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SOME INEQUALITIES FOR g–GENERALIZED EUCLIDEAN BEREZIN NUMBER

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Abstract. In this paper, we present several Berezin number inequalities involving upper bounds of the g-generalized Euclidean Berezin number. For example, we show Berezin number inequalities for finite sums of n operators. Among other inequalities, if $T_1,\ldots,T_n$, $S_1,\ldots,S_n$ are operators in $\mathbb{B}(\mathscr{H}(\Omega))$, then we obtain

$$\mathbf{ber}^r\left(\sum_{i=1}^{n}(T_i+S_i)\right)\leqslant 2^{r-2}\mathbf{ber}(\eta),$$

where $\eta=\sum_{i=1}^{n}\left(f^{2r}|T_i|+f^{2r}|S_i|+g^{2r}|T_i^*|+g^{2r}|S_i^*|\right)$, f and g are nonnegative continuous functions on $[0,\infty)$ such that $f(t)g(t)=t$, $(t\geqslant 0)$, and $r\geqslant 2$. Moreover, we present some results involving hyponormal operators.

1. Introduction

A functional Hilbert space $\mathscr{H}=\mathscr{H}(\Omega)$ is a Hilbert space of complex valued functions on a (nonempty) set Ω, which has the property that point evaluations are continuous i.e. for each $\tau\in\Omega$ the map $f\mapsto f(\tau)$ is a continuous linear functional on $\mathscr{H}$. The Riesz representation theorem ensure that for each $\tau\in\Omega$ there is a unique element $k_\tau\in\mathscr{H}$ such that $f(\tau)=\langle f,k_\tau\rangle$ for all $f\in\mathscr{H}$. The collection $\{k_\tau:\tau\in\Omega\}$ is called the reproducing kernel of $\mathscr{H}$. If $\{e_n\}$ is an orthonormal basis for a functional Hilbert space $\mathscr{H}$, then the reproducing kernel of $\mathscr{H}$ is given by $k_\tau(z)=\sum_n\overline{e_n(\tau)}e_n(z)$; (see [19, Problem 37]). For $\tau\in\Omega$, let $\hat{k}_{\mathscr{H},\tau}=\frac{k_\tau}{\|k_\tau\|}$ be the normalized reproducing kernel of $\mathscr{H}$. For a bounded linear operator T on $\mathscr{H}$, the function $\widetilde{T}$ defined on Ω by $\widetilde{T}(\tau)=\langle T\hat{k}_{\mathscr{H},\tau},\hat{k}_{\mathscr{H},\tau}\rangle$ is the Berezin symbol of T, which firstly have been introduced by Berezin in [7, 8]. For the complex Hilbert space $\mathscr{H}=\mathscr{H}(\Omega)$, let $\mathbb{B}(\mathscr{H}(\Omega))$ be the space of the Banach algebra of all bounded linear operators defined on $\mathscr{H}$. In the case when $\dim\mathscr{H}=n$, we identify $\mathbb{B}(\mathscr{H}(\Omega))$ with the matrix algebra $\mathbb{M}_n(\mathbb{C})$ of all $n\times n$ matrices with entries in the complex field. An operator $T\in\mathbb{B}(\mathscr{H}(\Omega))$ is called positive

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if $\langle Tk,k\rangle \geqslant 0$ for all $k\in\mathscr{H}$, and we then write $T\geqslant 0$. An operator $T\in\mathbb{B}(\mathscr{H}(\Omega))$ is called that T is hyponormal if $[T^*,T]:=T^*T-TT^*\geqslant 0$.

The Berezin set and the Berezin number of an operator $T\in\mathbb{B}(\mathscr{H}(\Omega))$ are defined by

$$\mathbf{Ber}(T):=\{\widetilde{T}(\tau):\tau\in\Omega\} \qquad \text{and} \qquad \mathbf{ber}(T):=\sup\{|\widetilde{T}(\tau)|:\tau\in\Omega\},$$

respectively, (see [22]). In [5], the authors show that $\mathbf{ber}(T)=\sup_{\theta\in\mathbb{R}}\mathbf{ber}\left(\Re(\mathrm{e}^{i\theta}T)\right)$. The Berezin symbol and Berezin number has large application in the study of various questions of operator theory in the functional Hilbert space, quantum physics and non-commutative geometry. These are the important tools to study operators on Hardy and Bergman spaces, especially for Toeplitz and Hankel operators. Recall that the Hardy space $\mathscr{H}_2(\mathbb{D})$ of the unit disk $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$ is a RKHS of analytic functions on $\mathbb{D}$ with reproducing kernel $k_\tau(z)=\frac{1}{1-\bar{\tau}z}$ (see, Paulsen and Raghupati [27]). Since, the collection of normalized reproducing kernel of $\mathscr{H}$ is a subset of the unit sphere of $\mathscr{H}$, so the numerical radius and the Berezin number of an operator on $\mathscr{H}$ may not be equal. The Berezin number inequalities have been studied by many mathematicians over the years, interested readers can see [9, 18, 23, 24, 26]. Namely, the Berezin symbol have been investigated in detail for the Toeplitz and Hankel operators on the Hardy and Bergman spaces; it is widely applied in the various questions of analysis and uniquely determines the operator (i.e., for all $\tau\in\Omega,\widetilde{S}(\tau)=\widetilde{T}(\tau)$ implies $S=T$). For further information about Berezin symbol we refer the reader to [3, 6, 4, 16, 17] and references therein.

The Berezin number of operators S,T satisfy the following properties:
(i) $\mathbf{ber}(\alpha \mathrm{T})=|\alpha|\mathbf{ber}(\mathrm{T})$ for all $\alpha\in\mathbb{C}$;
(ii) $\mathbf{ber}(T+S)\leqslant\mathbf{ber}(T)+\mathbf{ber}(S)$.

Recall that the numerical radius of $T\in\mathbb{B}(\mathscr{H}(\Omega))$ is defined by

$$w(T):=\sup\{|\langle Tx,x\rangle|:x\in\mathscr{H},\|x\|=1\}.$$

It is clear that

$$\mathbf{ber}(T)\leqslant w(T)\leqslant\|T\| \qquad \text{for all } T\in\mathbb{B}(\mathscr{H}(\Omega)). \tag{1}$$

Recently, in [5] the authors proved that

$$\mathbf{ber}^2(T)\leqslant\frac{1}{2}\mathbf{ber}(T^*T+TT^*) \qquad \text{for all } T\in\mathbb{B}(\mathscr{H}(\Omega)). \tag{2}$$

Moreover, they showed the Berezin number inequality involving the product of two operators as follows

$$\mathbf{ber}(T^*S)\leqslant\frac{1}{2}\mathbf{ber}(T^*T+S^*S) \qquad \text{for all } S,T\in\mathbb{B}(\mathscr{H}(\Omega)). \tag{3}$$

Assume that $T_i\in\mathbb{B}(\mathscr{H}(\Omega))$ $(1\leqslant i\leqslant n)$. The generalized Euclidean Berezin number of $T_1,\ldots,T_n$ defined in [3] as follows:

$$\mathbf{ber_p}(T_1,\cdots,T_n):=\sup_{\tau\in\Omega}\left(\sum_{i=1}^{n}|\widetilde{T_i}(\tau)|^p\right)^{\frac{1}{p}} \quad \text{for all} \quad p\geqslant 1.$$

In the case $p=2$, we have the Euclidean Berezin number and denote by

$$\mathbf{ber_e}(T_1,\ldots,T_n) := \sup_{\tau\in\Omega}\left(\sum_{i=1}^{n}|\widetilde{T_i}(\tau)|^2\right)^{\frac{1}{2}}.$$

For $p=1$ if $T_1=\cdots=T_n=T$, then $\mathbf{ber_1}(T,\cdots,T)=n\mathbf{ber}(T)$.

In [10], the authors obtained some generalized Euclidean Berezin number inequalities. Among of them, they obtained the following inequality

$$\mathbf{ber_p}(T_1,\cdots,T_n) \leqslant \frac{1}{2}\left(\mathbf{ber}\sum_{i=1}^{n}(|T_i|+|T_i^*|)^p\right)^{\frac{1}{p}}, \quad (p>1) \tag{4}$$

Recall that, a function $g:[0,\infty)\to[0,\infty)$ is convex if $g((1-\mu)a+\mu b)\leqslant(1-\mu)g(a)+\mu g(b)$ for all $\mu\in[0,1]$ and $a,b\in[0,\infty)$. For a convex function $g:[0,\infty)\to[0,\infty)$ such that $g(0)=0$, we have the superadditivity property, i.e.

$$g(x)+g(y)\leqslant g(x+y) \quad \text{for all } x,y\in[0,\infty). \tag{5}$$

The above inequality is reversed if g is concave. Utilizing the concept of the definition convex functions, we have the definition of the operator convex functions. We say that a function g is operator convex on an interval J if

$$g((1-\mu)A+\mu B)\leqslant(1-\mu)g(A)+\mu g(B)$$

for all self-adjoint operators A and B with spectra in J. For instance, the function $g(t)=t^p$, $(t>0)$ is operator convex for all $-1\leqslant p\leqslant 0$ and $1\leqslant p\leqslant 2$. For more information see [28] and references therein.

Recently, the authors in [15] defined the g-generalized Euclidean Berezin number for operators $T_1,\cdots,T_n$ in $\mathbb{B}(\mathscr{H}(\Omega))$ as follows:

$$\mathbf{ber}_g(T_1,\ldots,T_n) := \sup_{\tau\in\Omega} g^{-1}\left(\sum_{i=1}^{n} g\left(|\widetilde{T_i}(\tau)|\right)\right),$$

where $g:[0,\infty)\to[0,\infty)$ is a continuous increasing convex function such that $g(0)=0$.

It is clear for $g(t)=t^p$ $(p\geqslant 1)$ we have $\mathbf{ber}_g(\cdot)=\mathbf{ber_p}(\cdot)$ and for $g(t)=t^2$ we have $\mathbf{ber}_g(\cdot)=\mathbf{ber_e}(\cdot)$; see also [1].

The g-generalized Euclidean Berezin number $\mathbf{ber}_g(\cdot)$ has the following properties:

(i) $\mathbf{ber}_g(T_1,\ldots,T_n)=0$ if and only if $T_i=0$ $(i=1,\ldots,n)$;

(ii) $\mathbf{ber}_g(\alpha T_1,\ldots,\alpha T_n)=|\alpha|\mathbf{ber}_g(T_1,\ldots,T_n)$ for all $\alpha\in\mathbb{C}$ if g is multiplicative;

(iii) $\mathbf{ber}_g(T_1,T_2,\ldots,T_n)=\mathbf{ber}_g(T_1^*,T_2^*,\ldots,T_n^*)$;

(iv) $\mathbf{ber}_g(T_1+S_1,\ldots,T_n+S_n)\leqslant\mathbf{ber}_g(T_1,\ldots,T_n)+\mathbf{ber}_g(S_1,\ldots,S_n)$, if g is geometrically convex i.e. $g(\sqrt{xy})\leqslant\sqrt{g(x)g(y)}$, where $T_i,S_i\in\mathbb{B}(\mathscr{H}(\Omega))$ $(i=1,\ldots,n)$.

In this paper, we obtain generalizations of (2). For this purpose, we use of the Berezin number inequality for the finite sum of operators, which is shown in the first part of this paper. Some ideas of the first section are stimulated by [2]. Moreover, we present some upper bounds for g-generalized Euclidean Berezin number. Further, we obtain a refinement of (4). We also present some results involving hyponormal operators.

2. Inequalities of the Berezin number

In this section, we obtain some extensions of the known Berezin number inequalities. First, we prove some Berezin number inequalities for finite sum of operators. To prove our results, we need the several known lammas.

LEMMA 1. [20] *Let a,b be real numbers. Then*
(a) $|a+b|^r+|a-b|^r \geqslant 2(|a|^r+|b|^r),\ r\geqslant 2$;
(b) $a^r+b^r \leqslant (a+b)^r \leqslant 2^{r-1}(a^r+b^r),\ a,b>0$ *and* $r\geqslant 1$.

The following lemma known as Minkowski's inequality.

LEMMA 2. [20] *Let a_i,b_i $(i=1,\cdots,n)$ be positive real numbers and $r\geqslant 1$. Then*

$$\left(\sum_{i=1}^n (a_i+b_i)^r\right)^{\frac{1}{r}} \leqslant \left(\sum_{i=1}^n a_i^r\right)^{\frac{1}{r}} + \left(\sum_{i=1}^n b_i^r\right)^{\frac{1}{r}}.$$

LEMMA 3. [30] *Let $T\in\mathbb{B}(\mathscr{H}(\Omega))$ be positive and $x\in\mathscr{H}$ be a unite vector. Then*

(1) $\langle Tx,x\rangle^r \leqslant \langle T^r x,x\rangle$ *for* $r\geqslant 1$.

(2) $\langle Tx,x\rangle^r \geqslant \langle T^r x,x\rangle$ *for* $0\leqslant r\leqslant 1$.

LEMMA 4. [28] *Let $T\in\mathbb{B}(\mathscr{H}(\Omega))$ be self-adjoint operator and $x\in\mathscr{H}$. Then*

$$|\langle Tx,x\rangle| \leqslant \langle |T|x,x\rangle.$$

LEMMA 5. [25] (Mixed-Schwarz inequality) *Let $T\in\mathbb{B}(\mathscr{H}(\Omega))$ and $x,y\in\mathscr{H}$ be any vectors. If f,g are nonnegative continuous functions on $[0,\infty)$ which are satisfying the relation $f(t)g(t)=t$, $(t\geqslant 0)$, then*

$$|\langle Tx,y\rangle|^2 \leqslant \langle f^2(|T|)x,x\rangle\langle g^2(|T^*|)y,y\rangle.$$

In particular,

$$|\langle Tx,y\rangle|^2 \leqslant \langle |T|x,x\rangle\langle |T^*|y,y\rangle.$$

Now, we are in a position to state the first result of this section.

THEOREM 1. *Let $T_i, S_i \in \mathbb{B}(\mathscr{H}(\Omega))$ $(i=1,\ldots,n)$, and f,g be nonnegative continuous functions on $[0,\infty)$, which are satisfying the relation $f(t)g(t)=t$, $(t \geqslant 0)$. Then for all $r \geqslant 2$*

$$\mathbf{ber}^r\left(\sum_{i=1}^{n}(T_i+S_i)\right) \leqslant 2^{r-2}\mathbf{ber}(\eta), \tag{6}$$

where $\eta = \sum_{i=1}^{n}\left(f^{2r}|T_i| + f^{2r}|S_i| + g^{2r}|T_i^*| + g^{2r}|S_i^*|\right)$.

Proof. Let $\hat{k}_{\mathscr{H},\tau}$ be the normalized reproducing kernel of $\mathscr{H}(\Omega)$. Then

$$\begin{aligned}
&\left|\left\langle \sum_{i=1}^{n}(T_i+S_i)\hat{k}_{\mathscr{H},\tau},\hat{k}_{\mathscr{H},\tau}\right\rangle\right|^r \\
&\leqslant \sum_{i=1}^{n}\left|\langle (T_i+S_i)\hat{k}_{\mathscr{H},\tau},\hat{k}_{\mathscr{H},\tau}\rangle\right|^r \\
&= \sum_{i=1}^{n}\left|\langle T_i\hat{k}_{\mathscr{H},\tau},\hat{k}_{\mathscr{H},\tau}\rangle + \langle S_i\hat{k}_{\mathscr{H},\tau},\hat{k}_{\mathscr{H},\tau}\rangle\right|^r \\
&\leqslant \sum_{i=1}^{n}\left(|\langle T_i\hat{k}_{\mathscr{H},\tau},\hat{k}_{\mathscr{H},\tau}\rangle| + |\langle S_i\hat{k}_{\mathscr{H},\tau},\hat{k}_{\mathscr{H},\tau}\rangle|\right)^r \qquad \text{(by the triangle inequality)} \\
&\leqslant \left[\Big(\sum_{i=1}^{n}|\langle T_i\hat{k}_{\mathscr{H},\tau},\hat{k}_{\mathscr{H},\tau}\rangle|^r\Big)^{\frac{1}{r}} + \Big(\sum_{i=1}^{n}|\langle S_i\hat{k}_{\mathscr{H},\tau},\hat{k}_{\mathscr{H},\tau}\rangle|^r\Big)^{\frac{1}{r}}\right]^r \text{(by Minkowski's inequality)} \\
&\leqslant 2^{r-1}[\sum_{i=1}^{n}|\langle T_i\hat{k}_{\mathscr{H},\tau},\hat{k}_{\mathscr{H},\tau}\rangle|^r + (\sum_{i=1}^{n}|\langle S_i\hat{k}_{\mathscr{H},\tau},\hat{k}_{\mathscr{H},\tau}\rangle|^r] \qquad \text{(by Lemma 1(b))} \\
&\leqslant 2^{r-1}\left[\sum_{i=1}^{n}\langle f^2(|T_i|)\hat{k}_{\mathscr{H},\tau},\hat{k}_{\mathscr{H},\tau}\rangle^{\frac{r}{2}}\langle g^2(|T_i^*|)\hat{k}_{\mathscr{H},\tau},\hat{k}_{\mathscr{H},\tau}\rangle^{\frac{r}{2}}\right. \\
&\qquad \left. + \sum_{i=1}^{n}\langle f^2(|S_i|)\hat{k}_{\mathscr{H},\tau},\hat{k}_{\mathscr{H},\tau}\rangle^{\frac{r}{2}}\langle g^2(|S_i^*|)\hat{k}_{\mathscr{H},\tau},\hat{k}_{\mathscr{H},\tau}\rangle^{\frac{r}{2}}\right] \qquad \text{(by Lemma 5)} \\
&\leqslant 2^{r-2}[\sum_{i=1}^{n}(\langle f^2(|T_i|)\hat{k}_{\mathscr{H},\tau},\hat{k}_{\mathscr{H},\tau}\rangle^r\langle g^2(|T_i^*|)\hat{k}_{\mathscr{H},\tau},\hat{k}_{\mathscr{H},\tau}\rangle^r) \\
&\qquad + \sum_{i=1}^{n}(\langle f^2(|S_i|)\hat{k}_{\mathscr{H},\tau},\hat{k}_{\mathscr{H},\tau}\rangle^r\langle g^2(|S_i^*|)\hat{k}_{\mathscr{H},\tau},\hat{k}_{\mathscr{H},\tau}\rangle^r)] \\
&\qquad\qquad \text{(by the arithmetic-geometric inequality)} \\
&\leqslant 2^{r-2}\left[\sum_{i=1}^{n}(\langle f^{2r}(|T_i|)\hat{k}_{\mathscr{H},\tau},\hat{k}_{\mathscr{H},\tau}\rangle + \langle g^{2r}(|T_i^*|)\hat{k}_{\mathscr{H},\tau},\hat{k}_{\mathscr{H},\tau}\rangle)\right. \\
&\qquad \left. + \sum_{i=1}^{n}(\langle f^{2r}(|S_i|)\hat{k}_{\mathscr{H},\tau},\hat{k}_{\mathscr{H},\tau}\rangle + \langle g^{2r}(|S_i^*|)\hat{k}_{\mathscr{H},\tau},\hat{k}_{\mathscr{H},\tau}\rangle)\right] \qquad \text{(by Lemma 3)}
\end{aligned}$$

$$
\begin{aligned}
&= 2^{r-2}\Bigg[\sum_{i=1}^{n}\bigg(\left\langle f^{2r}(|T_i|)\hat{k}_{\mathscr{H},\tau},\hat{k}_{\mathscr{H},\tau}\right\rangle+\left\langle g^{2r}(|T_i^*|)\hat{k}_{\mathscr{H},\tau},\hat{k}_{\mathscr{H},\tau}\right\rangle \\
&\qquad +\sum_{i=1}^{n}\left\langle f^{2r}(|S_i|)\hat{k}_{\mathscr{H},\tau},\hat{k}_{\mathscr{H},\tau}\right\rangle+\left\langle g^{2r}(|S_i^*|)\hat{k}_{\mathscr{H},\tau},\hat{k}_{\mathscr{H},\tau}\right\rangle\bigg)\Bigg] \\
&\leqslant 2^{r-2}\left[\sum_{i=1}^{n}\left\langle\left(f^{2r}(|T_i|)+f^{2r}(|S_i|)+g^{2r}(|T_i^*|)+g^{2r}(|S_i^*|)\right)\hat{k}_{\mathscr{H},\tau},\hat{k}_{\mathscr{H},\tau}\right\rangle\right] \quad \text{(by (5))}.
\end{aligned}
$$

Taking the supremum over all $\tau\in\Omega$, we get the desired result. □

COROLLARY 1. *Let $T,S\in\mathbb{B}(\mathscr{H}(\Omega))$ and let f,g be nonnegative continuous functions on $[0,\infty)$, which are satisfying the relation $f(t)g(t)=t$, $(t\geqslant 0)$. Then for $r\geqslant 2$*

$$
\mathbf{ber}^r(T+S)\leqslant 2^{r-2}\mathbf{ber}\left(f^{2r}|T|+f^{2r}|S|+g^{2r}|T^*|+g^{2r}|S^*|\right). \tag{7}
$$

In particular,

$$
\mathbf{ber}^r(T)\leqslant \frac{1}{2}\mathbf{ber}\left(|T|^r+|T^*|^r\right). \tag{8}
$$

Proof. By putting $T_i=S_i=0$ $(i=2,3,\ldots,n)$ in (6), we have (7). Moreover for $T=S$ and $f(t)=g(t)=\sqrt{t}$ in (7), we have (8). □

REMARK 1. Note that, by assuming $r=2$ in (8), we have $\mathbf{ber}^2(T)\leqslant\frac{1}{2}\mathbf{ber}\left(|T|^2+|T^*|^2\right)$. So (8) is an extension of (2).

REMARK 2. If $T\in\mathbb{B}(\mathscr{H}(\Omega))$ and $r\geqslant 2$, then we have

$$
\begin{aligned}
\mathbf{ber}^r(T) &\leqslant \frac{1}{2}\mathbf{ber}(|T|^r+|T^*|^r) \\
&\leqslant \frac{1}{2}\left(\mathbf{ber}(|T|^r)+\mathbf{ber}(|T^*|^r)\right) \qquad \text{(by the subadditivity of } \mathbf{ber}(\cdot)) \\
&\leqslant \frac{1}{2}\left(w(|T|^r)+w(|T^*|^r)\right) \qquad \text{(by the inequality (1))} \\
&= \frac{1}{2}\left(\|\,|T|^r\|+\|\,|T^*|^r\|\right) \qquad \text{(since } |T|^r \text{ and } |T^*|^r \text{ are normal)} \\
&= \frac{1}{2}\left(\|\,|T|\|^r+\|\,|T^*|\|^r\right) \qquad \text{(by the functional calculus)} \\
&= \frac{1}{2}\left(\|T\|^r+\|T^*\|^r\right) \qquad \text{(since } \|\,|X|\,\|=\|X\|) \\
&= \|T\|^r \qquad \text{(since } \|X\|=\|X^*\|).
\end{aligned}
$$

Hence,

$$
\mathbf{ber}(T)\leqslant\sqrt[r]{\frac{1}{2}\mathbf{ber}\left(|T|^r+|T^*|^r\right)}\leqslant\|T\|. \tag{9}
$$

The above inequality shows that the inequality (8) is a refinement of the inequality (1).

EXAMPLE 1. Let $\{e_1, e_1\}$ be the standard orthonormal basis for $\mathbb{C}^2$. Consider $\mathbb{C}^2$ as a RKHS on the set $\{1,2\}$. If we put the matrix $T = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ and $r = 5$ in the inequality (9), we have

$$\mathbf{ber}(T) = 2 \lneqq \sqrt[5]{\frac{1}{2}\mathbf{ber}(|T|^5 + |T^*|^5)} \approx 2.4540 \lneqq \|T\| \approx 2.6173.$$

THEOREM 2. *Let $T, S \in \mathbb{B}(\mathscr{H}(\Omega))$ and f, g be nonnegative continuous functions on $[0,\infty)$, which are satisfying the relation $f(t)g(t) = t$, $(t \geqslant 0)$. Then for $r \geqslant 2$*

$$\begin{aligned} &\boldsymbol{ber}^r(T+S) \\ &\leqslant 2^{r-3}\boldsymbol{ber}\left(f^{2r}(|T+S|) + f^{2r}(|T-S|) + g^{2r}(|(T+S)^*|) + g^{2r}(|(T-S)^*|)\right). \end{aligned} \tag{10}$$

Proof. Let $\hat{k}_{\mathscr{H},\tau} \in \mathscr{H}(\Omega)$. Then

$$\begin{aligned}
&|\langle (T+S)\hat{k}_{\mathscr{H},\tau}, \hat{k}_{\mathscr{H},\tau}\rangle|^r \\
&= |\langle T\hat{k}_{\mathscr{H},\tau}, \hat{k}_{\mathscr{H},\tau}\rangle + \langle S\hat{k}_{\mathscr{H},\tau}, \hat{k}_{\mathscr{H},\tau}\rangle|^r \\
&\leqslant 2^{r-1}\left(|\langle T\hat{k}_{\mathscr{H},\tau}, \hat{k}_{\mathscr{H},\tau}\rangle|^r + |\langle S\hat{k}_{\mathscr{H},\tau}, \hat{k}_{\mathscr{H},\tau}\rangle|^r\right) \\
&\leqslant 2^{r-2}\left(|\langle (T+S)\hat{k}_{\mathscr{H},\tau}, \hat{k}_{\mathscr{H},\tau}\rangle|^r + |\langle (T-S)\hat{k}_{\mathscr{H},\tau}, \hat{k}_{\mathscr{H},\tau}\rangle|^r\right) \\
&\qquad\qquad \text{(by Lemma 1(a))} \\
&\leqslant 2^{r-2}\Big[\left\langle f^2(|T+S|)\hat{k}_{\mathscr{H},\tau}, \hat{k}_{\mathscr{H},\tau}\right\rangle^{\frac{r}{2}} \left\langle g^2(|(T+S)^*|)\hat{k}_{\mathscr{H},\tau}, \hat{k}_{\mathscr{H},\tau}\right\rangle^{\frac{r}{2}} \\
&\quad + \left\langle f^2(|T-S|)\hat{k}_{\mathscr{H},\tau}, \hat{k}_{\mathscr{H},\tau}\right\rangle^{\frac{r}{2}} \left\langle g^2(|(T-S)^*|)\hat{k}_{\mathscr{H},\tau}, \hat{k}_{\mathscr{H},\tau}\right\rangle^{\frac{r}{2}}\Big]. \\
&\qquad \text{(by Lemma 5)} \\
&\leqslant 2^{r-2}\Big[\left\langle f^r(|T+S|)\hat{k}_{\mathscr{H},\tau}, \hat{k}_{\mathscr{H},\tau}\right\rangle \left\langle g^r(|(T+S)^*|)\hat{k}_{\mathscr{H},\tau}, \hat{k}_{\mathscr{H},\tau}\right\rangle \\
&\quad + \left\langle f^r(|T-S|)\hat{k}_{\mathscr{H},\tau}, \hat{k}_{\mathscr{H},\tau}\right\rangle \left\langle g^r(|(T-S)^*|)\hat{k}_{\mathscr{H},\tau}, \hat{k}_{\mathscr{H},\tau}\right\rangle\Big]. \\
&\qquad\qquad \text{(by Lemma 3)} \\
&\leqslant 2^{r-3}\Big[\left\langle f^{2r}(|T+S|)\hat{k}_{\mathscr{H},\tau}, \hat{k}_{\mathscr{H},\tau}\right\rangle + \left\langle g^{2r}(|(T+S)^*|)\hat{k}_{\mathscr{H},\tau}, \hat{k}_{\mathscr{H},\tau}\right\rangle \\
&\quad + \left\langle f^{2r}(|T-S|)\hat{k}_{\mathscr{H},\tau}, \hat{k}_{\mathscr{H},\tau}\right\rangle + \left\langle g^{2r}(|(T-S)^*|)\hat{k}_{\mathscr{H},\tau}, \hat{k}_{\mathscr{H},\tau}\right\rangle\Big] \\
&\qquad\qquad \text{(by the arithmetic-geometric mean inequality and Lemma 3)} \\
&= 2^{r-3}\left\langle f^{2r}(|T+S|) + f^{2r}(|T-S|) + g^{2r}(|(T+S)^*|) + g^{2r}(|(T-S)^*|)\hat{k}_{\mathscr{H},\tau}, \hat{k}_{\mathscr{H},\tau}\right\rangle.
\end{aligned}$$

Thus, by taking the supremum on over all $\tau \in \Omega$, we get the desired inequality. □

By replacing T by S and assuming $f(t)=g(t)=\sqrt{t}$ in (10), we have the next result.

COROLLARY 2. *Let $T\in\mathbb{B}(\mathscr{H}(\Omega))$ and $r\geqslant 2$. Then*

$$\boldsymbol{ber}^r(T)\leqslant 2^{r-3}\boldsymbol{ber}(|T|^r+|T^*|^r). \tag{11}$$

THEOREM 3. *Let $T,S\in\mathbb{B}(\mathscr{H}(\Omega))$ be self-adjoint and $r\geqslant 2$. Then*

$$\boldsymbol{ber}^r(T+S)\leqslant 2^{r-2}\boldsymbol{ber}(|T+S|^r+|T-S|^r). \tag{12}$$

In particular,

$$\boldsymbol{ber}^r(T)\leqslant 2^{r-2}\boldsymbol{ber}(|T|^r). \tag{13}$$

Proof. With letting $f(t)=g(t)=\sqrt{t}$ in the inequality (10), we get

$$\begin{aligned}&\mathbf{ber}^r(T+S)\\&\leqslant 2^{r-3}\mathbf{ber}\left(|T+S|^r+|T-S|^r+|(T+S)^*|^r+|(T-S)^*|^r\right)\\&=2^{r-3}\mathbf{ber}\left(2|T+S|^r+2|T-S|^r\right)\\&=2^{r-2}\mathbf{ber}\left(|T+S|^r+|T-S|^r\right).\end{aligned}$$

For (13), it is sufficient to replace T by S in (12). $\square$

In the following theorem, we obtain the Berezin number inequality for the products of two operators in $\mathbb{B}(\mathscr{H}(\Omega))$.

THEOREM 4. *Let $T,S\in\mathbb{B}(\mathscr{H}(\Omega))$. Then*

$$\boldsymbol{ber}^r(T^*S)\leqslant\frac{1}{2}\boldsymbol{ber}\left(|T|^r+(S^*|T^*|S)^r\right)\quad\textit{for all } r\geqslant 1. \tag{14}$$

Proof. If $\hat{k}_{\mathscr{H},\tau}\in\mathscr{H}(\Omega)$, then

$$\begin{aligned}\left|\langle S^*T\hat{k}_{\mathscr{H},\tau},\hat{k}_{\mathscr{H},\tau}\rangle\right|^r&=\left|\langle T\hat{k}_{\mathscr{H},\tau},S\hat{k}_{\mathscr{H},\tau}\rangle\right|^r\\&\leqslant\langle|T|\hat{k}_{\mathscr{H},\tau},\hat{k}_{\mathscr{H},\tau}\rangle^{\frac{r}{2}}\langle|T^*|S\hat{k}_{\mathscr{H},\tau},S\hat{k}_{\mathscr{H},\tau}\rangle^{\frac{r}{2}}\qquad\text{(by Lemma 13)}\\&=\langle|T|\hat{k}_{\mathscr{H},\tau},\hat{k}_{\mathscr{H},\tau}\rangle^{\frac{r}{2}}\langle S^*|T^*|S\hat{k}_{\mathscr{H},\tau},\hat{k}_{\mathscr{H},\tau}\rangle^{\frac{r}{2}}\\&\leqslant\langle|T|^r\hat{k}_{\mathscr{H},\tau},\hat{k}_{\mathscr{H},\tau}\rangle^{\frac{1}{2}}\langle(S^*|T^*|S)^r\hat{k}_{\mathscr{H},\tau},\hat{k}_{\mathscr{H},\tau}\rangle^{\frac{1}{2}}\qquad\text{(by Lemma 10)}\\&\leqslant\frac{1}{2}\left[\langle|T|^r\hat{k}_{\mathscr{H},\tau},\hat{k}_{\mathscr{H},\tau}\rangle+\langle(S^*|T^*|S)^r\hat{k}_{\mathscr{H},\tau},\hat{k}_{\mathscr{H},\tau}\rangle\right]\\&\qquad\text{(by the arithmetic-geometric mean inequality)}\\&=\frac{1}{2}\left\langle\left[|T|^r+(S^*|T^*|S)^r\right]\hat{k}_{\mathscr{H},\tau},\hat{k}_{\mathscr{H},\tau}\right\rangle.\end{aligned}$$

Now, by taking the supremum on over all $\tau\in\Omega$, we reach the desired result. $\square$

REMARK 3. Let $T,S \in \mathbb{B}(\mathscr{H}(\Omega))$ and $r \geqslant 1$. Similar to Remark 1, we have

$$\mathbf{ber}(T^*S) \leqslant \sqrt[r]{\frac{1}{2}\mathbf{ber}\left(|T|^r + (S^*|T^*|S)^r\right)} \leqslant \frac{\|T\|}{2}\sqrt[r]{1+\|S\|^{2r}}. \tag{15}$$

Moreover, if we put $r=2$ and $S=I$ in (14), then we get the inequality (2) as follows

$$\mathbf{ber}^2(T) \leqslant \frac{1}{2}\mathbf{ber}(T^*T+TT^*).$$

EXAMPLE 2. If we put $T=\begin{pmatrix}3&2\\2&5\end{pmatrix}$, $S=\begin{pmatrix}1&1\\0&2\end{pmatrix}$ and $r=3$ in the inequality (15), then

$$\mathbf{ber}(T^*S)=12 \lneqq \sqrt[3]{\frac{1}{2}\mathbf{ber}\left(|T|^3+(S^*|T^*|S)^3\right)} \approx 16.05 \lneqq \frac{\|T\|}{2}\sqrt[3]{1+\|S\|^6} \approx 16.376.$$

3. Some upper bounds for g-generalized Euclidean Berezin number

In this section, we make use of some properties of g-generalized Euclidean Berezin number and, then we obtain relations between the g-generalized Euclidean Berezin number and the Euclidean Berezin number. Moreover, we present some refined inequalities. Specially, we obtain refinements of (2) and (4). First, we need the following well-known lemmas.

LEMMA 6. [14] (Hermit-Hadamard inequality) *Let $f:[a,b]\to\mathbb{R}$ be a convex function and let $0\leqslant\mu\leqslant 1$. Then*

$$f\left(\frac{a+b}{2}\right) \leqslant l(\mu) \leqslant \frac{1}{b-a}\int_a^b f(t)dt \leqslant \frac{f(a)+f(b)}{2},$$

where $l(\mu)=(1-\mu)f\left(\frac{(1-\mu)a+(1+\mu)b}{2}\right)+\mu f\left(\frac{(2-\mu)a+\mu b}{2}\right)$.

THEOREM 5. *Suppose that $T_j \in \mathbb{B}(\mathscr{H}(\Omega))$ $(j=1,2,\dots,n)$ and $g:[0,\infty)\to[0,\infty)$ be an increasing operator convex function such that $g(0)=0$. If $0\leqslant\alpha,\mu\leqslant 1$, then*

$$\begin{aligned}
&\boldsymbol{ber}_g(T_1,\dots,T_n)\\
&\leqslant g^{-1}\Bigg(\boldsymbol{ber}\Bigg(\sum_{j=1}^n\Bigg[(1-\mu)g\Bigg(\frac{(1-\mu)|T_j|^{2\alpha}+(1+\mu)|T^*j|^{2(1-\alpha)}}{2}\Bigg)\\
&\qquad\qquad +\mu g\Bigg(\frac{(2-\mu)|T_j|^{2\alpha}+\mu|T^*j|^{2(1-\alpha)}}{2}\Bigg)\Bigg]\Bigg)\Bigg)\\
&\leqslant g^{-1}\Bigg(\boldsymbol{ber}\Bigg(\int_0^1\sum_{j=1}^n g\left(t|T_j|^{2\alpha}+(1-t)|T_j^*|^{2(1-\alpha)}\right)dt\Bigg)\Bigg)\\
&\leqslant g^{-1}\Bigg(\frac{1}{2}\boldsymbol{ber}\Bigg(\sum_{j=1}^n g\left(|T_j|^{2\alpha}\right)+g\left(|T_j^*|^{2(1-\alpha)}\right)\Bigg)\Bigg).
\end{aligned}$$

Proof. Let $\hat{k}_{\mathscr{H},\tau} \in \mathscr{H}$. Using Lemmas 6 and 5, we have

$$\begin{aligned}
& g\left(\left|\langle T_j \hat{k}_{\mathscr{H},\tau}, \hat{k}_{\mathscr{H},\tau}\rangle\right|\right) \leqslant g\left(\sqrt{\langle |T_j|^{2\alpha} T \hat{k}_{\mathscr{H},\tau}, \hat{k}_{\mathscr{H},\tau}\rangle \left\langle |T_j^*|^{2(1-\alpha)} T \hat{k}_{\mathscr{H},\tau}, \hat{k}_{\mathscr{H},\tau}\right\rangle}\right) \\
& \leqslant (1-\mu)\left\langle g\left(\frac{(1-\mu)|T_j|^{2\alpha} + (1+\mu)|T_j^*|^{2(1-\alpha)}}{2}\right)\hat{k}_{\mathscr{H},\tau}, \hat{k}_{\mathscr{H},\tau}\right\rangle \\
& \quad + \mu\left\langle g\left(\frac{(2-\mu)|T_j|^{2\alpha} + \mu|T_j^*|^{2(1-\alpha)}}{2}\right)\hat{k}_{\mathscr{H},\tau}, \hat{k}_{\mathscr{H},\tau}\right\rangle \\
& = \left\langle \left[(1-\mu) g(\frac{(1-\mu)|T_j|^{2\alpha} + (1+\mu)|T_j^*|^{2(1-\alpha)}}{2}) + \mu g(\frac{(2-\mu)|T_j|^{2\alpha} + \mu|T_j^*|^{2(1-\alpha)}}{2})\right]\hat{k}_{\mathscr{H},\tau}, \hat{k}_{\mathscr{H},\tau}\right\rangle.
\end{aligned}$$

Since g is operator convex, the operator version Lemma 6 implies that

$$\begin{aligned}
& \left\langle \left[(1-\mu) g\left(\frac{(1-\mu)|T_j|^{2\alpha} + (1+\mu)|T_j^*|^{2(1-\alpha)}}{2}\right) + \mu g\left(\frac{(2-\mu)|T_j|^{2\alpha} + \mu|T_j^*|^{2(1-\alpha)}}{2}\right)\right]\hat{k}_{\mathscr{H},\tau}, \hat{k}_{\mathscr{H},\tau}\right\rangle \\
& \leqslant \left\langle \left(\int_0^1 g\left(t|T_j|^{2\alpha} + (1-t)|T_j^*|^{2(1-\alpha)}\right) dt\right) \hat{k}_{\mathscr{H},\tau}, \hat{k}_{\mathscr{H},\tau}\right\rangle \\
& \leqslant \left\langle \left(\frac{g\left(|T_j|^{2\alpha}\right) + g\left(|T_j^*|^{2(1-\alpha)}\right)}{2}\right)\hat{k}_{\mathscr{H},\tau}, \hat{k}_{\mathscr{H},\tau}\right\rangle.
\end{aligned}$$

Thus

$$\begin{aligned}
& \sum_{j=1}^n g\left(|\langle T_j \hat{k}_{\mathscr{H},\tau}, \hat{k}_{\mathscr{H},\tau}\rangle|\right) \\
& \leqslant \sum_{j=1}^n \left\langle \left[(1-\mu) g(\frac{(1-\mu)|T_j|^{2\alpha} + (1+\mu)|T_j^*|^{2(1-\alpha)}}{2}) + \mu g\left(\frac{(2-\mu)|T_j|^{2\alpha} + \mu|T_j^*|^{2(1-\alpha)}}{2})\right]\hat{k}_{\mathscr{H},\tau}, \hat{k}_{\mathscr{H},\tau}\right\rangle \\
& \leqslant \sum_{j=1}^n \left\langle \left(\int_0^1 g(t|T_j|^{2\alpha} + (1-t)|T_j^*|^{2(1-\alpha)}) dt\right) \hat{k}_{\mathscr{H},\tau}, \hat{k}_{\mathscr{H},\tau}\right\rangle \\
& \leqslant \sum_{j=1}^n \left\langle \left(\frac{g(|T_j|^{2\alpha}) + g(|T_j^*|^{2(1-\alpha)})}{2}\right)\hat{k}_{\mathscr{H},\tau}, \hat{k}_{\mathscr{H},\tau}\right\rangle.
\end{aligned}$$

This is equivalent to

$$\begin{aligned}
& \sum_{j=1}^n g\left(\left|\langle T_j \hat{k}_{\mathscr{H},\tau}, \hat{k}_{\mathscr{H},\tau}\rangle\right|\right) \\
& \leqslant \left\langle \sum_{j=1}^n \left[(1-\mu) g\left(\frac{(1-\mu)|T_j|^{2\alpha} + (1+\mu)|T_j^*|^{2(1-\alpha)}}{2}\right)\right.\right. \\
& \qquad \left.\left. + \mu g\left(\frac{(2-\mu)|T_j|^{2\alpha} + \mu|T_j^*|^{2(1-\alpha)}}{2}\right)\right]\hat{k}_{\mathscr{H},\tau}, \hat{k}_{\mathscr{H},\tau}\right\rangle \\
& \leqslant \left\langle \left(\int_0^1 \sum_{j=1}^n g((t|T_j|^{2\alpha} + (1-t)|T_j^*|^{2(1-\alpha)}) dt\right)\hat{k}_{\mathscr{H},\tau}, \hat{k}_{\mathscr{H},\tau}\right\rangle \\
& \leqslant \frac{1}{2}\left\langle \sum_{j=1}^n \left(g(|T_j|^{2\alpha}) + g(|T_j^*|^{2(1-\alpha)})\right)\hat{k}_{\mathscr{H},\tau}, \hat{k}_{\mathscr{H},\tau}\right\rangle.
\end{aligned}$$

It follows from g^{-1} is increasing, we have

$$
\begin{aligned}
& g^{-1}\left(\sum_{j=1}^{n} g\left(\left|\langle T_j\hat{k}_{\mathscr{H},\tau},\hat{k}_{\mathscr{H},\tau}\rangle\right|\right)\right) \\
& \leqslant g^{-1}\left(\left\langle \sum_{j=1}^{n}\left[(1-\mu)g\left(\frac{(1-\mu)|T_j|^{2\alpha}+(1+\mu)|T_j^*|^{2(1-\alpha)}}{2}\right)\right.\right.\right. \\
& \qquad \left.\left.\left. +\mu g\left(\frac{(2-\mu)|T_j|^{2\alpha}+\mu|T_j^*|^{2(1-\alpha)}}{2}\right)\right]\hat{k}_{\mathscr{H},\tau},\hat{k}_{\mathscr{H},\tau}\right\rangle\right) \\
& \leqslant g^{-1}\left(\left\langle\left(\int_0^1\sum_{j=1}^{n} g\left(t|T_j|^{2\alpha}+(1-t)|T_j^*|^{2(1-\alpha)}\right)dt\right)\hat{k}_{\mathscr{H},\tau},\hat{k}_{\mathscr{H},\tau}\right\rangle\right) \\
& \leqslant g^{-1}\left(\frac{1}{2}\left\langle\sum_{j=1}^{n}\left(g\left(|T_j|^{2\alpha}\right)+g\left(|T_j^*|^{2(1-\alpha)}\right)\right)\hat{k}_{\mathscr{H},\tau},\hat{k}_{\mathscr{H},\tau}\right\rangle\right).
\end{aligned}
$$

By taking the supremum over all $\tau\in\Omega$, we get desired result. □

COROLLARY 3. *Suppose that $T_j\in\mathbb{B}(\mathscr{H}(\Omega))$ $(j=1,2,\ldots,n)$ and $0\leqslant\mu\leqslant 1$. Then for $1\leqslant p\leqslant 2$,*

$$
\begin{aligned}
& \boldsymbol{ber}_p(T_1,\ldots T_n) \\
& \leqslant\left(\boldsymbol{ber}\left(\sum_{j=1}^{n}\left[(1-\mu)\left(\frac{(1-\mu)|T_j|+(1+\mu)|T_j^*|}{2}\right)^p+\mu\left(\frac{(2-\mu)|T_j|+\mu|T_j^*|}{2}\right)^p\right]\right)\right)^{\frac{1}{p}} \\
& \leqslant\left(\boldsymbol{ber}\left(\int_0^1\sum_{j=1}^{n}(t|T_j|+(1-t)|T_j^*|)^p dt\right)\right)^{\frac{1}{p}} \\
& \leqslant\left(\frac{1}{2}\boldsymbol{ber}\left(\sum_{j=1}^{n}(((|T_j|^p+(|T_j^*|^p)\right)\right)^{\frac{1}{p}}.
\end{aligned}
$$

Proof. This immediately follows from Theorem 5 by assuming $g(t)=t^p,\ p\geqslant 1$. □

REMARK 4. By putting $n=1,\ T_1=T,\ \alpha=\frac{1}{2}$ and $g(t)=t^2$ in Theorem 5, we have a refinement of (2) as follows:

$$
\begin{aligned}
\mathbf{ber}^2(T) & \leqslant\frac{1}{2}\mathbf{ber}\left(\left(\frac{|T|+3|T^*|}{4}\right)^2+\left(\frac{3|T|+|T^*|}{4}\right)^2\right) \\
& \leqslant\mathbf{ber}\left(\int_0^1(t|T|+(1-t)|T^*|)^2 dt\right) \\
& \leqslant\frac{1}{2}\mathbf{ber}\left(|T|^2+|T^*|^2\right). \qquad (16)
\end{aligned}
$$

EXAMPLE 3. Consider for $\mathbb{C}^2$ the standard orthonormal basis $\{e_1, e_1\}$ as a RKHS on the set $\{1,2\}$. If we put the matrix $T = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$ in the inequality (16), then we have

$$\begin{aligned}\mathbf{ber}^2(T) = 4 &\lneqq \frac{1}{2}\mathbf{ber}\left(\left(\frac{|T|+3|T^*|}{4}\right)^2 + \left(\frac{3|T|+|T^*|}{4}\right)^2\right) \approx 4.17 \\ &\lneqq \mathbf{ber}\left(\int_0^1 (t|T|+(1-t)|T^*|)^2 dt\right) \approx 4.23 \\ &\lneqq \frac{1}{2}\mathbf{ber}\left(|T|^2+|T^*|^2\right) = 4.5.\end{aligned}$$

In the next theorem, we present g-generalized Euclidean Berezin number for product of operators.

THEOREM 6. *Let $T_j, S_j \in \mathbb{B}(\mathscr{H}(\Omega))$ $(j=1,\ldots,n)$ and $g:[0,\infty)\to[0,\infty)$ be an increasing operator convex function such that $g(0)=0$. Then*

$$\begin{aligned}\boldsymbol{ber}_g(T_1^*S_1,\ldots,T_n^*S_n) &\leqslant g^{-1}\left(\boldsymbol{ber}\left(\int_0^1\left(\sum_{j=1}^n g\left(t|S_j|^2+(1-t)|T_j|^2\right)\right)dt\right)\right) \\ &\leqslant g^{-1}\left(\frac{1}{2}\boldsymbol{ber}\left(\sum_{j=1}^n \left(g\left(|S_j|^2\right)+g\left(|T_j|^2\right)\right)\right)\right).\end{aligned}$$

Proof. Let $\hat{k}_{\mathscr{H},\tau}\in\mathscr{H}$ and $j=1,\ldots,n$. Then, by the Cauchy-Schwarz inequality we have

$$\begin{aligned}g\left(|\langle T^*jS_j\hat{k}_{\mathscr{H},\tau},\hat{k}_{\mathscr{H},\tau}\rangle|\right) &= g\left(|\langle S_j\hat{k}_{\mathscr{H},\tau},T_j\hat{k}_{\mathscr{H},\tau}\rangle|\right) \\ &\leqslant g\left(\|T_j\hat{k}_{\mathscr{H},\tau}\|\|S_j\hat{k}_{\mathscr{H},\tau}\|\right) \\ &\leqslant g\left(\sqrt{\left\langle|T_j|^2\hat{k}_{\mathscr{H},\tau},\hat{k}_{\mathscr{H},\tau}\right\rangle\left\langle|S_j|^2\hat{k}_{\mathscr{H},\tau},\hat{k}_{\mathscr{H},\tau}\right\rangle}\right) \\ &\leqslant g\left(\frac{\left\langle|T_j|^2\hat{k}_{\mathscr{H},\tau},\hat{k}_{\mathscr{H},\tau}\right\rangle+\left\langle|S_j|^2\hat{k}_{\mathscr{H},\tau},\hat{k}_{\mathscr{H},\tau}\right\rangle}{2}\right) \\ &\leqslant \int_0^1 g\left(t\langle|T_j|^2\hat{k}_{\mathscr{H},\tau},\hat{k}_{\mathscr{H},\tau}\rangle+(1-t)\langle|S_j|^2\hat{k}_{\mathscr{H},\tau},\hat{k}_{\mathscr{H},\tau}\rangle\right)dt.\end{aligned} \tag{17}$$

Moreover, the convexity of g implies that

$$\begin{aligned}&g\left(t\langle|T_j|^2\hat{k}_{\mathscr{H},\tau},\hat{k}_{\mathscr{H},\tau}\rangle+(1-t)\langle|S_j|^2\hat{k}_{\mathscr{H},\tau},\hat{k}_{\mathscr{H},\tau}\rangle\right) \\ &= g\left(\left\langle\left[t|T_j|^2+(1-t)|S_j|^2\right]\hat{k}_{\mathscr{H},\tau},\hat{k}_{\mathscr{H},\tau}\right\rangle\right) \\ &\leqslant \left\langle g\left(t|T_j|^2+(1-t)|S_j|^2\right)\hat{k}_{\mathscr{H},\tau},\hat{k}_{\mathscr{H},\tau}\right\rangle \\ &\leqslant t\left\langle g\left(|T_j|^2\right)\hat{k}_{\mathscr{H},\tau},\hat{k}_{\mathscr{H},\tau}\right\rangle+(1-t)\left\langle g\left(|S_j|^2\right)\hat{k}_{\mathscr{H},\tau},\hat{k}_{\mathscr{H},\tau}\right\rangle.\end{aligned}$$

So,

$$\begin{aligned}
&\int_0^1 g\left(t\langle |T_j|^2\hat{k}_{\mathscr{H},\tau},\hat{k}_{\mathscr{H},\tau}\rangle+(1-t)\langle |S_j|^2\hat{k}_{\mathscr{H},\tau},\hat{k}_{\mathscr{H},\tau}\rangle\right)dt\\
&\leqslant \left\langle \int_0^1 \left(g(t|T_j|^2+(1-t)|S_j|^2)dt\right)\hat{k}_{\mathscr{H},\tau},\hat{k}_{\mathscr{H},\tau}\rangle\right\rangle\\
&\leqslant \left\langle \left(\frac{g(|T_j|^2)+g(|S_j|^2)}{2}\right)\hat{k}_{\mathscr{H},\tau},\hat{k}_{\mathscr{H},\tau}\right\rangle. \qquad (18)
\end{aligned}$$

With combining the inequalities (17) and (18), we get

$$\begin{aligned}
g(|\langle S^* jT_j\hat{k}_{\mathscr{H},\tau},\hat{k}_{\mathscr{H},\tau}\rangle|) &\leqslant \left\langle \int_0^1 \left(g(t|T_j|^2+(1-t)|S_j|^2)\right)\hat{k}_{\mathscr{H},\tau},\hat{k}_{\mathscr{H},\tau}\right\rangle\\
&\leqslant \left\langle \left(\frac{g(|T_j|^2)+g(|S_j|^2)}{2}\right)\hat{k}_{\mathscr{H},\tau},\hat{k}_{\mathscr{H},\tau}\right\rangle.
\end{aligned}$$

This implies that

$$\begin{aligned}
\sum_{j=1}^n g(|\langle S_j^*T_j\hat{k}_{\mathscr{H},\tau},\hat{k}_{\mathscr{H},\tau}\rangle|) &\leqslant \sum_{j=1}^n \left\langle \int_0^1 \left(g(t|T_j|^2+(1-t)|S_j|^2)dt\right)\hat{k}_{\mathscr{H},\tau},\hat{k}_{\mathscr{H},\tau}\right\rangle\\
&\leqslant \sum_{j=1}^n \left\langle \left(\frac{g(|T_j|^2)+g(|S_j|^2)}{2}\right)\hat{k}_{\mathscr{H},\tau},\hat{k}_{\mathscr{H},\tau}\right\rangle.
\end{aligned}$$

Since g^{-1} is increasing,

$$\begin{aligned}
&g^{-1}\left(\sum_{j=1}^n g(|\langle S_j^*T_j\hat{k}_{\mathscr{H},\tau},\hat{k}_{\mathscr{H},\tau}\rangle|)\right)\\
&\leqslant g^{-1}\left(\sum_{j=1}^n \left\langle \int_0^1 \left(g(t|T_j|^2+(1-t)|S_j|^2)dt\right)\hat{k}_{\mathscr{H},\tau},\hat{k}_{\mathscr{H},\tau}\right\rangle\right)\\
&\leqslant g^{-1}\left(\frac{1}{2}\sum_{j=1}^n \left\langle \left(g(|T_j|^2)+g(|S_j|^2)\right)\hat{k}_{\mathscr{H},\tau},\hat{k}_{\mathscr{H},\tau}\right\rangle\right).
\end{aligned}$$

Now, by taking the supremum over all $\tau\in\Omega$, we get the desired result. □

COROLLARY 4. *Let $T_j,S_j\in\mathbb{B}(\mathscr{H}(\Omega))$ $(j=1,\dots,n)$ and $1\leqslant p\leqslant 2$. Then*

$$\begin{aligned}
\boldsymbol{ber}_p(T_1^*S_1,\dots,T_n^*S_n) &\leqslant \boldsymbol{ber}^{\frac{1}{p}}\left(\int_0^1\left(\sum_{j=1}^n\left(t|S_j|^2+(1-t)|T_j|^2\right)^p\right)dt\right)\\
&\leqslant \frac{1}{2^{\frac{1}{p}}}\boldsymbol{ber}^{\frac{1}{p}}\left(\sum_{j=1}^n\left(|S_j|^{2p}+|T_j|^{2p}\right)\right).
\end{aligned}$$

In particular,

$$\begin{aligned} \boldsymbol{ber}^p(T^*S) &\leqslant \boldsymbol{ber}\left(\int_0^1 \left(t|S|^2+(1-t)|T|^2\right)^p dt\right) \\ &\leqslant \frac{1}{2}\boldsymbol{ber}\left(|S|^{2p}+|T|^{2p}\right). \end{aligned} \tag{19}$$

Proof. Since $g(t)=t^p$, $(1\leqslant p\leqslant 2)$ is an increasing operator convex function, so by letting $g(t)=t^p$, $(1\leqslant p\leqslant 2)$ in Theorem 6 we get the first result. Moreover, by assuming $n=1$, $T_1=T$ and $S_1=S$ in the first inequality, we have the second result. □

REMARK 5. Note that, the inequality (19) is a refinement of (3).

4. One application

The reproducing kernel Hilbert space $\mathscr{H}=\mathscr{H}(\Omega)$ is said to be standard (see Nordgren and Rosental [26]) if the underlying set Ω is a subset of a topological space and its boundary $\partial\Omega$ is nonempty and has the property that $\{\hat{k}_{\mathscr{H},\tau_n}\}$ convergence weakly to 0 whenever $\{\tau_n\}$ is a sequence in Ω that convergence to a point in $\partial\Omega$. The common reproducing kernel Hilbert space of analytic functions, including the Hardy, Bergman and Fock Hilbert spaces, are standard in this sense. For a compact operator K, on the standard reproducing kernel Hilbert space $\mathscr{H}$, it is easy to see that

$$\lim_{n\to\infty}\widetilde{K}(\tau_n)=0$$

whenever $\{\tau_n\}$, convergence to a point of $\partial\Omega$ (since compact operators send weakly convergent sequences to strongly convergent ones). In this sense, the Berezin symbol of a compact operator on a standard reproducing kernel Hilbert space vanishes on the boundary. For $T\in\mathbb{B}(\mathscr{H}(\Omega))$, its Berezin norm is defined also by

$$\|T\|_{ber}:=\sup_{\tau\in\Omega}\left\|T\hat{k}_{\mathscr{H},\tau}\right\|.$$

For more information and relationship about the Berezin set, numerical range, Berezin number and numerical radius, we refer the readers to Karaev [24].

In this section, in terms of Berezin numbers we give a new necessary condition for hyponormality of operators on the reproducing kernel of Hilbert space $\mathscr{H}(\Omega)$. For any operator $T\in\mathbb{B}(\mathscr{H}(\Omega))$, we denote

$$\delta(T):=\inf_{\tau\in\Omega}\left|\widetilde{T}(\tau)\right| \qquad \text{and} \qquad \Delta(T):=\inf_{\tau\in\Omega}\left\|T\hat{k}_{\mathscr{H},\tau}\right\|.$$

It is obvious that $\delta(T)\leqslant\Delta(T)$. We begin with the following simple proposition where $[T^*,T]=T^*T-TT^*$.

PROPOSITION 1. If $T \in \mathbb{B}(\mathscr{H}(\Omega))$ is a hyponormal operator, then

$$\mathbf{ber}([T^*,T]) \leqslant \|T\|_{\mathbf{ber}}^2 - \Delta^2(T^*), \tag{20}$$

or equivalently

$$\mathbf{ber}([T^*,T]) \leqslant \mathbf{ber}(T^*T) - \Delta^2(T^*). \tag{21}$$

Proof. It follows from $[T^*,T] \geqslant 0$ that $\widetilde{[T^*,T]}(\tau) \geqslant 0$ for all $\tau \in \Omega$. Then we have

$$\begin{aligned}\mathbf{ber}([T^*,T]) &= \sup_{\tau\in\Omega} \left\langle (T^*T - TT^*)\hat{k}_{\mathscr{H},\tau}, \hat{k}_{\mathscr{H},\tau} \right\rangle \\ &= \sup_{\tau\in\Omega} \left[\left\langle T\hat{k}_{\mathscr{H},\tau}, T\hat{k}_{\mathscr{H},\tau} \right\rangle - \left\langle T^*\hat{k}_{\mathscr{H},\tau}, T^*\hat{k}_{\mathscr{H},\tau} \right\rangle \right] \\ &\leqslant \sup_{\tau\in\Omega} \|T\hat{k}_{\mathscr{H},\tau}\|^2 - \inf_{\tau\in\Omega} \|T^*\hat{k}_{\mathscr{H},\tau}\|^2 \\ &= \|T\|_{\mathbf{ber}}^2 - \Delta^2(T^*),\end{aligned}$$

which proves the inequality (20). □

We remark that if $\delta(T) \neq 0$, then

$$\begin{aligned}\left(\frac{\Delta^4(T)}{2\Delta^2(T) - \delta^2(T)} \right)^{\frac{1}{2}} &\leqslant \left(\frac{\Delta^4(T)}{2\delta^2(T) - \delta^2(T)} \right)^{\frac{1}{2}} \\ &= \left(\frac{\Delta^4(T)}{\delta^2(T)} \right)^{\frac{1}{2}} = \frac{\Delta^2(T)}{\delta(T)} = \frac{\Delta(T)}{\delta(T)}\Delta(T) \\ &\leqslant \frac{\Delta(T)}{\delta(T)} \|T\|_{\mathbf{ber}},\end{aligned}$$

hence

$$\eta(T) := \left(\frac{\Delta^4(T)}{2\Delta^2(T) - \delta^2(T)} \right)^{\frac{1}{2}} \leqslant \frac{\Delta(T)}{\delta(T)} \|T\|_{\mathbf{ber}},$$

where $\frac{\Delta(T)}{\delta(T)} \geqslant 1$. We also denote $\alpha := \|T\|_{\mathbf{ber}}$.

Our next result essentially improves the inequality (20) for some class of operators.

THEOREM 7. *Let $T \in \mathbb{B}(\mathscr{H}(\Omega))$ be an operator such that $\delta(T) \neq 0$ and*

$$\Delta(T) \leqslant \sqrt{\alpha^2 - \alpha\sqrt{\alpha^2 - \delta^2(T)}}. \tag{22}$$

If T is hyponormal, then

$$\boldsymbol{ber}([T^*,T]) \leqslant \|T\|_{\boldsymbol{ber}} (\|T\|_{\boldsymbol{ber}}^2 - \delta^2(T))^{\frac{1}{2}}, \tag{23}$$

which is better than the estimate (20) in Proposition 1.

Proof. Since T is hyponormal, the Berezin symbol of the self-commutator $[T^*,T]$ is positive. Then, by considering that $|\widetilde{T^*}(\tau)| = |\widetilde{T}(\tau)|$ and $T\hat{k}_{\mathscr{H},\tau} - \widetilde{T}(\tau)\hat{k}_{\mathscr{H},\tau} \perp \hat{k}_{\mathscr{H},\tau}$, we have:

$$\begin{aligned}
0 \leqslant \widetilde{[T^*,T]}(\tau) &= \langle T\hat{k}_{\mathscr{H},\tau}, T\hat{k}_{\mathscr{H},\tau}\rangle - \langle T^*\hat{k}_{\mathscr{H},\tau}, T^*\hat{k}_{\mathscr{H},\tau}\rangle \\
&= \left\langle T\hat{k}_{\mathscr{H},\tau}, T\hat{k}_{\mathscr{H},\tau} - \widetilde{T}(\tau)\hat{k}_{\mathscr{H},\tau}\right\rangle + \left\langle T\hat{k}_{\mathscr{H},\tau}, \widetilde{T}(\tau)\hat{k}_{\mathscr{H},\tau}\right\rangle \\
&\quad - \left\langle T^*\hat{k}_{\mathscr{H},\tau}, T^*\hat{k}_{\mathscr{H},\tau} - \widetilde{T^*}(\tau)\hat{k}_{\mathscr{H},\tau}\right\rangle - \left\langle T^*\hat{k}_{\mathscr{H},\tau}, \widetilde{T^*}(\tau)\hat{k}_{\mathscr{H},\tau}\right\rangle \\
&= \left\langle T\hat{k}_{\mathscr{H},\tau}, T\hat{k}_{\mathscr{H},\tau} - \widetilde{T}(\tau)\hat{k}_{\mathscr{H},\tau}\right\rangle + \left|\widetilde{T}(\tau)\right|^2 \\
&\quad - \left\langle T^*\hat{k}_{\mathscr{H},\tau}, T^*\hat{k}_{\mathscr{H},\tau} - \widetilde{T^*}(\tau)\hat{k}_{\mathscr{H},\tau}\right\rangle - \left|\widetilde{T^*}(\tau)\right|^2 \\
&= \left\langle T\hat{k}_{\mathscr{H},\tau}, T\hat{k}_{\mathscr{H},\tau} - \widetilde{T}(\tau)\hat{k}_{\mathscr{H},\tau}\right\rangle - \left\langle T^*\hat{k}_{\mathscr{H},\tau}, T^*\hat{k}_{\mathscr{H},\tau} - \widetilde{T^*}(\tau)\hat{k}_{\mathscr{H},\tau}\right\rangle \\
&\leqslant \left\langle T\hat{k}_{\mathscr{H},\tau}, T\hat{k}_{\mathscr{H},\tau} - \widetilde{T}(\tau)\hat{k}_{\mathscr{H},\tau}\right\rangle,
\end{aligned}$$

and hence by the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\widetilde{[T^*,T]}(\tau) &\leqslant \left\|T\hat{k}_{\mathscr{H},\tau}\right\| \left\|T\hat{k}_{\mathscr{H},\tau} - \widetilde{T}(\tau)\hat{k}_{\mathscr{H},\tau}\right\| \\
&\leqslant \left\|T\hat{k}_{\mathscr{H},\tau}\right\| \left(\left\|T\hat{k}_{\mathscr{H},\tau}\right\|^2 - \left|\widetilde{T}(\tau)\right|^2\right)^{\frac{1}{2}} \\
&\leqslant \sup_{\tau\in\Omega}\left\|T\hat{k}_{\mathscr{H},\tau}\right\| \left(\sup_{\tau\in\Omega}\left\|T\hat{k}_{\mathscr{H},\tau}\right\|^2 - \inf_{\tau\in\Omega}\left|\widetilde{T}(\tau)\right|^2\right)^{\frac{1}{2}} \\
&= \|T\|_{\mathbf{ber}}\left(\|T\|_{\mathbf{ber}}^2 - \delta^2(T)\right)^{\frac{1}{2}} \qquad \text{for all } \tau\in\Omega,
\end{aligned}$$

so it follows that

$$\mathbf{ber}([T^*,T]) \leqslant \|T\|_{\mathbf{ber}}(\|T\|_{\mathbf{ber}}^2 - \delta^2(T))^{\frac{1}{2}}$$

as required. Now, we show that for operators $T \in \mathbb{B}(\mathscr{H}(\Omega))$ with the condition (22), the estimate (23) is better than (20). Indeed, $x = \Delta^2(T)$. It is easy to see that the condition (22) implies that

$$x^2 - 2\alpha^2 x + \alpha^2\delta^2(T) \geqslant 0,$$

that is

$$(\Delta^2(T))^2 - 2\alpha^2\Delta^2(T) + \alpha^2\delta^2(T) \geqslant 0.$$

Whence

$$\alpha^2 \leqslant \frac{\Delta^4(T)}{2\Delta^2(T) - \delta^2(T)}$$

or equivalently

$$\|T\|_{\mathbf{ber}} \leqslant \left(\frac{\Delta^4(T)}{2\Delta^2(T) - \delta^2(T)} \right)^{\frac{1}{2}}. \tag{24}$$

An elementary calculus shows that the estimate (24) is equivalent to the inequality

$$\|T\|_{\mathbf{ber}}(\|T\|_{\mathbf{ber}}^2 - \delta^2(T))^{\frac{1}{2}} \leqslant \|T\|_{\mathbf{ber}}^2 - \Delta^2(T^*),$$

which shows that (23) is better than (20). The theorem is proven. □

Notice that if $\Delta(T) = 0$, then the estimates (20) and (23) are the same, since $\Delta(T) \geqslant \delta(T)$. For example, if $\mathscr{H} = \mathscr{H}(\Omega)$ is a standard reproducing kernel Hilbert space, then for every compact operator K on $\mathscr{H}$, $\Delta(T) = 0$.

For another necessary conditions for Toeplitz operators the reader can consult in the works [11, 12, 13, 21, 29].

Declarations

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