H})$. Similarly, we can get

$$\Theta(\mathcal{P}_2\mathscr{A}\mathcal{P}_1) = \mathcal{P}_2\Theta(\mathcal{P}_2\mathscr{A}\mathcal{P}_1)\mathcal{P}_1 \tag{3.12}$$

for all $\mathscr{A} \in B(\mathcal{H})$. Since $\mathscr{M}^*\mathscr{A}\mathscr{M}^* = \mathscr{M}^*\mathcal{P}_1\mathscr{A}\mathcal{P}_2\mathscr{M}^*$, so we have from the last two relations that

$$\begin{aligned}\Theta(\mathscr{M}^*\mathcal{P}_1\mathscr{A}\mathcal{P}_2\mathscr{M}^*) = \Theta(\mathscr{M}^*)\mathcal{P}_1\mathscr{A}\mathcal{P}_2\mathscr{M}^* - \mathscr{M}^*\Theta(\mathcal{P}_1\mathscr{A}\mathcal{P}_2)\mathscr{M}^*\\ +\mathscr{M}^*\mathcal{P}_1\mathscr{A}\mathcal{P}_2\Theta(\mathscr{M}^*)\end{aligned} \tag{3.13}$$

for all $\mathscr{A} \in B(\mathcal{H})$. On the other hand,

$$\begin{aligned}\Theta(\mathscr{M}^*\mathscr{A}\mathscr{M}^*) = \Theta(\mathscr{M}^*)\mathscr{A}\mathscr{M}^* - \mathscr{M}^*\Theta(\mathscr{A})\mathscr{M}^*\\ +\mathscr{M}^*\mathscr{A}\Theta(\mathscr{M}^*)\end{aligned} \tag{3.14}$$

for all $\mathscr{A} \in B(\mathcal{H})$. Observe from (3.12) that $\Theta(\mathscr{M}^*) = \mathcal{P}_2\Theta(\mathscr{M}^*)\mathcal{P}_1$. Thus, it follows from (3.14) that

$$\begin{aligned}\Theta(\mathscr{M}^*\mathscr{A}\mathscr{M}^*) = \Theta(\mathscr{M}^*)\mathcal{P}_1\mathscr{A}\mathcal{P}_2\mathscr{M}^* - \mathscr{M}^*\Theta(\mathscr{A})\mathscr{M}^*\\ +\mathscr{M}^*\mathcal{P}_1\mathscr{A}\mathcal{P}_2\Theta(\mathscr{M}^*)\end{aligned} \tag{3.15}$$

for all $\mathscr{A} \in B(\mathcal{H})$. In view of (3.13) and (3.15), we have $\mathscr{M}^*\Theta(\mathcal{P}_1\mathscr{A}\mathcal{P}_2)\mathscr{M}^* = \mathscr{M}^*\Theta(\mathscr{A})\mathscr{M}^*$ for all $\mathscr{A} \in B(\mathcal{H})$. Multiply both sides of last relation with $\mathscr{M}$, we get $\mathcal{P}_1\Theta(\mathcal{P}_1\mathscr{A}\mathcal{P}_2)\mathcal{P}_2 = \mathcal{P}_1\Theta(\mathscr{A})\mathcal{P}_2$ for all $\mathscr{A} \in B(\mathcal{H})$. In view of (3.11), we have $\Theta(\mathcal{P}_1\mathscr{A}\mathcal{P}_2) = \mathcal{P}_1\Theta(\mathscr{A})\mathcal{P}_2$ for all $\mathscr{A} \in B(\mathcal{H})$. In a similar manner, we can obtain $\Theta(\mathcal{P}_2\mathscr{A}\mathcal{P}_1) = \mathcal{P}_2\Theta(\mathscr{A})\mathcal{P}_1$ for all $\mathscr{A} \in B(\mathcal{H})$. This completes the lemma. □

LEMMA 3.5. *$\Theta(\sum_{i,j=2}^2 \mathscr{A}_{ij}) = \sum_{i,j=2}^2 \Theta(\mathscr{A}_{ij})$ for all $\mathscr{A}_{ij} \in \mathcal{A}_{ij}$.*

Proof. Let $\tau = \Theta(\sum_{i,j=2}^2 \mathscr{A}_{ij})$. Then, in view of Lemma 3.4, we have

$$\tau_{ij} = \mathcal{P}_i\Theta\left(\sum_{i,j=2}^2 \mathscr{A}_{ij}\right)\mathcal{P}_j = \left(\mathcal{P}_i\Theta(\sum_{i,j=2}^2 \mathscr{A}_{ij})\mathcal{P}_j\right) = \Theta(\mathscr{A}_{ij}).$$

Hence the lemma. □

LEMMA 3.6. *$\Theta(2\mathscr{A}_{ij}) = 2\Theta(\mathscr{A}_{ij})$ for all $\mathscr{A}_{ij} \in \mathcal{A}_{ij}$ and $1 \leqslant i \neq j \leqslant 2$.*

Proof. It follows from Lemma 3.2 and Lemma 3.5 that

$$\Theta(I+\mathscr{A}_{ij}) = \Theta(\mathcal{P}_1+\mathcal{P}_2+\mathscr{A}_{ij}) = \Theta(\mathscr{A}_{ij}). \tag{3.16}$$

Thus, we have

$$\begin{aligned} \Theta(2\mathscr{A}_{ij}) &= \Theta((I+\mathscr{A}_{ij})^2) \\ &= \Theta(I+\mathscr{A}_{ij})(I+\mathscr{A}_{ij}) + (I+\mathscr{A}_{ij})\Theta(I+\mathscr{A}_{ij}) \\ &= \Theta(\mathscr{A}_{ij})(I+\mathscr{A}_{ij}) + (I+\mathscr{A}_{ij})\Theta(\mathscr{A}_{ij}) \\ &= 2\Theta(\mathscr{A}_{ij}). \end{aligned} \tag{3.17}$$

Hence the proof. □

LEMMA 3.7. *$\Theta(\mathscr{A}_{ij}+\mathscr{B}_{ij}) = \Theta(\mathscr{A}_{ij}) + \Theta(\mathscr{B}_{ij})$ for all $\mathscr{A}_{ij}, \mathscr{B}_{ij} \in \mathcal{A}_{ij}$ and $1 \leqslant i \neq j \leqslant 2$.*

Proof. The following can be observed from Lemma 3.5 and (3.16) that

$$\begin{aligned} \Theta(\mathscr{A}_{ij}+\mathscr{B}_{ij}) &= \Theta(I+\mathscr{A}_{ij}+\mathscr{B}_{ij}) \\ &= \Theta\left(\left(I+\frac{1}{2}\mathscr{A}_{ij}\right)(I-\mathscr{B}_{ij})\left(I+\frac{1}{2}\mathscr{A}_{ij}\right)\right) \\ &= \frac{1}{2}\Theta(\mathscr{A}_{ij})(I-\mathscr{B}_{ij})\left(I+\frac{1}{2}\mathscr{A}_{ij}\right) \\ &\quad -\left(I+\frac{1}{2}\mathscr{A}_{ij}\right)\Theta(I-\mathscr{B}_{ij})\left(I+\frac{1}{2}\mathscr{A}_{ij}\right) \\ &\quad +\frac{1}{2}\Theta\left(I+\frac{1}{2}\mathscr{A}_{ij}\right)(I-\mathscr{B}_{ij})\Theta(\mathscr{A}_{ij}) \\ &= \frac{1}{2}\Theta(\mathscr{A}_{ij})(I-\mathscr{B}_{ij})\left(I+\frac{1}{2}\mathscr{A}_{ij}\right) \\ &\quad +\left(I+\frac{1}{2}\mathscr{A}_{ij}\right)\Theta(\mathscr{B}_{ij})\left(I+\frac{1}{2}\mathscr{A}_{ij}\right) \\ &\quad +\frac{1}{2}\Theta\left(I+\frac{1}{2}\mathscr{A}_{ij}\right)(I-\mathscr{B}_{ij})\Theta(\mathscr{A}_{ij}) \\ &= \frac{1}{2}\Theta(\mathscr{A}_{ij}) + \Theta(\mathscr{B}_{ij}) + \frac{1}{2}\Theta(\mathscr{A}_{ij}) \\ &= \Theta(\mathscr{A}_{ij}) + \Theta(\mathscr{B}_{ij}). \end{aligned}$$

Hence proved. □

LEMMA 3.8. *$\Theta(\mathscr{A}_{ii}+\mathscr{B}_{ii}) = \Theta(\mathscr{A}_{ii}) + \Theta(\mathscr{B}_{ii})$ for all $\mathscr{A}_{ii}, \mathscr{B}_{ii} \in \mathcal{A}_{ii}$ with $1 \leqslant i \neq j \leqslant 2$.*

Proof. Let $\mathscr{A}_{ii} \in \mathcal{A}_{ii}$ and $\mathscr{B}_{ij} \in \mathcal{A}_{ij}$. Then it follows from Lemma 3.4 and 3.5 that

$$\begin{aligned}
\Theta(\mathscr{A}_{ii}) - \Theta(\mathscr{A}_{ii}\mathscr{B}_{ij}) &= \Theta(\mathscr{A}_{ii} - \mathscr{A}_{ii}\mathscr{B}_{ij}) \\
&= \Theta((\mathcal{P}_i - \mathscr{B}_{ij})\mathscr{A}_{ii}(\mathcal{P}_i - \mathscr{B}_{ij})) \\
&= \Theta(\mathcal{P}_i - \mathscr{B}_{ij})\mathscr{A}_{ii}(\mathcal{P}_i - \mathscr{B}_{ij}) - (\mathcal{P}_i - \mathscr{B}_{ij})\Theta(\mathscr{A}_{ii}) \\
&\quad \times (\mathcal{P}_i - \mathscr{B}_{ij}) + (\mathcal{P}_i - \mathscr{B}_{ij})\mathscr{A}_{ii}\Theta(\mathcal{P}_i - \mathscr{B}_{ij}) \\
&= -\Theta(\mathscr{B}_{ij})\mathscr{A}_{ii}(\mathcal{P}_i - \mathscr{B}_{ij}) - \Theta(\mathscr{A}_{ii})(\mathcal{P}_i - \mathscr{B}_{ij}) \\
&\quad + \mathscr{A}_{ii}\Theta(\mathcal{P}_i - \mathscr{B}_{ij}) \\
&= -\Theta(\mathscr{A}_{ii}) + \Theta(\mathscr{A}_{ii})\mathscr{B}_{ij} + \mathscr{A}_{ii}\Theta(\mathscr{B}_{ij})
\end{aligned}$$

Thus we have

$$\Theta(\mathscr{A}_{ii}) - \Theta(\mathscr{A}_{ii}\mathscr{B}_{ij}) = -\Theta(\mathscr{A}_{ii}) + \Theta(\mathscr{A}_{ii})\mathscr{B}_{ij} + \mathscr{A}_{ii}\Theta(\mathscr{B}_{ij}) \tag{3.18}$$

for all $\mathscr{A}_{ii} \in \mathcal{A}_{ii}$ and $\mathscr{B}_{ij} \in \mathcal{A}_{ij}$. Also, $\mathscr{A}_{ii} + \mathscr{A}_{ii}\mathscr{B}_{ij} = (\mathcal{P}_i + \mathscr{B}_{ij})\mathscr{A}_{ii}(\mathcal{P}_i + \mathscr{B}_{ij})$ for all $\mathscr{A}_{ii} \in \mathcal{A}_{ii}$ and $\mathscr{B}_{ij} \in \mathcal{A}_{ij}$. The same arguments as above can be used to determine

$$\Theta(\mathscr{A}_{ii}) + \Theta(\mathscr{A}_{ii}\mathscr{B}_{ij}) = -\Theta(\mathscr{A}_{ii}) - \Theta(\mathscr{A}_{ii})\mathscr{B}_{ij} + \mathscr{A}_{ii}\Theta(\mathscr{B}_{ij}) \tag{3.19}$$

for all $\mathscr{A}_{ii} \in \mathcal{A}_{ii}$ and $\mathscr{B}_{ij} \in \mathcal{A}_{ij}$. Combination of (3.18) and (3.19) gives

$$\Theta(\mathscr{A}_{ii}\mathscr{B}_{ij}) = -\Theta(\mathscr{A}_{ii})\mathscr{B}_{ij} \tag{3.20}$$

for all $\mathscr{A}_{ii} \in \mathcal{A}_{ii}$ and $\mathscr{B}_{ij} \in \mathcal{A}_{ij}$. In view of (3.20), for any $\mathscr{A}_{ii}, \mathscr{B}_{ii} \in \mathcal{A}_{ii}$ and $\mathscr{C}_{ij} \in \mathcal{A}_{ij}$, we have

$$\Theta((\mathscr{A}_{ii} + \mathscr{B}_{ii})\mathscr{C}_{ij}) = -\Theta(\mathscr{A}_{ii} + \mathscr{B}_{ii})\mathscr{C}_{ij}. \tag{3.21}$$

On the other hand,

$$\begin{aligned}
\Theta((\mathscr{A}_{ii} + \mathscr{B}_{ii})\mathscr{C}_{ij}) &= \Theta(\mathscr{A}_{ii}\mathscr{C}_{ij}) + \Theta(\mathscr{B}_{ii}\mathscr{C}_{ij}) && (3.22) \\
&= -\Theta(\mathscr{A}_{ii})\mathscr{C}_{ij} - \Theta(\mathscr{B}_{ii})\mathscr{C}_{ij} \\
&= -((\Theta(\mathscr{A}_{ii}) + \Theta(\mathscr{B}_{ii}))\mathscr{C}_{ij}) && (3.23)
\end{aligned}$$

In view of (3.21) and (3.22), we have

$$\Theta(\mathscr{A}_{ii} + \mathscr{B}_{ii})\mathscr{C}_{ij} = (\Theta(\mathscr{A}_{ii}) + \Theta(\mathscr{B}_{ii}))\mathscr{C}_{ij}$$

for all $\mathscr{A}_{ii}, \mathscr{B}_{ii} \in \mathcal{A}_{ii}$ and $\mathscr{C}_{ij} \in \mathcal{A}_{ij}$. Therefore, $\Theta(\mathscr{A}_{ii} + \mathscr{B}_{ii}) = \Theta(\mathscr{A}_{ii}) + \Theta(\mathscr{B}_{ii})$ for all $\mathscr{A}_{ii}, \mathscr{B}_{ii} \in \mathcal{A}_{ii}$. This completes the lemma. □

LEMMA 3.9. *Θ is additive.*

Proof. It follows from Lemmas 3.3, 3.5, 3.6, 3.7 and Lemma 3.8 that Θ is additive. This completes the proof. □

Declarations

Conflict of interest. The author declare that he has no competing interests.

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