Operators and Matrices
Volume 20, Number 1 (2026), 27–46
doi:10.7153/oam-2026-20-03

A NEW VIEW ON MULTIPLIERS FOR K–FRAMES IN HILBERT SPACES

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(*Communicated by D. Han*)

Abstract. In this paper, we give several sufficient conditions under which a K-frame multiplier is invertible and particularly, an equivalent characterization on the invertibility of K-frame multipliers from the perspective of operator theory is presented. We then show that a K-right inverse (resp. K-left inverse) of a K-frame multiplier can be written as a multiplier induced by canonical K-duals, and by means of the corresponding K-duals, we also characterize conditions for a K-right inverse (resp. K-left inverse) to be represented by a multiplier. These results have potential applications in operator sampling theory and signal reconstruction, where K-frames arise naturally.

1. Introduction

Frames, introduced by Duffin and Schaeffer [8] and popularized by Daubechies et al. [6], have been generalized to K-frames [10], which allow representation of signals under operator constraints. K-frames have attracted many scholars' interest [11, 13–15, 18], since some behaviors of them are quite different from the ones shown in frame theory.

We will collect some notations and definitions in order to continue with this section.

Let ℓ^∞ be the family $\ell^\infty = \{\{b_i\}_{i=1}^\infty : \max_i |b_i| < \infty\}$ with norm $\|\{b_i\}_{i=1}^\infty\|_\infty = \max_i |b_i|$. The sequence $m = \{m_i\}_{i=1}^\infty \in \ell^\infty$ is said to be ℓ^∞-invertible, if every m_i is invertible with $m^{-1} = \{m_i^{-1}\}_{i=1}^\infty \in \ell^\infty$. Also $\overline{m} := \{\overline{m}_i\}_{i=1}^\infty$.

Throughout the paper, $\mathscr{N}$, $\mathscr{P}$ and $\mathscr{Q}$ are complex separable Hilbert spaces, the symbol $\mathscr{L}\mathscr{B}(\mathscr{N},\mathscr{P})$ is designated as the family of linear bounded operators from $\mathscr{N}$ to $\mathscr{P}$, and $\mathscr{L}\mathscr{B}(\mathscr{N},\mathscr{N})$ is simply denoted by $\mathscr{L}\mathscr{B}(\mathscr{N})$. For $K \in \mathscr{L}\mathscr{B}(\mathscr{N})$, we denote by $Ran(K)$ the range of K. We use, as usual, the symbol $\mathrm{Id}_{\mathscr{N}}$ and $\pi_{\mathscr{N}'}$ to denote respectively the identity operator on $\mathscr{N}$ and the orthogonal projection onto the closed subspace $\mathscr{N}' \subset \mathscr{N}$.

Let K be a linear and bounded operator on $\mathscr{N}$. Recall that a sequence $\mathscr{F} = \{f_i\}_{i=1}^\infty$ with each element taken from $\mathscr{N}$ is said to be a K-*frame* for $\mathscr{N}$, if there are

Mathematics subject classification (2020): Primary 42C15, 42C40; Secondary 47B40.
Keywords and phrases: K-frame multiplier, canonical K-dual, K-left (right) inverse, pseudo-inverse.
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Paper OaM-20-03

two constants $0 < C_{\mathscr{F}} \leqslant D_{\mathscr{F}} < \infty$, called, respectively, the *lower and upper K-frame bounds*, so that for all $x \in \mathscr{N}$ we have

$$C_{\mathscr{F}}\|K^*x\|^2 \leqslant \sum_{i=1}^{\infty} |\langle x, f_i\rangle|^2 \leqslant D_{\mathscr{F}}\|x\|^2. \tag{1.1}$$

Particularly, we call $\mathscr{F} = \{f_i\}_{i=1}^{\infty}$ a *Parseval K-frame* for $\mathscr{N}$, if

$$\|K^*x\|^2 = \sum_{i=1}^{\infty} |\langle x, f_i\rangle|^2, \quad \forall x \in \mathscr{N}. \tag{1.2}$$

Taking $K = \mathrm{Id}_{\mathscr{N}}$ in above concept, then a K-frame (resp. Parseval K-frame) turns to be a *frame* (resp. *Parseval frame*). If only the inequality to the right in (1.1) is assumed to hold, then $\mathscr{F}$ is called a *Bessel sequence* with *Bessel bound $D_{\mathscr{F}}$*, induced by which there is always a self-adjoint and positive operator, called the *frame operator* of $\mathscr{F}$, given by

$$S_{\mathscr{F}} : \mathscr{N} \to \mathscr{N}, \quad S_{\mathscr{F}}x = \sum_{i=1}^{\infty} \langle x, f_i\rangle f_i, \quad \forall x \in \mathscr{N}. \tag{1.3}$$

Recall also that a Bessel sequence $\mathscr{G} = \{g_i\}_{i=1}^{\infty}$ for $\mathscr{N}$ is said to be a *K-dual* of $\mathscr{F}$, a K-frame for $\mathscr{N}$, if

$$Kx = \sum_{i=1}^{\infty} \langle x, g_i\rangle f_i, \quad \forall x \in \mathscr{N}. \tag{1.4}$$

It is easy to check that $\widetilde{\mathscr{F}} = \{\widetilde{f_i} := K^*(S_{\mathscr{F}}|_{Ran(K)})^{-1}\pi_{S_{\mathscr{F}}(Ran(K))} f_i\}_{i=1}^{\infty}$ is a Bessel sequence for $\mathscr{N}$ when K admits closed range, which is said to be the *canonical K-dual* of $\mathscr{F}$. It is worth pointing out that a K-frame and its canonical K-dual may be not biorthogonal in general (see [22] for details).

Associated with two Bessel sequences $\mathscr{F} = \{f_i\}_{i=1}^{\infty}$ and $\mathscr{G} = \{g_i\}_{i=1}^{\infty}$, and a sequence $m = \{m_i\}_{i=1}^{\infty} \in \ell^{\infty}$, there is a linear bounded operator, called a *Bessel multiplier* [1], given below

$$\mathscr{M}_{m,\mathscr{F},\mathscr{G}} : \mathscr{N} \to \mathscr{N}, \quad \mathscr{M}_{m,\mathscr{F},\mathscr{G}}x = \sum_{i=1}^{\infty} m_i\langle x, g_i\rangle f_i, \quad \forall x \in \mathscr{N}. \tag{1.5}$$

The multiplier $\mathscr{M}_{m,\mathscr{F},\mathscr{G}}$ will turn to be a *frame multiplier*, when $\mathscr{F}$ and $\mathscr{G}$ in (1.5) are frames. In recent years, more and more researchers are diving into the investigation of Bessel and frame multipliers since, as interesting mathematical objects, they have showed important applications in many theoretical and applied areas [3, 4, 9, 12, 16, 23]. Some other variations of multipliers such as generalized g-frame multipliers (first introduced in [19], and improved in [17] and other works), continuous frame multipliers [2], p-Bessel sequence multipliers [20], and fusion frame multipliers [21], etc, also emerged.

Recently, the authors in [22], drawing on the idea of [1], brought forward the concept of K-frame multipliers (taking the Bessel sequences in (1.5) instead of K-frames), and examine some of their properties by introducing the so-called canonical K-dual and K-right (left) inverse. It is worth remarking that some useful techniques available in classical frame theory are invalid in K-frame theory, meaning that the discussion of K-frame multipliers is more complicated, and that new behaviors of K-frame multipliers may arise due to the maneuverability of the operator K. This makes the study of multipliers under this framework is interesting. The purpose of this paper is to investigate the invertibility and the representations of the inverses of K-frame multipliers with a new view. Unlike [22], which provided only weak invertibility conditions, we establish operator-theoretic necessary and sufficient conditions and explicit multiplier representations of inverses.

2. Motivation and operators

2.1. Motivation

In [1] the author told us that the multiplier $\mathscr{M}_{m,\Phi,\Psi}$ induced by two Riesz bases Φ,Ψ and a semi-normalized sequence m is naturally invertible. And the inverse can be represented by a multiplier, where the canonical duals $\widetilde{\Phi}$ and $\widetilde{\Psi}$ of Φ and Ψ are involved, that is $\mathscr{M}^{-1}_{m,\Phi,\Psi} = \mathscr{M}_{m^{-1},\widetilde{\Psi},\widetilde{\Phi}}$. Of course, there also exist other invertible frame multipliers whose inverses are just $\mathscr{M}^{-1}_{m,\Phi,\Psi} = \mathscr{M}_{m^{-1},\widetilde{\Psi},\widetilde{\Phi}}$ under certain conditions. We note, however, that the validity of above results require either the biorthogonality of Φ (Ψ) and $\widetilde{\Phi}$ ($\widetilde{\Psi}$), or rely on the reconstruction of frame elements in the whole space.

As mentioned before, the biorthogonality between $\mathscr{F}$ and $\widetilde{\mathscr{F}}$, and the reconstruction of the K-frame elements in whole space are not available in general, leading to the fact that the inverse of a K-frame multiplier may not be written as a multiplier induced by the corresponding canonical K-duals. Fortunately, the authors in [22] recently introduced the so-called K-right inverse and K-left inverse, a weak version of invertibility (see Definition 4.1 for details). So, settling for second best, this motivates us to examine the following questions:

[**Q1**] Are there K-frame multipliers $\mathscr{M}_{m,\mathscr{F},\mathscr{G}}$ whose K-right inverses (or K-left inverses) are precisely $\mathscr{M}_{m^{-1},\widetilde{\mathscr{G}},\widetilde{\mathscr{F}}}$ (or $\mathscr{M}_{m^{-1},\widetilde{\mathscr{F}},\widetilde{\mathscr{G}}}$)?

If the answer to [**Q1**] is affirmative, then we consider:

[**Q2**] Are there other K-frame multipliers whose K-right inverses (or K-left inverses) can be expressed as multipliers?

The absence of biorthogonality represents a key distinction between K-frames and (standard) frames, as it undermines the foundation of the traditional inverse multiplier representation. The interest of [**Q1**] and [**Q2**] lies in overcoming this limitation by exploring the possibility of representing inverses through the concept of weak invertibility. Resolving these questions will enhance both the practical utility and theoretical completeness of K-frame multipliers, thereby supporting their application in broader scenarios such as partial reconstruction problems and constrained optimization. Consequently, investigating [**Q1**] and [**Q2**] will not only contribute to understanding the fundamental properties of K-frame multipliers but will also promote the development of related computational methods.

The following simple observation shows that if two Bessel sequences (along with a sequence in ℓ^∞) form an invertible multiplier, then they naturally become to be K-frames.

PROPOSITION 2.1. *Let $\mathscr{F} = \{f_i\}_{i=1}^\infty$ and $\mathscr{G} = \{g_i\}_{i=1}^\infty$ be two Bessel sequences for $\mathscr{N}$, and assume that $0 \neq m = \{m_i\}_{i=1}^\infty \in \ell^\infty$. If $\mathscr{M}_{m,\mathscr{F},\mathscr{G}}$ is invertible on $\mathscr{N}$, then both $\mathscr{F}$ and $\mathscr{G}$ are K-frames for $\mathscr{N}$ for any linear bounded operator K.*

Proof. We prove first that $\mathscr{G}$ is a K-frame for $\mathscr{N}$, and it is only required to show the lower K-frame bound condition. Since there exists $\Gamma \in \mathscr{L}\mathscr{B}(\mathscr{N})$ such that $\Gamma \mathscr{M}_{m,\mathscr{F},\mathscr{G}} = \mathrm{Id}_{\mathscr{N}}$, it follows that $K^*\Gamma\mathscr{M}_{m,\mathscr{F},\mathscr{G}} = K^*$. Hence for each $x \in \mathscr{N}$, we have

$$\begin{aligned} \|K^*x\|^2 = \|K^*\Gamma\mathscr{M}_{m,\mathscr{F},\mathscr{G}}x\|^2 &\leqslant \|K^*\Gamma\|^2 \left\| \sum_{i=1}^\infty m_i\langle x, g_i\rangle f_i \right\|^2 \\ &\leqslant \|K\|^2\|\Gamma\|^2\|m\|_\infty^2 D_{\mathscr{F}} \sum_{i=1}^\infty |\langle x, g_i\rangle|^2, \end{aligned}$$

giving that $\frac{1}{D_{\mathscr{F}}\|m\|_\infty^2\|K\|^2\|\Gamma\|^2}\|K^*x\|^2 \leqslant \sum_{i=1}^\infty |\langle x, g_i\rangle|^2$, as desired. Noting also that $\mathscr{M}_{m,\mathscr{F},\mathscr{G}}\Gamma = \mathrm{Id}_{\mathscr{N}}$, leading to $K^*\Gamma^*\mathscr{M}_{\overline{m},\mathscr{G},\mathscr{F}} = K^*$. A similar discussion as above can show the left-hand side inequality of a K-frame for $\mathscr{F}$. □

From the visual point of view, the converse of Proposition 2.1 is generally not true. So it is natural to investigate the following problems:

- Provide example to show that not all K-frame multipliers are invertible.

- Find sufficient, and necessary and sufficient conditions for a K-frame multiplier to be invertible.

This paper pays attention to above problems. In Section 3 we provide a counterexample to show that a K-frame multiplier is not necessarily invertible (even if on a subspace), and then several sufficient conditions under which a K-frame multiplier is invertible are given and, particularly, a necessary and sufficient condition for a K-frame multiplier to be invertible is presented from the perspective of operator theory. Section 4 gives affirmative answers to [**Q1**] and [**Q2**], and plus a bit more.

2.2. Operators

Let $\mathscr{F} = \{f_i\}_{i=1}^\infty$ be a Bessel sequence for $\mathscr{N}$, then it can induce a linear bounded operator, namely, the *analysis operator* of $\mathscr{F}$, defined in the following manner

$$U_{\mathscr{F}} : \mathscr{N} \to \ell^2, \quad U_{\mathscr{F}}x = \{\langle x, f_i\rangle\}_{i=1}^\infty. \tag{2.1}$$

It is easy to know that, $U_{\mathscr{F}}^*$, the adjoint operator of $U_{\mathscr{F}}$, is given by

$$U_{\mathscr{F}}^* : \ell^2 \to \mathscr{N}, \quad U_{\mathscr{F}}^*\{c_i\}_{i=1}^\infty = \sum_{i=1}^\infty c_i f_i. \tag{2.2}$$

Suppose $m = \{m_i\}_{i=1}^{\infty} \in \ell^{\infty}$. We can define an operator $D_m \in \mathscr{L}\mathscr{B}(\ell^2)$ by $D_m\{h_i\}_{i=1}^{\infty} = \{m_i h_i\}_{i=1}^{\infty}$, and it is easy to check that $\|D_m\| \leqslant \|m\|_{\infty}$ and $D_m^* = D_{\overline{m}}$. And further, if m_i is invertible for each i, then a simple calculation can show that D_m is invertible with $D_m^{-1} = D_{m^{-1}}$. Now, the Bessel multiplier $\mathscr{M}_{m,\mathscr{F},\mathscr{G}}$ in (1.5) can be represented as $\mathscr{M}_{m,\mathscr{F},\mathscr{G}} = U_{\mathscr{F}}^* D_m U_{\mathscr{G}}$.

PROPOSITION 2.2. (see [5]) *If $\Theta \in \mathscr{L}\mathscr{B}(\mathscr{N},\mathscr{P})$ admits closed range, then there is a unique operator $\Theta^{\dagger} \in \mathscr{L}\mathscr{B}(\mathscr{P},\mathscr{N})$, namely the pseudo-inverse operator, possessing the following properties*

$$\Theta\Theta^{\dagger}\Theta = \Theta, \quad \Theta^{\dagger}\Theta\Theta^{\dagger} = \Theta^{\dagger}, \quad (\Theta\Theta^{\dagger})^* = \Theta\Theta^{\dagger}, \quad (\Theta^{\dagger}\Theta)^* = \Theta^{\dagger}\Theta. \tag{2.3}$$

PROPOSITION 2.3. (see [7]) *Suppose $\Theta \in \mathscr{L}\mathscr{B}(\mathscr{N},\mathscr{Q})$ and $\Phi \in \mathscr{L}\mathscr{B}(\mathscr{P},\mathscr{Q})$. Then we have the following equivalent conditions:*

(1) *$Ran(\Theta) \subset Ran(\Phi)$.*

(2) *There is $\gamma > 0$ satisfying $\Theta\Theta^* \leqslant \gamma\Phi\Phi^*$.*

(3) *There is $\Lambda \in \mathscr{L}\mathscr{B}(\mathscr{N},\mathscr{P})$ such that $\Theta = \Phi\Lambda$.*

3. Invertibility of the multiplier $\mathscr{M}_{m,\mathscr{F},\mathscr{G}}$

Assume that $K \in \mathscr{L}\mathscr{B}(\mathscr{N})$ has closed range and that $\mathscr{F} = \{f_i\}_{i=1}^{\infty}$ is a K-frame for $\mathscr{N}$. Since $Ran(K)$ is closed, $KK^{\dagger}|_{Ran(K)} = \mathrm{Id}_{Ran(K)}$, and therefore $\mathrm{Id}_{Ran(K)} = (K^{\dagger}|_{Ran(K)})^* K^*$. Now for each $x \in Ran(K)$, $\|x\| = \|(K^{\dagger}|_{Ran(K)})^* K^* x\| \leqslant \|K^{\dagger}\|\,\|K^* x\|$, leading to $\langle S_{\mathscr{F}} x, x\rangle \geqslant C_{\mathscr{F}} \|K^* x\|^2 \geqslant C_{\mathscr{F}} \|K^{\dagger}\|^{-2}\|x\|^2$. Hence

$$C_{\mathscr{F}} \|K^{\dagger}\|^{-2}\|x\| \leqslant \|S_{\mathscr{F}} x\| \leqslant D_{\mathscr{F}}\|x\|, \quad \forall x \in Ran(K),$$

from which we conclude that $S_{\mathscr{F}}|_{Ran(K)} : Ran(K) \to S_{\mathscr{F}}(Ran(K))$ is invertible and satisfies

$$D_{\mathscr{F}}^{-1}\|z\| \leqslant \|(S_{\mathscr{F}}|_{Ran(K)})^{-1} z\| \leqslant C_{\mathscr{F}}^{-1}\|K^{\dagger}\|^2\|z\|, \quad \forall z \in S_{\mathscr{F}}(Ran(K)). \tag{3.1}$$

The reader can also check [25].

Following above fact, we present a simple result on the invertibility of $\mathscr{M}_{m,\mathscr{F},\mathscr{F}}$.

PROPOSITION 3.1. *Suppose that $\mathscr{F} = \{f_i\}_{i=1}^{\infty} \subseteq \mathscr{N}$, that $0 \neq m = \{m_i\}_{i=1}^{\infty} \in \ell^{\infty}$ is a positive sequence, and that there exists $\gamma > 0$ such that $m_i \geqslant \gamma$ for each i. Then the following statements are equivalent.*

(1) *$\mathscr{F}$ is a K-frame for $Ran(K)$.*

(2) *$\mathscr{F}' = \{f_i' := \sqrt{m_i} f_i\}_{i=1}^{\infty}$ is a K-frame for $Ran(K)$.*

(3) *$\mathscr{F}$ is a Bessel sequence for $Ran(K)$, and $\mathscr{M}_{m,\mathscr{F},\mathscr{F}}$ is bounded and invertible on $Ran(K)$.*

Proof. (1)$\Leftrightarrow$(2). For each $y \in Ran(K)$ and each positive integer i we have

$$\gamma|\langle y, f_i\rangle|^2 \leqslant |\langle y, \sqrt{m_i} f_i\rangle|^2 = |\langle y, f_i'\rangle|^2 \leqslant \|m\|_{\infty}|\langle y, f_i\rangle|^2, \tag{3.2}$$

which shows that $\mathscr{F}$ is a K-frame for $Ran(K)$ if and only if $\mathscr{F}'$ is a K-frame for $Ran(K)$.

(2)$\Rightarrow$(3). From (3.2) we conclude that if $\mathscr{F}'$ is a Bessel sequence for $Ran(K)$, then so is $\mathscr{F}$. Now for any $y \in Ran(K)$,

$$\mathscr{M}_{m,\mathscr{F},\mathscr{F}}y = \sum_{i=1}^{\infty} m_i \langle y, f_i\rangle f_i = \sum_{i=1}^{\infty} \langle y, \sqrt{m_i} f_i\rangle \sqrt{m_i} f_i = S_{\mathscr{F}'}|_{Ran(K)}y,$$

implying that $\mathscr{M}_{m,\mathscr{F},\mathscr{F}}$ is bounded and invertible on $Ran(K)$.

(3)$\Rightarrow$(1). Taking $\mathscr{G} = \{g_i := K^*(S_{\mathscr{F}'}|_{Ran(K)})^{-1}\pi_{S_{\mathscr{F}'(Ran(K))}} f_i'\}_{i=1}^{\infty}$. With a similar discussion as [22, Proposition 2.4], we can show that $Kx = \sum_{i=1}^{\infty}\langle x, g_i\rangle \pi_{Ran(K)} f_i'$ for any $x \in \mathscr{N}$ and thus, $K^*x = \sum_{i=1}^{\infty}\langle x, \pi_{Ran(K)} f_i'\rangle g_i$. To show that $\mathscr{F}$ is a K-frame for $Ran(K)$, it only needs to prove the inequality on the left. For any $y \in Ran(K)$ we get

$$\begin{aligned}\|K^*y\|^2 &= \sup_{\|z\|=1} |\langle K^*y, z\rangle|^2 = \sup_{\|z\|=1}\left|\sum_{i=1}^{\infty}\langle y, \pi_{Ran(K)} f_i'\rangle\langle g_i, z\rangle\right|^2 \\ &\leqslant \sup_{\|z\|=1}\sum_{i=1}^{\infty}|\langle y, \sqrt{m_i} f_i\rangle|^2 \sum_{i=1}^{\infty}|\langle g_i, z\rangle|^2 \leqslant D_{\mathscr{G}}\|m\|_{\infty}\sum_{i=1}^{\infty}|\langle y, f_i\rangle|^2,\end{aligned}$$

equivalently, $\frac{1}{D_{\mathscr{G}}\|m\|_{\infty}}\|K^*y\|^2 \leqslant \sum_{i=1}^{\infty}|\langle y, f_i\rangle|^2$, as desired. □

We now provide an example to show that not all K-frame multipliers $\mathscr{M}_{m,\mathscr{F},\mathscr{G}}$ are invertible, even if on subspaces.

EXAMPLE 3.2. Suppose that $\mathscr{N} = \mathbb{C}^3$ and that $\{e_i\}_{i=1}^3$ is an orthonormal basis of $\mathscr{N}$. We define $K \in \mathscr{L}\mathscr{B}(\mathscr{N})$ as follows

$$Ke_1 = e_1, \quad Ke_2 = e_1, \quad Ke_3 = e_2.$$

For each $x, y \in \mathscr{N}$ we have

$$\begin{aligned}\langle K^*x, y\rangle &= \langle x, Ky\rangle = \langle x, \langle y, e_1\rangle Ke_1 + \langle y, e_2\rangle Ke_2 + \langle y, e_3\rangle Ke_3\rangle \\ &= \langle x, \langle y, e_1 + e_2\rangle e_1 + \langle y, e_3\rangle e_2\rangle = \langle \langle x, e_1\rangle(e_1 + e_2) + \langle x, e_2\rangle e_3, y\rangle,\end{aligned}$$

giving that $K^*x = \langle x, e_1\rangle(e_1 + e_2) + \langle x, e_2\rangle e_3$. Taking

$$f_1 = e_1, \quad f_2 = e_1, \quad f_3 = e_2,$$

and

$$g_1 = e_2 - e_1, \quad g_2 = e_2 + e_1, \quad g_3 = e_2.$$

Then, for every $x \in \mathscr{N}$, we obtain

$$\|K^*x\|^2 = \|\langle x, e_1\rangle(e_1 + e_2) + \langle x, e_2\rangle e_3\|^2 = 2|\langle x, e_1\rangle|^2 + |\langle x, e_2\rangle|^2 = \sum_{i=1}^{3}|\langle x, f_i\rangle|^2,$$

and

$$\|K^*x\|^2 = 2|\langle x,e_1\rangle|^2 + |\langle x,e_2\rangle|^2 \leqslant \sum_{i=1}^{3} |\langle x,g_i\rangle|^2 = 2|\langle x,e_1\rangle|^2 + 3|\langle x,e_2\rangle|^2 \leqslant 3\|x\|^2,$$

showing that $\mathscr{F} = \{f_i\}_{i=1}^3$ and $\mathscr{G} = \{g_i\}_{i=1}^3$ are, respectively, a Parseval K-frame and a K-frame with K-frame bounds $C_{\mathscr{G}} = 1, D_{\mathscr{G}} = 3$ for $\mathscr{N}$. If we let $m_1 = m_2 = 1$ and $m_3 = 0$, and taking $e_1 \in Ran(K) = \overline{\text{span}}\{e_1,e_2\}$, then

$$\mathscr{M}_{m,\mathscr{F},\mathscr{G}} e_1 = \sum_{i=1}^{3} m_i \langle e_1,g_i\rangle f_i = \langle e_1,e_2-e_1\rangle e_1 + \langle e_1,e_2+e_1\rangle e_1 = -e_1+e_1 = 0.$$

Here e_1 is a nonzero element in $Ran(K)$ but is annihilated by the multiplier, hence invertibility fails even with K-frames.

As a matter of fact, even if only one K-frame is involved to construct $\mathscr{M}_{m,\mathscr{F},\mathscr{G}}$, the multiplier can be invertible on $Ran(K)$, when additional conditions are added.

THEOREM 3.3. *Suppose that $0 \neq K \in \mathscr{L}\mathscr{B}(\mathscr{N})$ has closed range, that $\mathscr{F} = \{f_i\}_{i=1}^\infty$ is a K-frame for $\mathscr{N}$ with a K-dual $\mathscr{G} = \{g_i\}_{i=1}^\infty$, and that $0 \neq m = \{m_i\}_{i=1}^\infty \in \ell^\infty$. If $\sqrt{D_{\mathscr{F}}D_{\mathscr{G}}}(\|m\|_\infty+1)(2\|K^\dagger\|+1) < 1$, then $\mathscr{M}_{m,\mathscr{F},\mathscr{G}}$ is invertible on $Ran(K)$.*

Proof. Taking $\mathscr{G}' = \{g_i' := (K^\dagger)^* g_i\}_{i=1}^\infty$. The duality gives

$$y = KK^\dagger y = \sum_{i=1}^{\infty} \langle K^\dagger y, g_i\rangle f_i = \sum_{i=1}^{\infty} \langle y,(K^\dagger)^* g_i\rangle f_i = \mathscr{M}_{\{1\},\mathscr{F},\mathscr{G}'} y$$

for each $y \in R(K)$. Thus

$$\begin{aligned}
\|\mathscr{M}_{m,\mathscr{F},\mathscr{G}} y - y\| &= \|\mathscr{M}_{m,\mathscr{F},\mathscr{G}} y - \mathscr{M}_{\{1\},\mathscr{F},\mathscr{G}'} y\| \\
&\leqslant \|\mathscr{M}_{m,\mathscr{F},\mathscr{G}} y - \mathscr{M}_{m,\mathscr{F},\mathscr{G}'} y\| + \|\mathscr{M}_{m,\mathscr{F},\mathscr{G}'} y - \mathscr{M}_{\{1\},\mathscr{F},\mathscr{G}'} y\| \\
&= \|\mathscr{M}_{m,\mathscr{F},\mathscr{G}-\mathscr{G}'} y\| + \|\mathscr{M}_{\{m_i-1\}_{i=1}^\infty,\mathscr{F},\mathscr{G}'} y\| \\
&\leqslant \sqrt{D_{\mathscr{F}}D_{\mathscr{G}}}\|m\|_\infty(1+\|K^\dagger\|)\|y\| + \sqrt{D_{\mathscr{F}}D_{\mathscr{G}}}(\|m\|_\infty+1)\|K^\dagger\|\|y\| \\
&< \sqrt{D_{\mathscr{F}}D_{\mathscr{G}}}(\|m\|_\infty+1)(2\|K^\dagger\|+1)\|y\| < \|y\|,
\end{aligned}$$

which shows that $\mathscr{M}_{m,\mathscr{F},\mathscr{G}}$ is invertible on $Ran(K)$. □

It is important to note that the conditions imposed on $D_{\mathscr{F}}$ and $D_{\mathscr{G}}$ are sufficient for the above result to hold. The theoretical guarantees derived in the result are not contingent upon the specific selection of $D_{\mathscr{F}}$ and $D_{\mathscr{G}}$, provided they satisfy the stated assumptions.

To state the next result, we need the following lemma, which is claimed in [23] as part of Proposition 2.4.

LEMMA 3.4. *Suppose that $U \in \mathscr{L}\mathscr{B}(\mathscr{N},\mathscr{P})$ is invertible, and that $V \in \mathscr{L}\mathscr{B}(\mathscr{N},\mathscr{P})$ satisfies $\|Ux - Vx\| \leqslant \gamma\|x\|$ for all $x \in \mathscr{N}$, where $0 \leqslant \gamma < \frac{1}{\|U^{-1}\|}$. Then V is invertible on $\mathscr{N}$ with*

$$\frac{1}{\gamma + \|U\|}\|y\| \leqslant \|V^{-1}y\| \leqslant \frac{1}{\frac{1}{\|U^{-1}\|} - \gamma}\|y\|, \quad \forall y \in \mathscr{P}.$$

The authors in [22, Theorem 3.6] proved the invertibility of $\mathscr{M}_{m,\mathscr{F},\mathscr{G}}$ on $Ran(K)$, when restricting $m = \{m_i\}_{i=1}^{\infty}$ to be a positive sequence. We show, however, that the conclusion remains true for a general sequence $m = \{m_i\}_{i=1}^{\infty}$.

THEOREM 3.5. *Suppose that $0 \neq K \in \mathscr{L}\mathscr{B}(\mathscr{N})$ admits closed range, that $\mathscr{F} = \{f_i\}_{i=1}^{\infty}$ is a K-frame for $\mathscr{N}$, that $\mathscr{G} = \{g_i\}_{i=1}^{\infty}$ is a Bessel sequences for $\mathscr{N}$, and that $0 \neq m = \{m_i\}_{i=1}^{\infty} \in \ell^{\infty}$. If $\mathscr{G} - \mathscr{F} = \{g_i - f_i\}_{i=1}^{\infty}$ is a Bessel sequence for $\mathscr{N}$ with*

$$\sqrt{D_{\mathscr{G}-\mathscr{F}}} < \min\left\{\frac{\sqrt{C_{\mathscr{F}}}}{\|K^{\dagger}\|}, \frac{C_{\mathscr{F}}}{\sqrt{D_{\mathscr{F}}}\|m\|_{\infty}\|K^{\dagger}\|^2} - \sqrt{D_{\mathscr{F}}}\left(1 + \frac{1}{\|m\|_{\infty}}\right)\right\}, \tag{3.3}$$

then

(1) *$\mathscr{G}$ is a K-frame for $Ran(K)$.*

(2) *$\mathscr{M}_{m,\mathscr{F},\mathscr{G}}$ is invertible on $Ran(K)$ and for each $z \in \mathscr{M}_{m,\mathscr{F},\mathscr{G}}(Ran(K))$, we have*

$$\begin{aligned}\frac{1}{\sqrt{D_{\mathscr{F}}}\|m\|_{\infty}(\sqrt{D_{\mathscr{G}-\mathscr{F}}} + \sqrt{D_{\mathscr{F}}})}\|z\| &\leqslant \|\mathscr{M}_{m,\mathscr{F},\mathscr{G}}^{-1}z\| \\ &\leqslant \frac{1}{\frac{C_{\mathscr{F}}}{\|K^{\dagger}\|^2} - D_{\mathscr{F}}(\|m\|_{\infty}+1) - \sqrt{D_{\mathscr{F}}D_{\mathscr{G}-\mathscr{F}}}\|m\|_{\infty}}\|z\|.\end{aligned}$$

Proof. (1) It is sufficient to show the inequality to the left of a K-frame, which, actually, can be achieved by the assumption $\sqrt{D_{\mathscr{G}-\mathscr{F}}} < \frac{\sqrt{C_{\mathscr{F}}}}{\|K^{\dagger}\|}$ and the following computation:

$$\begin{aligned}\|\{\langle y, g_i\rangle\}_{i=1}^{\infty}\| &= \|\{\langle y, f_i\rangle\}_{i=1}^{\infty} + \{\langle y, g_i - f_i\rangle\}_{i=1}^{\infty}\| \\ &\geqslant \|\{\langle y, f_i\rangle\}_{i=1}^{\infty}\| - \|\{\langle y, g_i - f_i\rangle\}_{i=1}^{\infty}\| \\ &\geqslant \sqrt{C_{\mathscr{F}}}\|K^*y\| - \sqrt{D_{\mathscr{G}-\mathscr{F}}}\|y\| \\ &\geqslant \sqrt{C_{\mathscr{F}}}\|K^*y\| - \sqrt{D_{\mathscr{G}-\mathscr{F}}}\|K^{\dagger}\|\|K^*y\| \\ &= (\sqrt{C_{\mathscr{F}}} - \sqrt{D_{\mathscr{G}-\mathscr{F}}}\|K^{\dagger}\|)\|K^*y\|\end{aligned}$$

for each $y \in Ran(K)$, where in the last inequality we apply the fact that

$$\|y\| = \|KK^{\dagger}y\| = \|(KK^{\dagger})^*y\| = \|(K^{\dagger})^*K^*y\| \leqslant \|K^{\dagger}\|\|K^*y\|.$$

(2) *Step* 1. For any $y \in Ran(K)$,

$$\begin{aligned}\|\mathscr{M}_{m,\mathscr{F},\mathscr{F}}y - \mathscr{M}_{\{1\},\mathscr{F},\mathscr{F}}y\| &= \left\|\sum_{i=1}^{\infty}(m_i - 1)\langle y, f_i\rangle f_i\right\| \\ &\leqslant D_{\mathscr{F}}\|\{m_i - 1\}_{i=1}^{\infty}\|\|y\| \leqslant D_{\mathscr{F}}(\|m\|_{\infty} + 1)\|\|y\|.\end{aligned}$$

From (3.3), $\frac{C_{\mathscr{F}}}{\sqrt{D_{\mathscr{F}}}\|m\|_\infty\|K^\dagger\|^2} - \sqrt{D_{\mathscr{F}}}\left(1+\frac{1}{\|m\|_\infty}\right) > \sqrt{D_{\mathscr{G}-\mathscr{F}}} > 0$, leading to

$$D_{\mathscr{F}}(\|m\|_\infty+1) < \frac{C_{\mathscr{F}}}{\|K^\dagger\|^2} \leqslant \frac{1}{\|(S_{\mathscr{F}}|_{Ran(K)})^{-1}\|} = \frac{1}{\|\mathscr{M}^{-1}_{\{1\},\mathscr{F},\mathscr{F}}\|}.$$

By Lemma 3.4 it follows that $\mathscr{M}_{m,\mathscr{F},\mathscr{F}}$ is invertible on $Ran(K)$ and

$$\frac{1}{D_{\mathscr{F}}(\|m\|_\infty+2)}\|h\| \leqslant \|\mathscr{M}^{-1}_{m,\mathscr{F},\mathscr{F}}h\| \leqslant \frac{1}{\frac{C_{\mathscr{F}}}{\|K^\dagger\|^2} - D_{\mathscr{F}}(\|m\|_\infty+1)}\|h\|,$$
$$\forall h \in \mathscr{M}_{m,\mathscr{F},\mathscr{F}}(Ran(K)).$$

Step 2. We obtain, for every $y \in Ran(K)$, that

$$\|\mathscr{M}_{m,\mathscr{F},\mathscr{G}}y - \mathscr{M}_{m,\mathscr{F},\mathscr{F}}y\| = \left\|\sum_{i=1}^{\infty} m_i\langle y, g_i - f_i\rangle f_i\right\| \leqslant \sqrt{D_{\mathscr{F}}D_{\mathscr{G}-\mathscr{F}}}\|m\|_\infty\|y\|.$$

From (3.3) we see that

$$\sqrt{D_{\mathscr{F}}D_{\mathscr{G}-\mathscr{F}}}\|m\|_\infty < \sqrt{D_{\mathscr{F}}}\|m\|_\infty\left(\frac{C_{\mathscr{F}}}{\sqrt{D_{\mathscr{F}}}\|m\|_\infty\|K^\dagger\|^2} - \sqrt{D_{\mathscr{F}}}\left(1+\frac{1}{\|m\|_\infty}\right)\right)$$
$$= \frac{C_{\mathscr{F}}}{\|K^\dagger\|^2} - D_{\mathscr{F}}(\|m\|_\infty+1) \leqslant \frac{1}{\|\mathscr{M}^{-1}_{m,\mathscr{F},\mathscr{F}}\|}.$$

Noting also that $\|\mathscr{M}_{m,\mathscr{F},\mathscr{F}}\| \leqslant \|m\|_\infty D_{\mathscr{F}}$, we obtain by Lemma 3.4 that $\mathscr{M}_{m,\mathscr{F},\mathscr{G}}$ is invertible on $Ran(K)$ and

$$\frac{1}{\sqrt{D_{\mathscr{F}}}\|m\|_\infty(\sqrt{D_{\mathscr{G}-\mathscr{F}}}+\sqrt{D_{\mathscr{F}}})}\|z\| \leqslant \|\mathscr{M}^{-1}_{m,\mathscr{F},\mathscr{G}}z\|$$
$$\leqslant \frac{1}{\frac{C_{\mathscr{F}}}{\|K^\dagger\|^2} - D_{\mathscr{F}}(\|m\|_\infty+1) - \sqrt{D_{\mathscr{F}}D_{\mathscr{G}-\mathscr{F}}}\|m\|_\infty}\|z\|$$

for each $z \in \mathscr{M}_{m,\mathscr{F},\mathscr{G}}(Ran(K))$. □

COROLLARY 3.6. *Suppose that $0 \neq K \in \mathscr{L}\mathscr{B}(\mathscr{H})$ admits closed range, that $\mathscr{F} = \{f_i\}_{i=1}^{\infty}$ is a K-frame for $\mathscr{H}$, that $\mathscr{G} = \{g_i\}_{i=1}^{\infty}$ is a Bessel sequences for $\mathscr{H}$, and that $0 \neq m = \{m_i\}_{i=1}^{\infty} \in \ell^\infty$. If there exists $\beta \in [0, \frac{C_{\mathscr{F}}}{\|K^\dagger\|^2})$ such that $\|m\|_\infty\sqrt{D_{\mathscr{F}}\beta} < \frac{C_{\mathscr{F}}}{\|K^\dagger\|^2} - (\|m\|_\infty+1)D_{\mathscr{F}}$ and $\sum_{i=1}^{\infty}|\langle y, g_i - f_i\rangle|^2 \leqslant \beta\|y\|^2$ for each $y \in Ran(K)$, then $\mathscr{G}$ is a K-frame for $Ran(K)$, and $\mathscr{M}_{m,\mathscr{F},\mathscr{G}}$ is invertible on $Ran(K)$ with*

$$\frac{1}{\sqrt{D_{\mathscr{F}}}\|m\|_\infty(\sqrt{\beta}+\sqrt{D_{\mathscr{F}}})}\|z\| \leqslant \|\mathscr{M}^{-1}_{m,\mathscr{F},\mathscr{G}}z\|$$
$$\leqslant \frac{1}{\frac{C_{\mathscr{F}}}{\|K^\dagger\|^2} - D_{\mathscr{F}}(\|m\|_\infty+1) - \sqrt{\beta D_{\mathscr{F}}}\|m\|_\infty}\|z\|$$

for each $z \in \mathscr{M}_{m,\mathscr{F},\mathscr{G}}(Ran(K))$.

Proof. The assumption $\sum_{i=1}^{\infty}|\langle y, g_i - f_i\rangle|^2 \leqslant \beta\|y\|^2$ for each $y \in Ran(K)$ means that the Bessel bound $D_{\mathscr{G}-\mathscr{F}}$ for the Bessel sequence $\mathscr{G}-\mathscr{F}$ satisfies $D_{\mathscr{G}-\mathscr{F}} \leqslant \beta$. Since $\beta \in [0, \frac{C_{\mathscr{F}}}{\|K^\dagger\|^2})$, we have $\sqrt{\beta} < \frac{\sqrt{C_{\mathscr{F}}}}{\|K^\dagger\|}$. By Theorem 3.5, $\mathscr{G}$ is a K-frame for $Ran(K)$. For the proof of the second part, observe that the given hypothesis $\|m\|_\infty\sqrt{D_{\mathscr{F}}\beta} < \frac{C_{\mathscr{F}}}{\|K^\dagger\|^2} - (\|m\|_\infty + 1)D_{\mathscr{F}}$ implies, upon division by $\|m\|_\infty\sqrt{D_{\mathscr{F}}}$, that $\sqrt{\beta} < \frac{C_{\mathscr{F}}}{\sqrt{D_{\mathscr{F}}}\|m\|_\infty\|K^\dagger\|^2} - \sqrt{D_{\mathscr{F}}}(1 + \frac{1}{\|m\|_\infty})$. Again by Theorem 3.5, $\mathscr{M}_{m,\mathscr{F},\mathscr{G}}$ is invertible on $Ran(K)$, and the norm bounds for its inverse follow by substituting β for $D_{\mathscr{G}-\mathscr{F}}$ in the bounds from Theorem 3.5. □

Inspired by [23, Proposition 4.2], alternative conditions under which $\mathscr{M}_{m,\mathscr{F},\mathscr{G}}$ is invertible on $Ran(K)$ are also given as follows. We will not include the proof here, since it is in spirit to Theorem 3.5.

PROPOSITION 3.7. *Suppose that $0 \neq K \in \mathscr{L}\mathscr{B}(\mathscr{N})$ admits closed range, that $\mathscr{F} = \{f_i\}_{i=1}^{\infty}$ is a K-frame for $\mathscr{N}$, that $\mathscr{G} = \{g_i\}_{i=1}^{\infty}$ is a Bessel sequences for $\mathscr{N}$ and that there is $0 \leqslant \beta < \frac{C_{\mathscr{F}}}{\|K^\dagger\|^2}$ such that $\sum_{i=1}^{\infty}|\langle y, g_i - f_i\rangle|^2 \leqslant \beta\|y\|^2$ for any $y \in Ran(K)$. Let now $m = \{m_i\}_{i=1}^{\infty} \in \ell^\infty$ satisfy $|m_i - 1| \leqslant \alpha < \frac{C_{\mathscr{F}} - \|K^\dagger\|^2\sqrt{\beta D_{\mathscr{F}}}}{\|K^\dagger\|^2(D_{\mathscr{F}} + \sqrt{\beta D_{\mathscr{F}}})}$ for every i. Then $\mathscr{G}$ is a K-frame for $Ran(K)$, $\mathscr{M}_{m,\mathscr{F},\mathscr{G}}$ is invertible on $Ran(K)$ and for each $z \in \mathscr{M}_{m,\mathscr{F},\mathscr{G}}(Ran(K))$,*

$$\frac{1}{(\alpha+1)(\sqrt{\beta D_{\mathscr{F}}} + D_{\mathscr{F}})}\|z\| \leqslant \|\mathscr{M}_{m,\mathscr{F},\mathscr{G}}^{-1}z\| \leqslant \frac{1}{\frac{C_{\mathscr{F}}}{\|K^\dagger\|^2} - \alpha D_{\mathscr{F}} - (\alpha+1)\sqrt{\beta D_{\mathscr{F}}}}\|z\|.$$

Let $\mathscr{G} = \{g_i\}_{i=1}^{\infty}$ be a K-frame for $\mathscr{N}$, and $m = \{m_i\}_{i=1}^{\infty} \in \ell^\infty$. From Proposition 3.7 we know that if $|m_i - 1| < \frac{C_{\mathscr{G}}}{D_{\mathscr{G}}\|K^\dagger\|^2}$ for every i, then $\mathscr{M}_{m,\mathscr{G},\mathscr{G}} : Ran(K) \to \mathscr{M}_{m,\mathscr{G},\mathscr{G}}(Ran(K))$ is invertible. Based on this fact, we present an equivalent characterization on the invertibility of K-frame multipliers from the operator-theoretic point of view.

PROPOSITION 3.8. *Let $0 \neq K \in \mathscr{L}\mathscr{B}(\mathscr{N})$ be a closed range operator, $\mathscr{F} = \{f_i\}_{i=1}^{\infty}$ and $= \{\Gamma_i\}_{i=1}^{\infty}$ be K-frames for $\mathscr{N}$. Assume that the sequence $m = \{m_i\}_{i=1}^{\infty} \in \ell^\infty$ satisfies $|m_i - 1| < \frac{C_{\mathscr{G}}}{D_{\mathscr{G}}\|K^\dagger\|^2}$ for each i, that $Ran(U_{\mathscr{F}}^*) \subset \mathscr{M}_{m,\mathscr{F},\mathscr{G}}(Ran(K))$, and that $Ran(U_{\mathscr{G}}^*) \subset \mathscr{M}_{m,\mathscr{G},\mathscr{G}}(Ran(K))$. Then $\mathscr{M}_{m,\mathscr{F},\mathscr{G}}$ is invertible on $Ran(K)$ if and only if there are $V \in \mathscr{L}\mathscr{B}(\ell^2, \mathscr{M}_{m,\mathscr{F},\mathscr{G}}(Ran(K)))$, and invertible operators $W, P \in \mathscr{L}\mathscr{B}(Ran(K), \mathscr{M}_{m,\mathscr{F},\mathscr{G}}(Ran(K)))$ such that*

$$P((\mathscr{M}_{m,\mathscr{G},\mathscr{G}}|_{Ran(K)})^{-1}U_{\mathscr{G}}^* + W^{-1}V(\mathrm{Id}_{\ell^2} - D_m U_{\mathscr{G}}(\mathscr{M}_{m,\mathscr{G},\mathscr{G}}|_{Ran(K)})^{-1}U_{\mathscr{G}}^*)) = U_{\mathscr{F}}^*. \quad (3.4)$$

Proof. Suppose first that $\mathscr{M}_{m,\mathscr{F},\mathscr{G}} : Ran(K) \to \mathscr{M}_{m,\mathscr{F},\mathscr{G}}(Ran(K))$ is invertible. Letting $V = U_{\mathscr{F}}^*$ and $W, P = \mathscr{M}_{m,\mathscr{F},\mathscr{G}}|_{Ran(K)}$. Then

$$\begin{aligned} &P((\mathscr{M}_{m,\mathscr{G},\mathscr{G}}|_{Ran(K)})^{-1}U_{\mathscr{G}}^* + W^{-1}V(\mathrm{Id}_{\ell^2} - D_m U_{\mathscr{G}}(\mathscr{M}_{m,\mathscr{G},\mathscr{G}}|_{Ran(K)})^{-1}U_{\mathscr{G}}^*)) \\ &= \mathscr{M}_{m,\mathscr{F},\mathscr{G}}|_{Ran(K)}((\mathscr{M}_{m,\mathscr{G},\mathscr{G}}|_{Ran(K)})^{-1}U_{\mathscr{G}}^* + (\mathscr{M}_{m,\mathscr{F},\mathscr{G}}|_{Ran(K)})^{-1}U_{\mathscr{F}}^* \\ &\quad - (\mathscr{M}_{m,\mathscr{F},\mathscr{G}}|_{Ran(K)})^{-1}U_{\mathscr{F}}^* D_m U_{\mathscr{G}}(\mathscr{M}_{m,\mathscr{G},\mathscr{G}}|_{Ran(K)})^{-1}U_{\mathscr{G}}^*) \end{aligned}$$

$$
\begin{aligned}
&= \mathcal{M}_{m,\mathcal{F},\mathcal{G}}|_{Ran(K)}\big((\mathcal{M}_{m,\mathcal{G},\mathcal{G}}|_{Ran(K)})^{-1}U_{\mathcal{G}}^{*} + (\mathcal{M}_{m,\mathcal{F},\mathcal{G}}|_{Ran(K)})^{-1}U_{\mathcal{F}}^{*} \\
&\quad - (\mathcal{M}_{m,\mathcal{F},\mathcal{G}}|_{Ran(K)})^{-1}(\mathcal{M}_{m,\mathcal{F},\mathcal{G}}|_{Ran(K)})(\mathcal{M}_{m,\mathcal{G},\mathcal{G}}|_{Ran(K)})^{-1}U_{\mathcal{G}}^{*}\big) \\
&= \mathcal{M}_{m,\mathcal{F},\mathcal{G}}|_{Ran(K)}\big((\mathcal{M}_{m,\mathcal{G},\mathcal{G}}|_{Ran(K)})^{-1}U_{\mathcal{G}}^{*} + (\mathcal{M}_{m,\mathcal{F},\mathcal{G}}|_{Ran(K)})^{-1}U_{\mathcal{F}}^{*} \\
&\quad - (\mathcal{M}_{m,\mathcal{G},\mathcal{G}}|_{Ran(K)})^{-1}U_{\mathcal{G}}^{*}\big) \\
&= (\mathcal{M}_{m,\mathcal{F},\mathcal{G}}|_{Ran(K)})(\mathcal{M}_{m,\mathcal{F},\mathcal{G}}|_{Ran(K)})^{-1}U_{\mathcal{F}}^{*} = U_{\mathcal{F}}^{*}.
\end{aligned}
$$

Assume now that (3.4) holds for the required assumption conditions. Then for each $y \in Ran(K)$ we obtain

$$
\begin{aligned}
\mathcal{M}_{m,\mathcal{F},\mathcal{G}}y = U_{\mathcal{F}}^{*}D_mU_{\mathcal{G}}y &= P\big((\mathcal{M}_{m,\mathcal{G},\mathcal{G}}|_{Ran(K)})^{-1}U_{\mathcal{G}}^{*} + W^{-1}V(\mathrm{Id}_{\ell^2} \\
&\qquad - D_mU_{\mathcal{G}}(\mathcal{M}_{m,\mathcal{G},\mathcal{G}}|_{Ran(K)})^{-1}U_{\mathcal{G}}^{*})\big)D_mU_{\mathcal{G}}y \\
&= P\big((\mathcal{M}_{m,\mathcal{G},\mathcal{G}}|_{Ran(K)})^{-1}U_{\mathcal{G}}^{*}D_mU_{\mathcal{G}}y + W^{-1}VD_mU_{\mathcal{G}}y \\
&\qquad - W^{-1}VD_mU_{\mathcal{G}}(\mathcal{M}_{m,\mathcal{G},\mathcal{G}}|_{Ran(K)})^{-1}U_{\mathcal{G}}^{*}D_mU_{\mathcal{G}}y\big) \\
&= P\big((\mathcal{M}_{m,\mathcal{G},\mathcal{G}}|_{Ran(K)})^{-1}(\mathcal{M}_{m,\mathcal{G},\mathcal{G}}|_{Ran(K)})y + W^{-1}VD_mU_{\mathcal{G}}y \\
&\qquad - W^{-1}VD_mU_{\mathcal{G}}(\mathcal{M}_{m,\mathcal{G},\mathcal{G}}|_{Ran(K)})^{-1}(\mathcal{M}_{m,\mathcal{G},\mathcal{G}}|_{Ran(K)})y\big) \\
&= P(y + W^{-1}VD_mU_{\mathcal{G}}y - W^{-1}VD_mU_{\mathcal{G}}y) = Py,
\end{aligned}
$$

from which it follows that $\mathcal{M}_{m,\mathcal{F},\mathcal{G}}|_{Ran(K)} = P$ is invertible. □

This parallels Douglas' lemma on operator range inclusions and provides a foundation for invertibility criteria in K-frame settings.

4. K-right (left) inverse of the multiplier $\mathcal{M}_{m,\mathcal{F},\mathcal{G}}$

The following definition is due to Shamsabadi and Arefijamaal in [22].

DEFINITION 4.1. Suppose that $\mathcal{F} = \{f_i\}_{i=1}^{\infty}$ and $\mathcal{G} = \{g_i\}_{i=1}^{\infty}$ are two Bessel sequences for $\mathcal{H}$, and that $m = \{m_i\}_{i=1}^{\infty} \in \ell^{\infty}$. One calls $\mathcal{M}_{m,\mathcal{F},\mathcal{G}}^{KRI} \in \mathcal{LB}(\mathcal{H})$ a K-*right inverse* of $\mathcal{M}_{m,\mathcal{F},\mathcal{G}}$, if for each $x \in \mathcal{H}$ we have

$$\mathcal{M}_{m,\mathcal{F},\mathcal{G}}\mathcal{M}_{m,\mathcal{F},\mathcal{G}}^{KRI}x = Kx. \tag{4.1}$$

Also, an operator $\mathcal{M}_{m,\mathcal{F},\mathcal{G}}^{KLI} \in \mathcal{LB}(\mathcal{H})$ is said to be a K-*left inverse* of $\mathcal{M}_{m,\mathcal{F},\mathcal{G}}$, if

$$\mathcal{M}_{m,\mathcal{F},\mathcal{G}}^{KLI}\mathcal{M}_{m,\mathcal{F},\mathcal{G}}x = Kx \tag{4.2}$$

holds for any $x \in \mathcal{H}$.

As shown in the example below, it is interesting that a K-frame multiplier admits both a K-left inverse and a K-right inverse, yet fails to be invertible.

EXAMPLE 4.2. Let $\mathscr{N} = \ell^2(\mathbb{N})$ with the standard orthonormal basis $\{e_i\}_{i=1}^{\infty}$. Define $K \in \mathscr{L}\mathscr{B}(\mathscr{N})$ by

$$Kx = \langle x, e_1 \rangle e_1, \quad \forall x \in \mathscr{N}.$$

It is straightforward to verify that K is self-adjoint. For each $i \in \mathbb{N}$, taking $f_i = g_i = e_i$. Then

$$\|K^*x\|^2 = |\langle x, e_1 \rangle|^2 \leqslant \|x\|^2 = \sum_{i=1}^{\infty} |\langle x, f_i \rangle|^2, \quad \forall x \in \mathscr{N},$$

showing that $\mathscr{F} = \{f_i\}_{i=1}^{\infty}$ and $\mathscr{G} = \{g_i\}_{i=1}^{\infty}$ are K-frames for $\mathscr{N}$. Consider the sequence $m = \{m_i\}_{i=1}^{\infty}$ where $m_i = \begin{cases} 1, \text{ if } i = 1, \\ 0, \text{ if } i \geqslant 2. \end{cases}$ The corresponding K-frame multiplier $\mathscr{M}_{m,\mathscr{F},\mathscr{G}}$ is given by

$$\mathscr{M}_{m,\mathscr{F},\mathscr{G}}x = \sum_{i=1}^{\infty} m_i \langle x, g_i \rangle f_i = \langle x, e_1 \rangle e_1, \quad \forall x \in \mathscr{N}.$$

Now, since

$$\mathscr{M}_{m,\mathscr{F},\mathscr{G}}\mathscr{M}_{m,\mathscr{F},\mathscr{G}}x = \mathscr{M}_{m,\mathscr{F},\mathscr{G}}(\langle x, e_1 \rangle e_1) = \langle \langle x, e_1 \rangle e_1, e_1 \rangle e_1 = \langle x, e_1 \rangle e_1 = Kx,$$

we find that $\mathscr{M}_{m,\mathscr{F},\mathscr{G}}$ is both a K-left inverse and a K-right inverse of itself. However, $\mathscr{M}_{m,\mathscr{F},\mathscr{G}}$ fails to be invertible, as $\mathscr{M}_{m,\mathscr{F},\mathscr{G}}e_2 = \langle e_2, e_1 \rangle e_1 = 0$.

We also show, by the following example, that a K-frame multiplier may admit neither a K-left inverse nor a K-right inverse.

EXAMPLE 4.3. Suppose that $\mathscr{N}$, $\mathscr{F} = \{f_i\}_{i=1}^{\infty}$ and $\mathscr{G} = \{g_i\}_{i=1}^{\infty}$ are the same as in Example 4.2. Define $K \in \mathscr{L}\mathscr{B}(\mathscr{N})$ by

$$Kx = (x_1, 0, 0, \cdots), \quad \forall x = (x_1, x_2, x_3, \cdots) \in \mathscr{N}.$$

It is easy to check that $K^* = K$. Since

$$\|K^*x\|^2 = \|Kx\|^2 = |x_1|^2 \leqslant \|x\|^2 = \sum_{i=1}^{\infty} |\langle x, f_i \rangle|^2, \quad \forall x \in \mathscr{N},$$

it follows that $\mathscr{F}$ and $\mathscr{G}$ are K-frames for $\mathscr{N}$. Define the sequence $m = \{m_i\}_{i=1}^{\infty}$ such that $m_1 = 0$, and $m_i = 1$ for $i \geqslant 2$. Now, the K-frame multiplier induced by m, $\mathscr{F}$ and $\mathscr{G}$ is

$$\mathscr{M}_{m,\mathscr{F},\mathscr{G}}x = \sum_{i=1}^{\infty} m_i \langle x, g_i \rangle f_i = \sum_{i=1}^{\infty} m_i \langle x, e_i \rangle e_i = (0, x_2, x_3, \cdots), \quad \forall x = (x_1, x_2, x_3, \cdots) \in \mathscr{N}.$$

Next, we show that $\mathscr{M}_{m,\mathscr{F},\mathscr{G}}$ has neither a K-left inverse nor a K-right inverse. Assume on the contrary that there exists $\mathscr{M}^{(1)} \in \mathscr{L}\mathscr{B}(\mathscr{N})$ such that $\mathscr{M}_{m,\mathscr{F},\mathscr{G}}\mathscr{M}^{(1)}x = Kx$ for all $x \in \mathscr{N}$. Let $\mathscr{M}^{(1)}x = y = (y_1, y_2, y_3, \cdots)$. Then for $x = e_1 = (1, 0, 0, \cdots)$,

$$(0, y_2, y_3, \cdots) = \mathscr{M}_{m,\mathscr{F},\mathscr{G}}y = Ke_1 = (1, 0, 0, \cdots),$$

a contradiction. Suppose now that there is $\mathscr{M}^{(2)} \in \mathscr{LB}(\mathscr{N})$ so that for each $x \in \mathscr{N}$, we have $\mathscr{M}^{(2)}\mathscr{M}_{m,\mathscr{F},\mathscr{G}}x = Kx$. Then

$$\mathscr{M}^{(2)}(0,0,0,\cdots) = \mathscr{M}^{(2)}\mathscr{M}_{m,\mathscr{F},\mathscr{G}}e_1 = Ke_1 = (1,0,0,\cdots),$$

and

$$\mathscr{M}^{(2)}(0,0,0,\cdots) = \mathscr{M}^{(2)}\mathscr{M}_{m,\mathscr{F},\mathscr{G}}(2e_1) = K(2e_1) = (2,0,0,\cdots),$$

leading to a contradiction.

REMARK 4.4. (1) It is clear that $\mathscr{M}_{m,\mathscr{F},\mathscr{G}}$ admits a K-left inverse, if $Ran(K^*) \subset Ran(\mathscr{M}_{\overline{m},\mathscr{G},\mathscr{F}})$ (one can show that $\mathscr{M}_{m,\mathscr{F},\mathscr{G}}$ has a K-right inverse when a modification range relationship is satisfied).

(2) If $\mathscr{M}_{m,\mathscr{F},\mathscr{G}}$ is invertible on $\mathscr{N}$, then $\mathscr{M}_{m,\mathscr{F},\mathscr{G}}^{-1}K$ and $K\mathscr{M}_{m,\mathscr{F},\mathscr{G}}^{-1}$ are respectively a K-right inverse and a K-left inverse of $\mathscr{M}_{m,\mathscr{F},\mathscr{G}}$.

(3) If $\mathscr{M}_{m,\mathscr{F},\mathscr{G}}^{KRI(1)}$ and $\mathscr{M}_{m,\mathscr{F},\mathscr{G}}^{KRI(2)}$ are both K-right inverses of $\mathscr{M}_{m,\mathscr{F},\mathscr{G}}$, then so is $\frac{\mathscr{M}_{m,\mathscr{F},\mathscr{G}}^{KRI(1)}+\mathscr{M}_{m,\mathscr{F},\mathscr{G}}^{KRI(2)}}{2}$ (the same result occurs for the case of K-left inverse).

We continue our investigation of K-left inverse and K-right inverse with the following simple observation.

PROPOSITION 4.5. *Suppose that $K \in \mathscr{LB}(\mathscr{N})$ has closed range, that $\mathscr{F} = \{f_i\}_{i=1}^{\infty}$ and $\mathscr{G} = \{g_i\}_{i=1}^{\infty}$ are, respectively, a K-frame and a Bessel sequence for $\mathscr{N}$, and that $m = \{m_i\}_{i=1}^{\infty} \in \ell^{\infty}$. The following statements hold.*

(1) If $\mathscr{M}_{m,\mathscr{F},\mathscr{G}}^{KLI}$ is a K-left inverse of $\mathscr{M}_{m,\mathscr{F},\mathscr{G}}$, then $(\mathscr{M}_{m,\mathscr{F},\mathscr{G}}^{KLI})^*$ is bounded below on $Ran(K)$.

(2) If $\mathscr{F} = \{f_i\}_{i=1}^{\infty}$ is a Parseval K-frame, then $(K^{\dagger})^*$ is a K-right inverse of $\mathscr{M}_{\{1\},\mathscr{F},\mathscr{F}}$.

Proof. (1) Since $\mathscr{M}_{m,\mathscr{F},\mathscr{G}}^{KLI}\mathscr{M}_{m,\mathscr{F},\mathscr{G}} = K$, we have $\mathscr{M}_{m,\mathscr{F},\mathscr{G}}^{KLI}\mathscr{M}_{m,\mathscr{F},\mathscr{G}}K^{\dagger} = KK^{\dagger}$, meaning that $\mathscr{M}_{m,\mathscr{F},\mathscr{G}}^{KLI}\mathscr{M}_{m,\mathscr{F},\mathscr{G}}K^{\dagger}|_{Ran(K)} = \mathrm{Id}_{Ran(K)}$. Hence

$$\begin{aligned}\|y\| &= \|(\mathscr{M}_{m,\mathscr{F},\mathscr{G}}K^{\dagger}|_{Ran(K)})^*(\mathscr{M}_{m,\mathscr{F},\mathscr{G}}^{KLI})^*y\| \\ &\leqslant \|\mathscr{M}_{m,\mathscr{F},\mathscr{G}}K^{\dagger}|_{Ran(K)}\|\,\|(\mathscr{M}_{m,\mathscr{F},\mathscr{G}}^{KLI})^*y\| \leqslant \|\mathscr{M}_{m,\mathscr{F},\mathscr{G}}K^{\dagger}\|\,\|(\mathscr{M}_{m,\mathscr{F},\mathscr{G}}^{KLI})^*y\|\end{aligned}$$

for each $y \in Ran(K)$, or equivalently, $\|\mathscr{M}_{m,\mathscr{F},\mathscr{G}}K^{\dagger}\|^{-1}\|y\| \leqslant \|(\mathscr{M}_{m,\mathscr{F},\mathscr{G}}^{KLI})^*y\|$.

(2) From [24, Lemma 7] we know that $\{K^{\dagger}f_i\}_{i=1}^{\infty}$ is a K-dual of $\mathscr{F}$. Now the result follows from the following calculation

$$\mathscr{M}_{\{1\},\mathscr{F},\mathscr{F}}(K^{\dagger})^*x = \sum_{i=1}^{\infty}\langle (K^{\dagger})^*x, f_i\rangle f_i = \sum_{i=1}^{\infty}\langle x, K^{\dagger}f_i\rangle f_i = Kx, \quad \forall x \in \mathscr{N}. \quad \square$$

In classical frame theory, the canonical dual frame enables perfect reconstruction of any element via the frame expansion. However, in the setting of K-frames, the concept of duality is adapted to the operator K. Specifically, for a K-frame $\mathscr{F} = \{f_i\}_{i=1}^{\infty}$ with K having closed range, the canonical K-dual $\widetilde{\mathscr{F}} = \{\widetilde{f_i}\}_{i=1}^{\infty}$ is defined as $\widetilde{f_i} = K^*(S_{\mathscr{F}}|_{Ran(K)})^{-1}\pi_{S_{\mathscr{F}}(Ran(K))}f_i$. Geometrically, this dual system facilitates the reconstruction of elements only within the range of K, i.e., $Kx = \sum_{i=1}^{\infty}\langle x, \widetilde{f_i}\rangle f_i$ for all $x \in \mathscr{N}$. Unlike classical dual frames, which are biorthogonal and allow full-space reconstruction, canonical K-duals are generally not biorthogonal and are constrained to the subspace $Ran(K)$. This limitation reflects the more flexible but structurally complex nature of K-frames, where the focus shifts from global to operator-constrained reconstruction.

We now ready to give a positive answer to [**Q1**] for specific K-frame multipliers.

THEOREM 4.6. *Suppose that $0 \neq K \in \mathscr{L}\mathscr{B}(\mathscr{N})$ has closed range, that $\mathscr{F} = \{f_i\}_{i=1}^{\infty}$ and $\mathscr{G} = \{g_i\}_{i=1}^{\infty}$ are K-frames for $\mathscr{N}$, and that $0 \neq \{m_i\}_{i=1}^{\infty} = \{c\}$. The following two statements hold.*

(1) *If $Ran(U_{\mathscr{F}}) \subset Ran(U_{\mathscr{G}}K)$, then $\mathscr{M}_{\{c\},\pi_{Ran(K)}\mathscr{F},K^{\dagger}\mathscr{G}}$ admits a K-right inverse, a multiplier induced by $\widetilde{\mathscr{F}}$ and $\widetilde{\mathscr{G}}$, which are respectively the canonical K-duals of $\mathscr{F}$ and $\mathscr{G}$. More precisely,*

$$\mathscr{M}^{KRI}_{\{c\},\pi_{Ran(K)}\mathscr{F},K^{\dagger}\mathscr{G}} = \mathscr{M}_{\{\frac{1}{c}\},\widetilde{\mathscr{G}},\widetilde{\mathscr{F}}}.$$

(2) *If $KK^{\dagger}U^*_{\mathscr{F}} = U^*_{\mathscr{G}}$, then $\mathscr{M}_{\{1\},K^{\dagger}\mathscr{F},\mathscr{G}}$ admits a K^*-left inverse on $S_{\mathscr{G}}(Ran(K))$, a multiplier on $S_{\mathscr{G}}(Ran(K))$ induced by $\widetilde{\mathscr{F}}$ and $\widetilde{\mathscr{G}}$, which are respectively the canonical K-duals of $\mathscr{F}$ and $\mathscr{G}$. That is,*

$$\mathscr{M}^{K^*LI}_{\{c\},K^{\dagger}\mathscr{F},\mathscr{G}} = \mathscr{M}_{\{\frac{1}{c}\},\widetilde{\mathscr{F}},\widetilde{\mathscr{G}}}$$

on $S_{\mathscr{G}}(Ran(K))$.

Proof. (1) A simple calculation gives

$$U^*_{\widetilde{\mathscr{G}}} = K^*(S_{\mathscr{G}}|_{Ran(K)})^{-1}\pi_{S_{\mathscr{G}}(Ran(K))}U^*_{\mathscr{G}}, \quad U_{\widetilde{\mathscr{F}}} = U_{\mathscr{F}}\pi_{S_{\mathscr{F}}(Ran(K))}((S_{\mathscr{F}}|_{Ran(K)})^{-1})^*K.$$

Therefore,

$$\begin{aligned}
&U^*_{\pi_{Ran(K)}\mathscr{F}}U_{K^{\dagger}\mathscr{G}}U^*_{\widetilde{\mathscr{G}}}U_{\widetilde{\mathscr{F}}}\\
&= \pi_{Ran(K)}U^*_{\mathscr{F}}U_{\mathscr{G}}(K^{\dagger})^*K^*(S_{\mathscr{G}}|_{Ran(K)})^{-1}\pi_{S_{\mathscr{G}}(Ran(K))}U^*_{\mathscr{G}}U_{\mathscr{F}}\pi_{S_{\mathscr{F}}(Ran(K))}((S_{\mathscr{F}}|_{Ran(K)})^{-1})^*K\\
&= \pi_{Ran(K)}U^*_{\mathscr{F}}U_{\mathscr{G}}(K^{\dagger})^*K^*(S_{\mathscr{G}}|_{Ran(K)})^{-1}\pi_{S_{\mathscr{G}}(Ran(K))}U^*_{\mathscr{G}}U_{\mathscr{F}}((S_{\mathscr{F}}|_{Ran(K)})^{-1})^*K.
\end{aligned}$$

The assumption $Ran(U_{\mathscr{F}}) \subset Ran(U_{\mathscr{G}}K)$ leads to $U_{\mathscr{F}} = U_{\mathscr{G}}K\Theta$ for some $\Theta \in \mathscr{L}\mathscr{B}(\mathscr{N})$, and it is easily seen that

$$\pi_{S_{\mathscr{G}}(Ran(K))}U^*_{\mathscr{G}}U_{\mathscr{G}}|_{Ran(K)} = S_{\mathscr{G}}|_{Ran(K)} : Ran(K) \to S_{\mathscr{G}}(Ran(K)).$$

Hence,

$$\begin{aligned}
&U^*_{\pi_{Ran(K)}\mathscr{F}}U_{K^\dagger\mathscr{G}}U^*_{\widetilde{\mathscr{G}}}U_{\widetilde{\mathscr{F}}}\\
&\quad=\pi_{Ran(K)}U^*_{\mathscr{F}}U_{\mathscr{G}}(K^\dagger)^*K^*(S_{\mathscr{G}}|_{Ran(K)})^{-1}\pi_{S_{\mathscr{G}}(Ran(K))}U^*_{\mathscr{G}}U_{\mathscr{G}}K\Theta((S_{\mathscr{F}}|_{Ran(K)})^{-1})^*K\\
&\quad=\pi_{Ran(K)}U^*_{\mathscr{F}}U_{\mathscr{G}}(K^\dagger)^*K^*(S_{\mathscr{G}}|_{Ran(K)})^{-1}(S_{\mathscr{G}}|_{Ran(K)})K\Theta((S_{\mathscr{F}}|_{Ran(K)})^{-1})^*K\\
&\quad=\pi_{Ran(K)}U^*_{\mathscr{F}}U_{\mathscr{G}}(K^\dagger)^*K^*K\Theta((S_{\mathscr{F}}|_{Ran(K)})^{-1})^*K\\
&\quad=\pi_{Ran(K)}U^*_{\mathscr{F}}U_{\mathscr{G}}K\Theta((S_{\mathscr{F}}|_{Ran(K)})^{-1})^*K\\
&\quad=\pi_{Ran(K)}U^*_{\mathscr{F}}U_{\mathscr{F}}((S_{\mathscr{F}}|_{Ran(K)})^{-1})^*K,
\end{aligned}$$

where the fourth equality follows from the observation that $(K^\dagger)^*K^*K=KK^\dagger K=K$. It is easy to check that

$$\pi_{Ran(K)}U^*_{\mathscr{F}}U_{\mathscr{F}}|_{S_{\mathscr{F}}(Ran(K))}=(S_{\mathscr{F}}|_{Ran(K)})^*:S_{\mathscr{F}}(Ran(K))\to Ran(K).$$

Thus,

$$\begin{aligned}
\mathscr{M}_{\{c\},\pi_{Ran(K)}\mathscr{F},K^\dagger\mathscr{G}}\mathscr{M}_{\{\frac{1}{c}\},\widetilde{\mathscr{G}},\widetilde{\mathscr{F}}}&=U^*_{\pi_{Ran(K)}\mathscr{F}}U_{K^\dagger\mathscr{G}}U^*_{\widetilde{\mathscr{G}}}U_{\widetilde{\mathscr{F}}}\\
&=(S_{\mathscr{F}}|_{Ran(K)})^*((S_{\mathscr{F}}|_{Ran(K)})^*)^{-1}K=K.
\end{aligned}$$

(2) Since $KK^\dagger U^*_{\mathscr{F}}=U^*_{\mathscr{G}}$, we have

$$\begin{aligned}
&U^*_{\widetilde{\mathscr{F}}}U_{\widetilde{\mathscr{G}}}U^*_{K^\dagger\mathscr{F}}U_{\mathscr{G}}h\\
&\quad=K^*(S_{\mathscr{F}}|_{Ran(K)})^{-1}\pi_{S_{\mathscr{F}}(Ran(K))}U^*_{\mathscr{F}}U_{\mathscr{G}}\pi_{S_{\mathscr{G}}(Ran(K))}((S_{\mathscr{G}}|_{Ran(K)})^*)^{-1}KK^\dagger U^*_{\mathscr{F}}U_{\mathscr{G}}h\\
&\quad=K^*(S_{\mathscr{F}}|_{Ran(K)})^{-1}\pi_{S_{\mathscr{F}}(Ran(K))}U^*_{\mathscr{F}}U_{\mathscr{G}}((S_{\mathscr{G}}|_{Ran(K)})^*)^{-1}\pi_{Ran(K)}U^*_{\mathscr{G}}U_{\mathscr{G}}h\\
&\quad=K^*(S_{\mathscr{F}}|_{Ran(K)})^{-1}\pi_{S_{\mathscr{F}}(Ran(K))}U^*_{\mathscr{F}}U_{\mathscr{G}}((S_{\mathscr{G}}|_{Ran(K)})^*)^{-1}\pi_{Ran(K)}U^*_{\mathscr{G}}U_{\mathscr{G}}|_{S_{\mathscr{G}}(Ran(K))}h\\
&\quad=K^*(S_{\mathscr{F}}|_{Ran(K)})^{-1}\pi_{S_{\mathscr{F}}(Ran(K))}U^*_{\mathscr{F}}U_{\mathscr{G}}((S_{\mathscr{G}}|_{Ran(K)})^*)^{-1}(S_{\mathscr{G}}|_{Ran(K)})^*h\\
&\quad=K^*(S_{\mathscr{F}}|_{Ran(K)})^{-1}\pi_{S_{\mathscr{F}}(Ran(K))}U^*_{\mathscr{F}}U_{\mathscr{F}}(KK^\dagger)^*h
\end{aligned}$$

for each $h\in S_{\mathscr{G}}(Ran(K))$. Clearly, $K^*KK^\dagger=K^*(K^\dagger)^*K^*=K^*(K^*)^\dagger K^*=K^*$. Hence,

$$\begin{aligned}
\mathscr{M}_{\{\frac{1}{c}\},\widetilde{\mathscr{F}},\widetilde{\mathscr{G}}}\mathscr{M}_{\{c\},K^\dagger\mathscr{F},\mathscr{G}}h&=U^*_{\widetilde{\mathscr{F}}}U_{\widetilde{\mathscr{G}}}U^*_{K^\dagger\mathscr{F}}U_{\mathscr{G}}h\\
&=K^*(S_{\mathscr{F}}|_{Ran(K)})^{-1}\pi_{S_{\mathscr{F}}(Ran(K))}U^*_{\mathscr{F}}U_{\mathscr{F}}KK^\dagger h\\
&=K^*(S_{\mathscr{F}}|_{Ran(K)})^{-1}\pi_{S_{\mathscr{F}}(Ran(K))}U^*_{\mathscr{F}}U_{\mathscr{F}}|_{Ran(K)}KK^\dagger h\\
&=K^*(S_{\mathscr{F}}|_{Ran(K)})^{-1}(S_{\mathscr{F}}|_{Ran(K)})KK^\dagger h=K^*KK^\dagger h=K^*h,
\end{aligned}$$

and we have the result. □

In [22, Theorem 3.4] the authors told us that the composition of K and the K-right (left) inverse of a K-frame multiplier is in the form of a multiplier. Our result shows, however, that the K-right (left) inverse itself can be represented by a multiplier without the participation of the operator K, which gives an affirmative answer to [**Q2**].

THEOREM 4.7. *Suppose that $0 \neq K \in \mathscr{LB}(\mathscr{N})$ is a closed range operator, that $\mathscr{F} = \{f_i\}_{i=1}^{\infty}$ and $\mathscr{G} = \{g_i\}_{i=1}^{\infty}$ are K-frames for $\mathscr{N}$, and that $m = \{m_i\}_{i=1}^{\infty}$ is ℓ^{∞}-invertible. The following assertions hold.*

(1) *If $\mathscr{M}_{m,\mathscr{F},\mathscr{G}}^{KRI}$ is a K-right inverse of $\mathscr{M}_{m,\mathscr{F},\mathscr{G}}$ and $Ran(\mathscr{M}_{m,\mathscr{F},\mathscr{G}}^{KRI}) \subset Ran((K^*)^{\dagger})$, then there exists a K-dual $\mathscr{F}^{\ddagger}$ of $\mathscr{F}$ such that for any K-dual $\mathscr{G}^{\ddagger}$ of $\mathscr{G}$ we obtain*

$$\mathscr{M}_{m,\mathscr{F},\mathscr{G}}^{KRI} = \mathscr{M}_{m^{-1},(K^*)^{\dagger}\mathscr{G}^{\ddagger},\mathscr{F}^{\ddagger}}.$$

(2) *If $\mathscr{M}_{m,\mathscr{F},\mathscr{G}}^{KLI}$ is a K-left inverse of $\mathscr{M}_{m,\mathscr{F},\mathscr{G}}$ and $Ran(\mathscr{M}_{m,\mathscr{F},\mathscr{G}}^{KLI}) \subset Ran((K^*)^{\dagger})$, then there exists a K^*-dual $\mathscr{G}^{\sharp}$ of $\mathscr{G}$ such that for any K-dual $\mathscr{F}^{\ddagger}$ of $\mathscr{F}$ we have*

$$\mathscr{M}_{m,\mathscr{F},\mathscr{G}}^{KLI} = \mathscr{M}_{m^{-1},(K^*)^{\dagger}\mathscr{F}^{\ddagger},\mathscr{G}^{\sharp}}.$$

Proof. (1) *Step* 1: Construct $\mathscr{F}^{\ddagger}$. For each $x \in \mathscr{N}$ we have $\mathscr{M}_{m,\mathscr{F},\mathscr{G}}\mathscr{M}_{m,\mathscr{F},\mathscr{G}}^{KRI}x = Kx$. That is

$$Kx = \sum_{i=1}^{\infty} m_i \langle \mathscr{M}_{m,\mathscr{F},\mathscr{G}}^{KRI}x, g_i\rangle f_i = \sum_{i=1}^{\infty} \langle x, \overline{m}_i (\mathscr{M}_{m,\mathscr{F},\mathscr{G}}^{KRI})^* g_i\rangle f_i,$$

meaning that $\mathscr{F}^{\ddagger} = \{f_i^{\ddagger} := \overline{m}_i(\mathscr{M}_{m,\mathscr{F},\mathscr{G}}^{KRI})^* g_i\}_{i=1}^{\infty}$ is a K-dual of $\mathscr{F}$.

Step 2: Relate operators. Since

$$U_{\mathscr{F}^{\ddagger}}x = \{\langle x, \overline{m}_i(\mathscr{M}_{m,\mathscr{F},\mathscr{G}}^{KRI})^* g_i\rangle\}_{i=1}^{\infty} = \{m_i\langle \mathscr{M}_{m,\mathscr{F},\mathscr{G}}^{KRI}x, g_i\rangle\}_{i=1}^{\infty} = D_m U_{\mathscr{G}} \mathscr{M}_{m,\Lambda,\Gamma}^{KRI}x,$$

or equivalently, $U_{\mathscr{F}^{\ddagger}} = D_m U_{\mathscr{G}} \mathscr{M}_{m,\mathscr{F},\mathscr{G}}^{KRI}$, it follows that

$$U_{\mathscr{G}} \mathscr{M}_{m,\mathscr{F},\mathscr{G}}^{KRI} = D_m^{-1} U_{\mathscr{F}^{\ddagger}} = D_{m^{-1}} U_{\mathscr{F}^{\ddagger}}.$$

For any K-dual $\mathscr{G}^{\ddagger} = \{g_i^{\ddagger}\}_{i=1}^{\infty}$ of $\mathscr{G}$ and any $x \in \mathscr{N}$, we get $K^*x = \sum_{i=1}^{\infty}\langle x, g_i\rangle g_i^{\ddagger}$. Since K has closed range, so does K^*. Now for each $h \in Ran((K^*)^{\dagger})$,

$$h = (K^*)^{\dagger}K^*h = \sum_{i=1}^{\infty} \langle h, g_i\rangle (K^*)^{\dagger} g_i^{\ddagger}.$$

That is, $(K^*)^{\dagger} U_{\mathscr{G}^{\ddagger}}^* U_{\mathscr{G}}|_{Ran((K^*)^{\dagger})} = \mathrm{Id}_{Ran((K^*)^{\dagger})}$.

Step 3: Represent the inverse. Noting that $Ran(\mathscr{M}_{m,\mathscr{F},\mathscr{G}}^{KRI}) \subset Ran((K^*)^{\dagger})$, we have

$$\begin{aligned}\mathscr{M}_{m,\mathscr{F},\mathscr{G}}^{KRI}x &= (K^*)^{\dagger} U_{\mathscr{G}^{\ddagger}}^* U_{\mathscr{G}} \mathscr{M}_{m,\mathscr{F},\mathscr{G}}^{KRI}x = (K^*)^{\dagger} U_{\mathscr{G}^{\ddagger}}^* D_{m^{-1}} U_{\mathscr{F}^{\ddagger}}x \\ &= (K^*)^{\dagger} \sum_{i=1}^{\infty} m_i^{-1}\langle x, f_i^{\ddagger}\rangle g_i^{\ddagger} = \sum_{i=1}^{\infty} m_i^{-1}\langle x, f_i^{\ddagger}\rangle (K^*)^{\dagger} g_i^{\ddagger} = \mathscr{M}_{m^{-1},(K^*)^{\dagger}\mathscr{G}^{\ddagger},\mathscr{F}^{\ddagger}}x,\end{aligned}$$

for each $x \in \mathscr{N}$, and we are done.

(2) A similar approach as described in (1) shows that $\mathscr{G}^{\sharp} = \{g_i^{\sharp} := m_i \mathscr{M}_{m,\mathscr{F},\mathscr{G}}^{KLI} f_i\}_{i=1}^{\infty}$ is a K^*-dual of $\mathscr{G}$, and that

$$U_{\mathscr{F}} \mathscr{M}_{m,\mathscr{F},\mathscr{G}}^{KLI} = D_{m^{-1}} U_{\mathscr{G}^{\sharp}}.$$

From this and taking into account the assumption that $Ran(\mathcal{M}^{KLI}_{m,\mathscr{F},\mathscr{G}}) \subset Ran((K^*)^\dagger)$, we get

$$\begin{aligned}\mathcal{M}^{KLI}_{m,\mathscr{F},\mathscr{G}}x &= (K^*)^\dagger U^*_{\mathscr{F}^\ddagger} U_{\mathscr{F}} \mathcal{M}^{KLI}_{m,\mathscr{F},\mathscr{G}}x = (K^*)^\dagger U^*_{\mathscr{F}^\ddagger} D_{m^{-1}} U_{\mathscr{G}^\sharp}x \\ &= \sum_{i=1}^{\infty} m_i^{-1}\langle x, g_i^\sharp\rangle (K^*)^\dagger f_i^\ddagger = \mathcal{M}_{m^{-1},(K^*)^\dagger\mathscr{F}^\ddagger,\mathscr{G}^\sharp}x\end{aligned}$$

for any K-dual $\mathscr{F}^\ddagger = \{f_i^\ddagger\}_{i=1}^\infty$ of $\mathscr{F}$ and any $x \in \mathscr{N}$, and we arrive at the conclusion. □

COROLLARY 4.8. *Suppose that $0 \neq K \in \mathscr{L}\mathscr{B}(\mathscr{N})$ is a closed range operator, that $\mathscr{F} = \{f_i\}_{i=1}^\infty$ and $\mathscr{G} = \{g_i\}_{i=1}^\infty$ are K-frames for $\mathscr{N}$, and that $m = \{m_i\}_{i=1}^\infty$ is ℓ^∞-invertible. If $\mathcal{M}^{KRI}_{m,\mathscr{F},\mathscr{G}}$ (resp. $\mathcal{M}^{KLI}_{m,\mathscr{F},\mathscr{G}}$) is a K-right (resp. K-left) inverse of $\mathcal{M}_{m,\mathscr{F},\mathscr{G}}$ with $Ran(\mathcal{M}^{KRI}_{m,\mathscr{F},\mathscr{G}}) \subset Ran((K^*)^\dagger)$ (resp. $Ran(\mathcal{M}^{KLI}_{m,\mathscr{F},\mathscr{G}}) \subset Ran((K^*)^\dagger)$), then $K\mathcal{M}^{KRI}_{m,\mathscr{F},\mathscr{G}}$ (resp. $\mathcal{M}^{KLI}_{m,\mathscr{F},\mathscr{G}}K$) can be written as a multiplier.*

Proof. We only prove the case of K-right inverse, since the proof of the K-left inverse setting is similar. By Theorem 4.7, there exists a K-dual $\mathscr{F}^\ddagger$ of $\mathscr{F}$ such that for any K-dual $\mathscr{G}^\ddagger$ of $\mathscr{G}$ we have $\mathcal{M}^{KRI}_{m,\mathscr{F},\mathscr{G}} = \mathcal{M}_{m^{-1},(K^*)^\dagger\mathscr{G}^\ddagger,\mathscr{F}^\ddagger}$. Therefore,

$$K\mathcal{M}^{KRI}_{m,\mathscr{F},\mathscr{G}} = K\mathcal{M}_{m^{-1},(K^*)^\dagger\mathscr{G}^\ddagger,\mathscr{F}^\ddagger} = \mathcal{M}_{m^{-1},K(K^*)^\dagger\mathscr{G}^\ddagger,\mathscr{F}^\ddagger}. \quad \square$$

The following result gives alternative conditions under which the K-right inverse and K-left inverse of a K-frame multiplier can be represented by a multiplier.

PROPOSITION 4.9. *Let $\mathscr{F} = \{f_i\}_{i=1}^\infty$ and $\mathscr{G} = \{g_i\}_{i=1}^\infty$ be K-frames for $\mathscr{N}$ with K-duals $\mathscr{F}^\ddagger = \{f_i^\ddagger\}_{i=1}^\infty$ and $\mathscr{G}^\ddagger = \{g_i^\ddagger\}_{i=1}^\infty$ respectively. Assume that $m = \{m_i\}_{i=1}^\infty$ is ℓ^∞-invertible, and that $\mathscr{F}$ and $\mathscr{G}$ are orthonormal in the sense that $\langle f_i, g_i\rangle = 1$ and $\langle f_j, g_i\rangle = 0$ if $i \neq j$. Then*

(1) *$\mathcal{M}_{m^{-1},\mathscr{F},\mathscr{F}^\ddagger}$ is a K-right inverse of $\mathcal{M}_{m,\mathscr{F},\mathscr{G}}$.*
(2) *$\mathcal{M}_{m^{-1},\mathscr{G},\mathscr{G}^\ddagger}$ is a K^*-left inverse of $\mathcal{M}_{m,\mathscr{F},\mathscr{G}}$.*

Proof. (1) For each $x \in \mathscr{N}$ we have

$$\begin{aligned}\mathcal{M}_{m,\mathscr{F},\mathscr{G}}\mathcal{M}_{m^{-1},\mathscr{F},\mathscr{F}^\ddagger}x &= \sum_{i=1}^{\infty} m_i \langle \mathcal{M}_{m^{-1},\mathscr{F},\mathscr{F}^\ddagger}x, g_i\rangle f_i = \sum_{i=1}^{\infty} m_i \left\langle \sum_{j=1}^{\infty} m_j^{-1}\langle x, f_j^\ddagger\rangle f_j, g_i \right\rangle f_i \\ &= \sum_{i=1}^{\infty} m_i \left(\sum_{j=1}^{\infty} m_j^{-1}\langle x, f_j^\ddagger\rangle \langle f_j, g_i\rangle \right) f_i = \sum_{i=1}^{\infty} m_i m_i^{-1}\langle x, f_i^\ddagger\rangle f_i = Kx,\end{aligned}$$

meaning that $\mathcal{M}_{m^{-1},\mathscr{F},\mathscr{F}^\ddagger}$ is a K-right inverse of $\mathcal{M}_{m,\mathscr{F},\mathscr{G}}$.

(2) The proof is similar to (1), which is left to the reader. □

The term "orthonormal" in Proposition 4.9 does not refer to each sequence $\mathscr{F}$ and $\mathscr{G}$ being orthonormal within itself. Instead, it describes a biorthogonality relationship between the two K-frames $\mathscr{F}$ and $\mathscr{G}$. Specifically, the assumption $\langle f_i, g_i\rangle = 1$ and $\langle f_j, g_i\rangle = 0$ for $i \neq j$ means that the sequences are biorthogonal to each other. In essence, each vector f_i from the first K-frame is orthogonal to every vector g_j from the second K-frame, except for its paired vector g_i, with which it has an inner product of 1. This condition is analogous to the relationship between a Riesz basis and its canonical dual basis. The purpose of this assumption is to ensure that the cross-frame multiplier $\mathscr{M}_{m,\mathscr{F},\mathscr{G}}$ simplifies nicely, as the biorthogonality forces the resulting frame operator to behave like a diagonal operator, which is crucial for proving the stated K-left and K-right inverse properties.

We provide an example illustrating Proposition 4.9.

EXAMPLE 4.10. Let $\mathscr{N} = \ell^2(\mathbb{N})$ with the standard orthonormal basis $\{e_i\}_{i=1}^{\infty}$. Taking $\mathscr{F} = \{f_i\}_{i=1}^{\infty}$ with $f_i = e_i$, and $\mathscr{G} = \{g_i\}_{i=1}^{\infty}$ with $g_i = e_i$. Then, for all i, j, $\langle f_i, g_i\rangle = \langle e_i, e_i\rangle = 1$, and $\langle f_j, g_i\rangle = \langle e_j, e_i\rangle = 0$ if $i \neq j$. Define $K : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ by $K(x_1, x_2, x_3, \ldots) = (x_1, 0, 0, \ldots)$.

Letting $f_i^{\ddagger} = K^* g_i = K g_i$ for each i. Then, $f_1^{\ddagger} = e_1$ and $f_i^{\ddagger} = 0$ for $i \geqslant 2$. Similarly, $g_1^{\ddagger} = e_1$ and $g_i^{\ddagger} = 0$ for $i \geqslant 2$, if we let $g_i^{\ddagger} = K^* f_i$ for all i. For each $x \in \mathscr{N}$, since

$$Kx = \sum_{i=1}^{\infty} \langle x, f_i^{\ddagger}\rangle f_i = \langle x, e_1\rangle e_1 = \sum_{i=1}^{\infty} \langle x, g_i^{\ddagger}\rangle g_i,$$

it follows that both $\mathscr{F}$ and $\mathscr{G}$ are K-frames for $\mathscr{N}$ with K-duals $\mathscr{F}^{\ddagger} = \{f_i^{\ddagger}\}_{i=1}^{\infty}$ and $\mathscr{G}^{\ddagger} = \{g_i^{\ddagger}\}_{i=1}^{\infty}$ respectively.

Pick $m_i = 1 + \frac{1}{i}$ for all i. Then, clearly, $m = \{m_i\}_{i=1}^{\infty}$ is ℓ^{∞}-invertible. A simple calculation gives

$$\mathscr{M}_{m,\mathscr{F},\mathscr{G}} x = \sum_{i=1}^{\infty} m_i \langle x, g_i\rangle f_i = \sum_{i=1}^{\infty} (1 + \frac{1}{i}) \langle x, e_i\rangle e_i,$$

and

$$\mathscr{M}_{m^{-1},\mathscr{F},\mathscr{F}^{\ddagger}} x = \sum_{i=1}^{\infty} m_i^{-1} \langle x, f_i^{\ddagger}\rangle f_i = \sum_{i=1}^{\infty} m_i^{-1} \langle x, K e_i\rangle e_i = m_1^{-1} \langle x, e_1\rangle e_1 = \frac{1}{2} \langle x, e_1\rangle e_1.$$

Noting that $\mathscr{M}_{m,\mathscr{F},\mathscr{G}} e_1 = \sum_{i=1}^{\infty} m_i \langle e_1, g_i\rangle f_i = m_1 \langle e_1, e_1\rangle e_1 = 2e_1$, we obtain

$$\begin{aligned} \mathscr{M}_{m,\mathscr{F},\mathscr{G}} \mathscr{M}_{m^{-1},\mathscr{F},\mathscr{F}^{\ddagger}} x &= \mathscr{M}_{m,\mathscr{F},\mathscr{G}} \left(\frac{1}{2} \langle x, e_1\rangle e_1\right) = \frac{1}{2} \langle x, e_1\rangle \mathscr{M}_{m,\mathscr{F},\mathscr{G}} e_1 \\ &= \frac{1}{2} \langle x, e_1\rangle (2e_1) = \langle x, e_1\rangle e_1 = Kx \end{aligned}$$

for each $x \in \mathscr{N}$, meaning that $\mathscr{M}_{m^{-1},\mathscr{F},\mathscr{F}^{\ddagger}}$ is a K-right inverse of $\mathscr{M}_{m,\mathscr{F},\mathscr{G}}$.

Obviously, $\mathscr{M}_{m^{-1},\mathscr{G},\mathscr{G}^{\ddagger}} x = \mathscr{M}_{m^{-1},\mathscr{F},\mathscr{F}^{\ddagger}} x = \frac{1}{2} \langle x, e_1\rangle e_1$. Thus

$$\begin{aligned} \mathscr{M}_{m^{-1},\mathscr{G},\mathscr{G}^{\ddagger}} \mathscr{M}_{m,\mathscr{F},\mathscr{G}} x &= \frac{1}{2} \langle \mathscr{M}_{m,\mathscr{F},\mathscr{G}} x, e_1\rangle e_1 = \frac{1}{2} \left\langle \sum_{i=1}^{\infty} m_i \langle x, e_i\rangle e_i, e_1 \right\rangle e_1 \\ &= \frac{1}{2} m_1 \langle x, e_1\rangle e_1 = \langle x, e_1\rangle e_1 = Kx = K^* x \end{aligned}$$

for any $x \in \mathscr{N}$, showing that $\mathscr{M}_{m^{-1},\mathscr{G},\mathscr{G}^\ddagger}$ is a K^*-left inverse of $\mathscr{M}_{m,\mathscr{F},\mathscr{G}}$.

5. Conclusion

In this paper, we have established several sufficient conditions for the invertibility of K-frame multipliers and provided an operator-theoretic characterization of their invertibility. We have shown that K-right inverses and K-left inverses of K-frame multipliers can be represented as multipliers induced by canonical K-duals, and we characterized the conditions under which such representations exist. These results provide a more comprehensive theoretical framework for analyzing K-frame multipliers, extending beyond the weak invertibility conditions previously available.

The theoretical framework developed here, particularly the explicit representations of inverses, provides mathematical tools that may be relevant to areas where K-frames are applied. These include problems in signal processing involving partial information or constrained reconstruction, and in operator sampling theory where the sampling process is naturally modeled by an operator K. Future work will focus on exploring the applicability of these theoretical results to such concrete problems and on developing corresponding computational methods.

Data availability. No data were used to support this study.

Conflicts of interest. The authors declare that they have no conflicts of interest.

Acknowledgements. The authors wish to express appreciation to the referees for constructive comments that have contributed significantly to the improvement of this work. This work was supported by the National Natural Science Foundation of China (Grant No. 12361028), the Fujian Alliance of Mathematics (Grant No. 2025SXLMMS06), and the Natural Science Foundation of Fujian Province, China (Grant No. 2025J011282).

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(Received October 7, 2024)

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