Operators
and
Matrices
Volume 20, Number 1 (2026), 15–26
doi:10.7153/oam-2026-20-02

SQUARE ROOTS OF m–COMPLEX SYMMETRIC OPERATORS

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Dedicated to my Professor Abdelaziz Tajmouati on the occasion of his 71st birthday.

(*Communicated by R. Curto*)

Abstract. A bounded linear operator $T : H \longrightarrow H$ is called m-complex symmetric if there exists a conjugation C on H such that $\Delta_m(T) = 0$, where $\Delta_m(T) = \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}T^{*j}CT^{m-j}C$. In this paper, we study the local spectral properties of the square root of an m-complex symmetric operator. First, we show that T^2 is m-complex symmetric if T is m-complex symmetric. Moreover, we study the transfer of some local spectral properties from the Hilbert adjoint T^* to T.

1. Introduction

Let $\mathscr{B}(H)$ denote the algebra of bounded linear operators on a complex separable Hilbert space H. Let T^* be the Hilbert adjoint of $T \in \mathscr{B}(H)$. A conjugation on H is an antilinear operator $C : H \longrightarrow H$ which is involutive and reverses the inner product, i.e. $C^2 = I$ and $\langle Cx, Cy\rangle = \langle y, x\rangle$ for all $x, y \in H$. For an integer $m \geqslant 1$, an operator $T \in \mathscr{B}(H)$ is said to be m-complex symmetric if there exists a conjugation C on H such that $\Delta_m(T) = 0$, where $\Delta_m(T) = \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}T^{*j}CT^{m-j}C$. It is interesting to note that

$$\Delta_{m+1}(T) = T^*\Delta_m(T) - \Delta_m(T)CTC. \tag{1}$$

For this reason, if T is m-complex symmetric, then T is n-complex symmetric for $n \geqslant m$. Furthermore, T is 1-complex symmetric if and only if there exists a conjugation C such that $CTC = T^*$. In this case, T is called complex symmetric. The class of complex symmetric operators contains all normal operators, Hankel operators, algebraic operators of order 2, truncated Toeplitz operators, and the Volterra integration operator (see [9, 10]).

In general, many spectral relations are unstable between an arbitrary operator T and its Hilbert adjoint T^*. For example, the (SVEP) (to be defined below) for an operator T is not transferred to T^*. It is enough to see that the right-unilateral shift R on $l^2(\mathbb{N})$ has the (SVEP), while its adjoint $R^* = L$ which coincides with the left-unilateral shift, does not satisfy the (SVEP) (see [2, page 137]). For this reason, many mathematicians considered some classes of operators for which T and T^* share the same

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Mathematics subject classification (2020): 47A10, 47A05, 47A11.

Keywords and phrases: Conjugation, square root of m-complex symmetric operators, local spectral properties.

spectral property. Among the papers that address this problem are [13, 14], for complex symmetric operators. One can find other related results for m-complex symmetric operators in [4–8].

We recall that an operator $T \in \mathscr{B}(H)$ has a square root if there exists an operator $S \in \mathscr{B}(H)$ such that $T = S^2$. It is well known that any positive operator T has a unique positive square root. But in general, an operator may not have a square root. Indeed, Halmos [11] showed that the unilateral shift on the Hardy space does not have a square root. For the study of square roots of a particular operator we refer the reader to [12, 15, 18, 19].

In this paper, we explore local spectral relations between an arbitrary operator $T \in \mathscr{B}(H)$ and its adjoint T^*, when the operator T is a square root of an m-complex symmetric operator (i.e., $\Delta_m(T^2) = 0$). In particular, we study the transfer from T^* to T of the (SVEP), property (β) and decomposability. We also examine the consequences of these relationships.

2. Preliminaries

Before beginning our discussion, we gather together some required results from [1, 2, 16] which are all known.

Let $T \in \mathscr{B}(H)$ denote by $\sigma(T), \sigma_{ap}(T)$ and $\sigma_{su}(T)$ the spectrum, the approximate point spectrum and the surjectivity spectrum respectively. An operator $T \in \mathscr{B}(H)$ is said to have the single valued extension property at $\lambda_0 \in \mathbb{C}$ (abbreviated (SVEP)) if $f \equiv 0$ is the unique analytic function $f : U \longrightarrow H$ on an open neighborhood U of λ_0 which satisfies the equation $(T - \lambda)f(\lambda) \equiv 0$ for all $\lambda \in U$. T is said to have the (SVEP) if T has the (SVEP) at every $\lambda \in \mathbb{C}$. The local resolvent $\rho_T(x)$ of $T \in \mathscr{B}(H)$ at $x \in H$ is the set of $\mu \in \mathbb{C}$, for which there exists an open neighborhood $U \subseteq \mathbb{C}$ of μ and an analytic function $f : U \longrightarrow H$ such that $(T - \lambda)f(\lambda) = x$ for all $\lambda \in U$. The set $\sigma_T(x) = \mathbb{C} \backslash \rho_T(x)$ is the local spectrum of T at x. For an arbitrary subset F of $\mathbb{C}$, the local spectral subspace of $T \in \mathscr{B}(H)$ is defined by $X_T(F) = \{x \in H : \sigma_T(x) \subset F\}$. Recall that an operator $T \in \mathscr{B}(H)$ satisfies Bishop's property (β), if for every open subset D of $\mathbb{C}$ and every sequence of analytic functions $f_n : D \longrightarrow H$ such that $\lim_{n\to\infty} \|(T - \lambda)f_n(\lambda)\|_K = 0$ uniformly on every compact subset K of D, we have $\lim_{n\to\infty} \|f_n(\lambda)\|_K = 0$ uniformly on K. An operator $T \in \mathscr{B}(H)$ is said to be decomposable if for every open cover $\{U, V\}$ of $\mathbb{C}$ there are T-invariant subspaces X and Y such that

$$H = X + Y, \quad \sigma(T_{/X}) \subset \overline{U}, \quad \text{and} \quad \sigma(T_{/Y}) \subset \overline{V}.$$

It is well known that,

$$\text{Decomposable} \ \Rightarrow \ \text{Bishop's property } (\beta) \ \Rightarrow \ \text{(SVEP)}.$$

In addition, T is decomposable if and only if T and T^* both have Bishop's property (β).

3. Square roots of m-complex symmetric operators

In this section, we will study the square root of m-complex symmetric operators. We begin with the following theorem, which shows that T^2 is m-complex symmetric for every m-complex symmetric operator $T \in \mathscr{B}(H)$.

THEOREM 1. *Let C be a conjugation on H and let $T \in \mathscr{B}(H)$. Then, for every $k \geqslant 1$, we have the following assertions:*

(i)

$$\begin{aligned}\Delta_{2k}(T^2) = &\sum_{i=0}^{k-1} 4^i \binom{k-1}{i} T^{*i} \Delta_{4k-2i}(T) C T^i C\\ &+ \sum_{i=0}^{k-1} 4^{k-i} \binom{k-1}{i} T^{*k-i} \Delta_{2k+2i}(T) C T^{k-i} C;\end{aligned}$$

(ii)

$$\begin{aligned}\Delta_{2k+1}(T^2) = &\sum_{i=0}^{k-1} 4^i \binom{k-1}{i} T^{*i} \Delta_{4k-2(i-1)}(T) C T^i C\\ &+ \sum_{i=0}^{k-1} 4^{k-i} \binom{k-1}{i} T^{*k-i} \Delta_{2k+2(i+1)}(T) C T^{k-i} C\\ &+ 2\sum_{i=0}^{k-1} \binom{k-1}{i} [4^i T^{*i} \Delta_{4k-2i+1}(T) C T^i C\\ &+ 4^{k-1} T^{*k-i} \Delta_{2k+2i+1}(T) C T^{k-i} C] CTC.\end{aligned}$$

Consequently, if T is m-complex symmetric, then T^2 is also m-complex symmetric.

Proof.

(i) Before starting our proof, some preparations are needed. First, let us prove that:

$$\Delta_{m+2}(T^2) = T^{*4}\Delta_m(T^2) - 2T^{*2}\Delta_m(T^2)CT^2C + \Delta_m(T^2)CT^4C.$$

By using (1), we have $\Delta_{m+1}(T^2) = T^{*2}\Delta_m(T^2) - \Delta_m(T^2)CT^2C$ and then

$$\begin{aligned}\Delta_{m+2}(T^2) &= T^{*2}\Delta_{m+1}(T^2) - \Delta_{m+1}(T^2)CT^2C\\ &= T^{*4}\Delta_m(T^2) - 2T^{*2}\Delta_m(T^2)CT^2C + \Delta_m(T^2)CT^4C.\end{aligned}$$

Now, assume that $\Delta_{m+i}(T) = \sum_{j=0}^{i}(-1)^j\binom{i}{j}T^{*i-j}\Delta_m(T)CT^jC$. We must show that $\Delta_{m+i+1}(T) = \sum_{j=0}^{i+1}(-1)^j\binom{i+1}{j}T^{*i+1-j}\Delta_m(T)CT^jC$.

Indeed, by using (1) we have:

$$\begin{aligned}
\Delta_{m+i+1}(T) =& T^*\Delta_{m+i}(T) - \Delta_{m+i}(T)CTC \\
=& \sum_{j=0}^{i}(-1)^j\binom{i}{j}T^{*i+1-j}\Delta_m(T)CT^jC - \sum_{j=0}^{i}(-1)^j\binom{i}{j}T^{*i-j}\Delta_m(T)CT^{j+1}C \\
=& T^{*i+1}\Delta_m(T) + \sum_{j=1}^{i}(-1)^j\binom{i}{j}T^{*i+1-j}\Delta_m(T)CT^jC \\
& + (-1)^{i+1}\Delta_m(T)CT^{i+1}C - \sum_{j=0}^{i-1}(-1)^j\binom{i}{j}T^{*i-j}\Delta_m(T)CT^{j+1}C \\
=& T^{*i+1}\Delta_m(T) + \sum_{j=0}^{i-1}(-1)^{j+1}\binom{i}{j+1}T^{*i-j}\Delta_m(T)CT^{j+1}C \\
& + (-1)^{i+1}\Delta_m(T)CT^{i+1}C + \sum_{j=0}^{i-1}(-1)^{j+1}\binom{i}{j}T^{*i-j}\Delta_m(T)CT^{j+1}C \\
=& T^{*i+1}\Delta_m(T) + \sum_{j=0}^{i-1}(-1)^{j+1}\left(\binom{i}{j+1}+\binom{i}{j}\right)T^{*i-j}\Delta_m(T)CT^{j+1}C \\
& + (-1)^{i+1}\Delta_m(T)CT^{i+1}C.
\end{aligned}$$

Hence, since $\binom{i}{j+1}+\binom{i}{j}=\binom{i+1}{j+1}$, we obtain.

$$\begin{aligned}
\Delta_{m+i+1}(T) =& T^{*i+1}\Delta_m(T) + \sum_{j=0}^{i-1}(-1)^{j+1}\binom{i+1}{j+1}T^{*i-j}\Delta_m(T)CT^{j+1}C \\
& + (-1)^{i+1}\Delta_m(T)CT^{i+1}C \\
=& T^{*i+1}\Delta_m(T) + \sum_{j=1}^{i}(-1)^j\binom{i+1}{j}T^{*i+1-j}\Delta_m(T)CT^jC \\
& + (-1)^{i+1}\Delta_m(T)CT^{i+1}C \\
=& \sum_{j=0}^{i+1}(-1)^j\binom{i+1}{j}T^{*i+1-j}\Delta_m(T)CT^jC.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\Delta_{m+4}(T) =& T^{*4}\Delta_m(T) - 4T^{*3}\Delta_m(T)CTC + 6T^{*2}\Delta_m(T)CT^2C - 4T^*\Delta_m(T)CT^3C \\
& + \Delta_m(T)CT^4C \\
=& \left[T^{*4}\Delta_m(T) - 2T^{*2}\Delta_m(T)CT^2C + \Delta_m(T)CT^4C\right] - 4[T^{*3}\Delta_m(T)CTC \\
& - 2T^{*2}\Delta_m(T)CT^2C + T^*\Delta_m(T)CT^3C] \\
=& \left[T^{*4}\Delta_m(T) - 2T^{*2}\Delta_m(T)CT^2C + \Delta_m(T)CT^4C\right] - 4T^*[T^{*2}\Delta_m(T) \\
& - 2T^*\Delta_m(T)CTC + \Delta_m(T)CT^2C]CTC \\
=& \left[T^{*4}\Delta_m(T) - 2T^{*2}\Delta_m(T)CT^2C + \Delta_m(T)CT^4C\right] - 4T^*\Delta_{m+2}(T)CTC.
\end{aligned}$$

Having disposed of these preliminary steps, we can now return to our proof. In fact, for $k=1$, we have

$$\begin{aligned}\Delta_2(T^2) &= T^{*4}-2T^{*2}CT^2C+CT^4C\\ &= \Delta_4(T)+4T^*CT^3C-8T^{*2}CT^2C+4T^{*3}CTC\\ &= \Delta_4(T)+4T^*\left[CT^2C-2T^*CTC+T^{*2}\right]CTC\\ &= \Delta_4(T)+4T^*\Delta_2(T)CTC.\end{aligned}$$

For $k=2$, we have

$$\begin{aligned}\Delta_4(T^2) =&T^{*4}\Delta_2(T^2)-2T^{*2}\Delta_2(T^2)CT^2C+\Delta_2(T^2)CT^4C\\ =&T^{*4}\Delta_4(T)-2T^{*2}\Delta_4(T)CT^2C+\Delta_4(T)CT^4C+4T^*[T^{*4}\Delta_2(T)\\ &-2T^{*2}\Delta_2(T)CT^2C+\Delta_2(T)CT^4C]CTC\\ =&\Delta_8(T)+4T^*\Delta_6(T)CTC+4T^*\Delta_6(T)CTC+4^2T^{*2}\Delta_4(T)CT^2C.\end{aligned}$$

For $k>2$ assume that then,

$$\begin{aligned}\Delta_{2k+2}(T^2) =&T^{*4}\Delta_{2k}(T^2)-2T^{*2}\Delta_{2k}(T^2)CT^2C+\Delta_{2k}(T^2)CT^4C\\ =&\sum_{i=0}^{k-1}4^i\binom{k-1}{i}T^{*i}\\ &\times\left(T^{*4}\Delta_{4k-2i}(T)-2T^{*2}\Delta_{4k-2i}(T)CT^2C+\Delta_{4k-2i}(T)CT^4C\right)CT^iC\\ &+\sum_{i=0}^{k-1}4^{k-i}\binom{k-1}{i}T^{*k-i}\\ &\times\left(T^{*4}\Delta_{2k+2i}(T)-2T^{*2}\Delta_{2k+2i}(T)CT^2C+\Delta_{2k+2i}(T)CT^4C\right)CT^{k-i}C\\ =&\sum_{i=0}^{k-1}4^i\binom{k-1}{i}T^{*i}\left(\Delta_{4k-2i+4}(T)+4T^*\Delta_{4k-2i+2}(T)CTC\right)CT^iC\\ &+\sum_{i=0}^{k-1}4^{k-i}\binom{k-1}{i}T^{*k-i}\left(\Delta_{2k+2i+4}(T)+4T^*\Delta_{2k+2i+2}(T)CTC\right)CT^{k-i}C\\ =&\sum_{i=0}^{k-1}4^i\binom{k-1}{i}T^{*i}\Delta_{4k-2i+4}(T)CT^iC\\ &+\sum_{i=0}^{k-1}4^{i+1}\binom{k-1}{i}T^{*i+1}\Delta_{4k-2i+2}(T)CT^{i+1}C\\ &+\sum_{i=0}^{k-1}4^{k-i}\binom{k-1}{i}T^{*k-i}\Delta_{2k+2i+4}(T)CT^{k-i}C\\ &+\sum_{i=0}^{k-1}4^{k-i+1}\binom{k-1}{i}T^{*k-i+1}\Delta_{2k+2i+2}(T)CT^{k-i+1}C\end{aligned}$$

$$
\begin{aligned}
&=\sum_{i=0}^{k-1}4^i\binom{k-1}{i}T^{*i}\Delta_{4k-2i+4}(T)CT^iC\\
&\quad+\sum_{i=1}^{k}4^i\binom{k-1}{i-1}T^{*i}\Delta_{4k-2i+4}(T)CT^iC\\
&\quad+\sum_{i=1}^{k}4^{k-i+1}\binom{k-1}{i-1}T^{*k-i+1}\Delta_{2k+2i+2}(T)CT^{k-i+1}C\\
&\quad+\sum_{i=0}^{k-1}4^{k-i+1}\binom{k-1}{i}T^{*k-i+1}\Delta_{2k+2i+2}(T)CT^{k-i+1}C\\
&=\Delta_{4k+4}(T)+\sum_{i=1}^{k-1}4^i\binom{k-1}{i}T^{*i}\Delta_{4k-2i+4}(T)CT^iC\\
&\quad+4^kT^{*k}\Delta_{2k+4}(T)CT^kC+\sum_{i=1}^{k-1}4^i\binom{k-1}{i-1}T^{*i}\Delta_{4k-2i+4}(T)CT^iC\\
&\quad+4T^*\Delta_{2k+2}(T)CTC+\sum_{i=1}^{k-1}4^{k-i+1}\binom{k-1}{i-1}T^{*k-i+1}\Delta_{2k+2i+2}(T)CT^{k-i+1}C\\
&\quad+4^{k+1}T^{*k+1}\Delta_{2k+2}(T)CT^{k+1}C\\
&\quad+\sum_{i=1}^{k-1}4^{k-i+1}\binom{k-1}{i}T^{*k-i+1}\Delta_{2k+2i+2}(T)CT^{k-i+1}C\\
&=\Delta_{4k+4}(T)+4^kT^{*k}\Delta_{2k+4}(T)CT^kC\\
&\quad+\sum_{i=1}^{k-1}4^i\left(\binom{k-1}{i}+\binom{k-1}{i-1}\right)T^{*i}\Delta_{4k-2i+4}(T)CT^iC\\
&\quad+4T^*\Delta_{4k+2}(T)CTC+4^{k+1}T^{*k+1}\Delta_{2k+2}(T)CT^{k+1}C\\
&\quad+\sum_{i=1}^{k-1}4^{k-i+1}\left(\binom{k-1}{i}+\binom{k-1}{i-1}\right)T^{*k-i+1}\Delta_{2k+2i+2}(T)CT^{k-i+1}C.
\end{aligned}
$$

Hence, since $\binom{k-1}{i-1}+\binom{k-1}{i}=\binom{k}{i}$, we obtain.

$$
\begin{aligned}
\Delta_{2k+2}(T^2)=&\Delta_{4k+4}(T)+4^kT^{*k}\Delta_{2k+4}(T)CT^kC\\
&+\sum_{i=1}^{k-1}4^i\binom{k}{i}T^{*i}\Delta_{4k-2i+4}(T)CT^iC\\
&+4T^*\Delta_{4k+2}(T)CTC+4^{k+1}T^{*k+1}\Delta_{2k+2}(T)CT^{k+1}C\\
&+\sum_{i=1}^{k-1}4^{k-i+1}\binom{k}{i}T^{*k-i+1}\Delta_{2k+2i+2}(T)CT^{k-i+1}C\\
=&\sum_{i=0}^{k}4^i\binom{k}{i}T^{*i}\Delta_{4k-2i+4}(T)CT^iC\\
&+\sum_{i=0}^{k}4^{k-i+1}\binom{k}{i}T^{*k-i+1}\Delta_{2k+2i+2}(T)CT^{k-i+1}C.
\end{aligned}
$$

Which is the desired conclusion.

(ii) is guaranteed by using assertion (i) and (1).

Since T is n-complex symmetric for every $n > m$ whenever it is m-complex symmetric, then by the above assertions, if T is m-complex symmetric, T^2 is also m-complex symmetric. $\square$

However, the converse is not true, as is shown in the following example.

EXAMPLE 1. Let $T = \begin{pmatrix} 0 & i & i \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ on $\mathbb{C}^3$ and let C be a conjugation on $\mathbb{C}^3$ given by $C(z_1, z_2, z_3) = (\overline{z_3}, \overline{z_2}, \overline{z_1})$. Then $T^* = \begin{pmatrix} 0 & 0 & 0 \\ -i & 0 & 0 \\ -i & 1 & 0 \end{pmatrix}$ and $CTC = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ i & i & 0 \end{pmatrix}$. By a simple computation we find that $\Delta_2(T^2) = T^{*4} - 2T^{*2}CT^2C + CT^4C = 0_3$. While $\Delta_2(T) = T^{*2} - 2T^*CTC + CT^2C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix} \neq 0_3$.

As a consequence of Theorem 1 and [6, Theorem 2.1], we have the following result.

COROLLARY 1. *Let $T \in \mathscr{B}(H)$ and let C be a conjugation on H. Then $\Delta_{2k}(T^2)$ is complex symmetric and $\Delta_{2k+1}(T^2)$ is skew complex symmetric, i.e $C\Delta_{2k}(T^2)C = \Delta_{2k}(T^2)^*$ and $C\Delta_{2k+1}(T^2)C = -\Delta_{2k+1}(T^2)^*$ respectively.*

Proof. It follows from [6, Theorem 2.1], that $C\Delta_m(T)C = \Delta_m(T)^*$ when m is even, and $C\Delta_m(T)C = -\Delta_m(T)^*$ when m is odd. Therefore

$$\begin{aligned} C\Delta_{2k}(T^2)^*C =& \sum_{i=0}^{k-1} 4^i \binom{k-1}{i} T^{*i} C\Delta_{4k-2i}(T)^* CCT^i C \\ &+ \sum_{i=0}^{k-1} 4^{k-i} \binom{k-1}{i} T^{*k-i} C\Delta_{2k+2i}(T)^* CCT^{k-i} C \\ =& \sum_{i=0}^{k-1} 4^i \binom{k-1}{i} T^{*i} \Delta_{4k-2i}(T) CT^i C \\ &+ \sum_{i=0}^{k-1} 4^{k-i} \binom{k-1}{i} T^{*k-i} \Delta_{2k+2i}(T) CT^{k-i} C \\ =& \Delta_{2k}(T^2). \end{aligned}$$

By applying the same argument again, we obtain that $C\Delta_{2k+1}(T^2)C = -\Delta_{2k+1}(T^2)^*$. $\square$

4. Local spectral properties of the square roots of m-complex symmetric operators

In this section, we will restrict our attention to the study of local spectral properties of a square root of an m-complex symmetric operator.

THEOREM 2. *Let $T \in \mathscr{B}(H)$ be a square root of an m-complex symmetric operator, i.e., $\Delta_m(T^2) = 0$ for a conjugation C on H. If T^* has the (SVEP), then T has also the (SVEP).*

Proof. Assume that $\Delta_m(T^2) = 0$. Then we have the following equation

$$\begin{aligned} 0 &= \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} (T^{*2} - \overline{\lambda}^2)^j (CT^2C - \overline{\lambda}^2)^{m-j} \\ &= \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} T^{*2j} C T^{2m-2j} C. \end{aligned}$$

That is $\Delta_m(T^2 - \lambda^2) = 0$ for every $\lambda \in \mathbb{C}$. Now let $f : G^* \longrightarrow H$ be an analytic function such that $(CTC - \overline{\lambda}) f(\overline{\lambda}) \equiv 0$ on G^*, where G is a domain in $\mathbb{C}$ and $G^* = \left\{\overline{\lambda} : \lambda \in G\right\}$. Then $(CT^2C - \overline{\lambda}^2) f(\overline{\lambda}) \equiv 0$ on G^*. Thus,

$$\begin{aligned} &\sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} (T^{*2} - \overline{\lambda}^2)^j (CT^2C - \overline{\lambda}^2)^{m-j} f(\overline{\lambda}) \\ =&(T^{*2} - \overline{\lambda}^2)^m f(\overline{\lambda}) + \left[\sum_{j=0}^{m-1} (-1)^{m-j} \binom{m}{j} (T^{*2} - \overline{\lambda}^2)^j (CT^2C - \overline{\lambda}^2)^{m-j-1} \right] \\ &\times (CT^2C - \overline{\lambda}^2) f(\overline{\lambda}) \\ =&0. \end{aligned}$$

We get that $(T^{*2} - \overline{\lambda}^2)^m f(\overline{\lambda}) \equiv 0$. Hence, $(T^* - \overline{\lambda})(T^* + \overline{\lambda})(T^{*2} - \overline{\lambda}^2)^{m-1} f(\overline{\lambda}) \equiv 0$. Since T^* has the (SVEP), it follows that $(T^* + \overline{\lambda})(T^{*2} - \overline{\lambda}^2)^{m-1} f(\overline{\lambda}) \equiv 0$. By a simple induction we obtain that $(T^* + \overline{\lambda})^m f(\overline{\lambda}) \equiv 0$. We set $\mu = -\lambda$, and let $g : -G^* \longrightarrow \mathbb{C}$ be an analytic function defined by $g(\overline{\mu}) = \overline{\lambda}$. Then

$$(T^* - \overline{\mu})^m (f \circ g)(\overline{\mu}) = (T^* + \overline{\lambda})^m f(\overline{\lambda}) \equiv 0.$$

Again, since T^* has the (SVEP), then $(T^* - \overline{\mu})^{m-1} (f \circ g)(\overline{\mu}) \equiv 0$ on $-G^*$. Repeating the same process one can write $(f \circ g)(\overline{\mu}) \equiv 0$ on $-G^*$. So $f(\overline{\lambda}) \equiv 0$ on G^*. As a result CTC has the (SVEP), and by using [3, Theorem 2.2] T has also the (SVEP). $\square$

COROLLARY 2. *Let $T \in \mathscr{B}(H)$ be a square root of an m-complex symmetric operator with a conjugation C. If T^* has the (SVEP), then we have the following inclusion*

$$\sigma_{T^*}(x) \subseteq \sigma_T(Cx)^* \cup -\sigma_T(Cx)^*.$$

Proof. If T^2 is m-complex symmetric, then according to [5, Theorem 4.10] we have $\sigma_{T^{*2}}(x) \subseteq \sigma_{T^2}(Cx)^*$ for all $x \in H$. In addition, T^* has the (SVEP), then Theorem 2, implies that T also has the (SVEP). By means of the local spectral mapping theorem in [16, Theorem 3.3.8] we have

$$\sigma_{T^*}(x)^2 = \sigma_{T^{*2}}(x) \subseteq \sigma_{T^2}(Cx)^* = \sigma_T(Cx)^{*2}.$$

On the other hand, $\sigma_{T^*}(x)^2 = \sigma_{T^*}(x) \cup -\sigma_{T^*}(x)$ and $\sigma_T(Cx)^{*2} = \sigma_T(Cx)^* \cup -\sigma_T(Cx)^*$. Consequently, $\sigma_{T^*}(x) \subseteq \sigma_T(Cx)^* \cup -\sigma_T(Cx)^*$. $\square$

Let us recall that an operator $T \in \mathscr{B}(H)$ has Dunford's boundedness condition (B) if T has the (SVEP) and for every $x, y \in H$, there exists a constant $K > 0$ such that $\|x\| \leqslant K\|x+y\|$ whenever $\sigma_T(x) \cap \sigma_T(y) = \emptyset$, where K is independent of x and y.

COROLLARY 3. *Let $T \in \mathscr{B}(H)$ be a square root of an m-complex symmetric operator with a conjugation C. If $\sigma(T) \cap -\sigma(T) = \emptyset$, then we have the following implication: T^* has the Dunfors's boundedness condition (B) then T also does.*

Proof. Since T^* has the (SVEP), then by Theorem 2, T also has the (SVEP). Now, let $x, y \in H$ such that $\sigma_T(x) \cap \sigma_T(y) = \emptyset$. Clearly $CH = H$ so, there exist x_1 and x_2 in H such that $x = Cx_1$ and $y = Cy_1$. Thus $\sigma_T(Cx_1)^* \cap \sigma_T(Cy_1)^* = \emptyset$ and $-\sigma_T(Cx_1)^* \cap -\sigma_T(Cy_1)^* = \emptyset$. By corollary 2 we have

$$\begin{aligned}
\sigma_{T^*}(x_1) \cap \sigma_{T^*}(y_1) &\subset (\sigma_T(Cx_1)^* \cup -\sigma_T(Cx_1)^*) \cap (\sigma_T(Cy_1)^* \cup -\sigma_T(Cy_1)^*). \\
&\subset (\sigma_T(Cx_1)^* \cap \sigma_T(Cy_1)^*) \cup (\sigma_T(Cx_1)^* \cap -\sigma_T(Cy_1)^*) \\
&\quad \cup (-\sigma_T(Cx_1)^* \cap \sigma_T(Cy_1)^*) \cup (\sigma_T(-Cx_1)^* \cap -\sigma_T(Cy_1)^*) \\
&= (\sigma_T(Cx_1)^* \cap -\sigma_T(Cy_1)^*) \cup (-\sigma_T(Cx_1)^* \cap \sigma_T(Cy_1)^*).
\end{aligned}$$

We have $\sigma(T) \cap -\sigma(T) = \emptyset$, consequently $\sigma_T(Cx_1)^* \cap -\sigma_T(Cy_1)^* \subset \sigma(T)^* \cap -\sigma(T)^* = \emptyset$ and $-\sigma_T(Cx_1)^* \cap \sigma_T(Cy_1)^* \subset \sigma(T)^* \cap -\sigma(T)^* = \emptyset$. We get that $\sigma_{T^*}(x_1) \cap \sigma_{T^*}(y_1) = \emptyset$. As T^* has the Dunford's boundedness condition (B), there exist a constant K such that $\|x_1\| \leqslant K\|x_1 + y_1\|$, where K is independent of x_1 and x_2. Thus $\|Cx_1\| \leqslant K\|Cx_1 + Cy_1\|$, i.e. $\|x\| \leqslant K\|x+y\|$. $\square$

COROLLARY 4. *Let $T \in \mathscr{B}(H)$ be a square root of an m-complex symmetric operator with a conjugation C. If T^* has the (SVEP), then we have the following inclusions*

$$CH_T(F) \subseteq H_{T^*}(F^*) \cup H_{T^*}(-F^*), \quad \textit{and} \quad CH_T(F) \subseteq H_{T^*}(F^* \cup -F^*).$$

Proof. Let $y \in CH_T(F)$ then $Cy \in H_T(F)$, and therefore $\sigma_T(Cy)^{*2} \subset F^{*2}$. By corollary 2, we get $\sigma_{T^*}(y)^2 \subset \sigma_T(Cy)^{*2} \subset F^{*2}$. Thus $\sigma_{T^*}(y) \subset F^* \cup -F^*$. Consequently $y \in H_{T^*}(F^*) \cup H_{T^*}(-F^*)$ and $y \in H_{T^*}(F^* \cup -F^*)$. $\square$

The following results describe the relation between the spectrum $\sigma(T)$, the approximate point spectrum $\sigma_{ap}(T)$ and the surjectivity spectrum $\sigma_{su}(T)$ when T^2 is m-complex symmetric. Further relations are given when T^* has the (SVEP).

THEOREM 3. *Let $T \in \mathscr{B}(H)$ be a square root of an m-complex symmetric operator with a conjugation C, then the following relations hold:*

(i) $\sigma_{ap}(T) \cup -\sigma_{ap}(T) \subseteq \sigma_{ap}(T^*)^* \cup -\sigma_{ap}(T^*)^*$ *and* $\sigma_p(T) \cup -\sigma_p(T) \subseteq \sigma_p(T^*)^* \cup -\sigma_p(T^*)^*$,

(ii) $\sigma_{su}(T^*)^* \cup -\sigma_{su}(T^*)^* \subseteq \sigma_{su}(T) \cup -\sigma_{su}(T)$,

(iii) $\sigma(T) \cup -\sigma(T) = \sigma_{su}(T) \cup -\sigma_{su}(T) = \sigma_{ap}(T^*)^* \cup -\sigma_{ap}(T^*)^*$.

Proof. (i) It suffices to show only the first relation. The same reasoning applies to establish the second. Indeed, if $\lambda \in \sigma_{ap}(T)$ then there exist a sequence $(x_n) \subset H$ such that $\|x_n\| = 1$ and $(T-\lambda)x_n \longrightarrow 0$ as $n \to \infty$. So, $(CTC - \overline{\lambda})Cx_n \longrightarrow 0$ as $n \to \infty$. Therefore $(CT^2C - \overline{\lambda}^2)Cx_n \longrightarrow 0$ as $n \to \infty$. Since T^2 is m-complex symmetric, then we have

$$\begin{aligned}0 &= \lim_{n\longrightarrow\infty}\left(\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}T^{*2j}CT^{2m-2j}C\right)Cx_n \\ &= \lim_{n\longrightarrow\infty}\left(\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}T^{*2j}\overline{\lambda}^{2m-2j}\right)Cx_n \\ &= \lim_{n\longrightarrow\infty}(T^{*2}-\overline{\lambda}^2)^m Cx_n\end{aligned}$$

If $\lim_{n\to\infty}\dfrac{(T^{*2}-\overline{\lambda}^2)^{m-1}Cx_n}{\|(T^{*2}-\overline{\lambda}^2)^{m-1}Cx_n\|} \neq 0$, then $\overline{\lambda}^2 \in \sigma_{ap}(T^{*2}) = \sigma_{ap}(T^*)^2$. Consequently $\overline{\lambda} \in \sigma_{ap}(T^*) \cup -\sigma_{ap}(T^*)$, i.e $\sigma_{ap}(T)^* \subseteq \sigma_{ap}(T^*) \cup -\sigma_{ap}(T^*)$.

Otherwise, $\lim_{n\to\infty}(T^{*2}-\overline{\lambda}^2)^{m-1}Cx_n = 0$. Now, if $\lim_{n\to\infty}\dfrac{(T^{*2}-\overline{\lambda}^2)^{m-2}Cx_n}{\|(T^{*2}-\overline{\lambda}^2)^{m-2}Cx_n\|} \neq 0$, we obtain that $\overline{\lambda}^2 \in \sigma_{ap}(T^{*2}) = \sigma_{ap}(T^*)^2$. Thus, $\overline{\lambda} \in \sigma_{ap}(T^*) \cup -\sigma_{ap}(T^*)$ which mean that $\sigma_{ap}(T)^* \subseteq \sigma_{ap}(T^*) \cup -\sigma_{ap}(T^*)$. Otherwise, $\lim_{n\to\infty}(T^{*2}-\overline{\lambda}^2)^{m-2}Cx_n = 0$. By induction we get that $\lim_{n\to\infty}(T^{*2}-\overline{\lambda}^2)Cx_n = 0$. Hence, $\sigma_{ap}(T)^* \subseteq \sigma_{ap}(T^*) \cup -\sigma_{ap}(T^*)$. We can see also that $-\sigma_{ap}(T)^* \subseteq \sigma_{ap}(T^*) \cup -\sigma_{ap}(T^*)$, which yields $-\sigma_{ap}(T)^* \cup \sigma_{ap}(T)^* \subseteq \sigma_{ap}(T^*) \cup -\sigma_{ap}(T^*)$.

(ii) It is known that $\sigma_{ap}(T) = \sigma_{su}(T^*)^*$ and $\sigma_{ap}(T^*)^* = \sigma_{su}(T)$. We deduce from the first inclusion in (i) that $\sigma_{su}(T^*)^* \cup -\sigma_{su}(T^*)^* \subseteq \sigma_{su}(T) \cup -\sigma_{su}(T)$.

(iii) It is clear that,

$$\sigma(T) = \sigma_{ap}(T) \cup \sigma_{su}(T) = \sigma_{su}(T^*)^* \cup \sigma_{su}(T) = \sigma_{ap}(T) \cup \sigma_{ap}(T^*)^*.$$

For this reason,

$$\begin{aligned}\sigma(T) \cup -\sigma(T) =& \sigma_{su}(T^*)^* \cup -\sigma_{su}(T^*)^* \cup \sigma_{su}(T) \cup -\sigma_{su}(T) \\ \subseteq& \sigma_{su}(T) \cup -\sigma_{su}(T).\end{aligned}$$

and,

$$\begin{aligned}\sigma(T)\cup-\sigma(T) =&\sigma_{ap}(T)\cup-\sigma_{ap}(T)\cup\sigma_{ap}(T^*)^*\cup-\sigma_{ap}(T^*)^*\\ \subseteq&\sigma_{ap}(T^*)^*\cup-\sigma_{ap}(T^*)^*.\end{aligned}$$

Therefore, $\sigma(T)\cup-\sigma(T)=\sigma_{su}(T)\cup-\sigma_{su}(T)=\sigma_{ap}(T^*)^*\cup-\sigma_{ap}(T^*)^*$. ☐

COROLLARY 5. *Let $T\in\mathscr{B}(H)$ be a square root of an m-complex symmetric operator with a conjugation C. If T^* has the (SVEP), then we have the following equality*

$$\sigma(T)\cup-\sigma(T)=\sigma_{ap}(T)\cup-\sigma_{ap}(T)=\sigma_{su}(T^*)^*\cup-\sigma_{su}(T^*)^*.$$

Proof. En particular, if T^* has the (SVEP) then by Theorem 2, T has also the (SVEP). In this case, $\sigma(T)=\sigma_{ap}(T)=\sigma_{su}(T)$. Therefore, $\sigma(T)\cup-\sigma(T)=\sigma_{ap}(T)\cup-\sigma_{ap}(T)=\sigma_{su}(T^*)^*\cup-\sigma_{su}(T^*)^*$. ☐

THEOREM 4. *Let $T\in\mathscr{B}(H)$ be a square root of an m-complex symmetric operator with a conjugation C. Then T^* has the Bishop's property (β) if and only if T is decomposable.*

Proof. The reverse implication is straightforward. Now assume that T^* has the Bishop's property (β). Hence, we need only to show that T has the Bishop's property (β). In fact, let D be an open subset of $\mathbb{C}$ and let $f_n: D^*\longrightarrow H$ be a sequence of analytic functions such that $\lim_{n\to\infty}\|(CTC-\overline{\lambda})f_n(\overline{\lambda})\|_K=0$ and so $\lim_{n\to\infty}\|(CT^2C-\overline{\lambda}^2)f_n(\overline{\lambda})\|_K=0$ for every compact set K in D^*, where $\|f(\overline{\lambda})\|_K=\sup_{\lambda\in K}\|f(\overline{\lambda})\|$ for an H-valued function f. Let K be a compact subset of D^*, then we obtain

$$\begin{aligned}&\lim_{n\to\infty}\|\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}T^{*2j}CT^{2m-2j}Cf_n(\overline{\lambda})\|_K\\ =&\lim_{n\to\infty}\|\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}(T^{*2}-\overline{\lambda}^2)^j(CT^2C-\overline{\lambda}^2)^{m-j}f_n(\overline{\lambda})\|_K\\ =&\lim_{n\to\infty}\|\sum_{j=0}^{m-1}(-1)^{m-j}\binom{m}{j}(T^{*2}-\overline{\lambda}^2)^j(CT^2C-\overline{\lambda}^2)^{m-j}f_n(\overline{\lambda})+(T^{*2}-\overline{\lambda}^2)^m f_n(\overline{\lambda})\|_K\\ =&0.\end{aligned}$$

Because $\Delta_m(T^2-\lambda^2)=0$ for every $\lambda\in\mathbb{C}$, it follows that $\lim_{n\to\infty}\|(T^{*2}-\overline{\lambda}^2)^m f_n(\overline{\lambda})\|_K=0$. Since T^* has the Bishop's property (β), we get

$$\lim_{n\to\infty}\|(T^*+\overline{\lambda})(T^{*2}-\overline{\lambda}^2)^{m-1}f_n(\overline{\lambda})\|_K=0.$$

By induction, $\lim_{n\to\infty}\|(T^*+\overline{\lambda})^m f_n(\overline{\lambda})\|_K=0$. Set $\mu=-\lambda$, define an analytic function $g:-D^*\longrightarrow H$ by $g(\overline{\mu})=\overline{\lambda}$. Then

$$\lim_{n\to\infty}\|(T^*-\overline{\mu})^m(f\circ g)(\overline{\mu})\|_{-K}=\lim_{n\to\infty}\|(T^*+\overline{\lambda})^m f_n(\overline{\lambda})\|_K=0.$$

Again, T^* has the Bishop's property (β), and $(f \circ g)(\overline{\mu})$ is analytic on $-D^*$, as a result $\lim_{n\to\infty}\|(f \circ g)(\overline{\mu})\|_{-K} = 0$, that is $\lim_{n\to\infty}\|f(\overline{\lambda})\|_K = 0$. So CTC has the Bishop's property (β). Consequently, by [3, Theorem 2.13] T has the Bishop's property (β). $\square$

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(Received January 24, 2024)

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