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WEIGHTED NORM ESTIMATES FOR COMMUTATORS OF SEVERAL SQUARE FUNCTIONS

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Abstract. In this article, by using the atomic decomposition of weighted Herz-type Hardy spaces, the authors obtain some strong type estimates for commutators of several square functions generated by BMO functions on these spaces. All these results are also new even for classical Herz-type Hardy spaces.

1. Introduction

As is known to all, Hardy space $H^p(\mathbb{R}^n)$ is a good substitute of the Lebesgue space $L^p(\mathbb{R}^n)$ with $p \in (0,1]$ in the study for the boundedness of some operators. For example, when $p \in (0,1]$, it is well known that some of the singular integrals (for example, the Riesz transforms) are not bounded on $L^p(\mathbb{R}^n)$; however, they are bounded on $H^p(\mathbb{R}^n)$, when $p \in (0,1]$. As a generalization of Lebesgue spaces, Herz spaces are essential function spaces in harmonic analysis. In 1995, Lu et al. [13, 14] introduced the weighted Herz spaces and Herz-type Hardy spaces and gave their atomic decompositions. Soon afterwards, the boundedness of some classical operators is also established on these spaces, see [6, 7, 15, 17, 22]. More conclusions of the Hardy-type spaces are referred to [3, 12, 19, 21, 23, 28, 29].

On the other hand, in recently years, the study of the square function on Lebesgue spaces and Hardy-type spaces has attracted steadily increasing interest. For examples, in 2007, Wilson [27] introduced a class of new square functions and obtained that they are bounded on the weighted Lebesgue $L^p_\omega(\mathbb{R}^n)$ with $p \in (1,\infty)$ and the weight ω being in the Muckenhoupt class $\mathbb{A}_p(\mathbb{R}^n)$. In 2014, Wang [25] proved that these square functions are bounded from weighted Herz spaces to itself. In 2015, Wang [26] further discussed the boundedness of these square functions on weighted Herz-Hardy spaces. More conclusions of square functions are referred to [1, 2, 4, 16, 20, 24, 30].

Motivated by Wang [26] and Lu [14], it is a natural problem to ask whether commutators of these square functions are bounded from weighted Herz-type Hardy spaces to Herz spaces? In this paper we shall answer this problem affirmatively.

Precisely, this paper is organized as follows.

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In Section 2, we recall some notions concerning Muckenhoupt weights, weighted Herz spaces and Herz-type Hardy spaces. Then we establish some strong type estimates for commutators of sveral square functions from $H\dot{K}_q^{\alpha,p}(\omega_1, \omega_2)$ to $\dot{K}_q^{\alpha,p}(\omega_1, \omega_2)$, (see Theorems 2.4, 2.5 and 2.6 below). All these results are also new even for classical Herz-type Hardy spaces.

Section 3 is devoted to proving Theorems 2.4, 2.5 and 2.6. In the process of the proof of Theorems 2.4, 2.5 and 2.6, it is worth pointing out that, establishing a more subtle pointwise estimate for $[b, \mathscr{S}_\beta](a_j)$ plays an important role, where a_j is a central atom, (see Theorems 2.4, 2.5 and 2.6 below for more details).

Finally, we make some conventions on notation. Let $\mathbb{Z}_+ := \{1, 2, \dots\}$ and $\mathbb{N} := \{0\} \cup \mathbb{Z}_+$. For any $\beta := (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$, let $|\beta| := \beta_1 + \cdots + \beta_n$ and $\partial^\beta := (\frac{\partial}{\partial x_1})^{\beta_1} \cdots (\frac{\partial}{\partial x_n})^{\beta_n}$. Throughout this paper the letter C will denote a *positive constant* that may vary from line to line but will remain independent of the main variables. The *symbol* $P \lesssim Q$ stands for the inequality $P \leqslant CQ$. If $P \lesssim Q \lesssim P$, we then write $P \sim Q$. For any sets E, we use $|E|$ as its *n-dimensional Lebesgue measure*, χ_E as its *characteristic function*. If there are no special instructions, any space $\mathscr{X}(\mathbb{R}^n)$ is denoted simply by $\mathscr{X}$. For instance, $L^2(\mathbb{R}^n)$ is simply denoted by L^2. For any index $q \in [1, \infty]$, q' denotes the *conjugate index* of q, namely, $1/q + 1/q' = 1$. For any set E of $\mathbb{R}^n$, measurable function ω, let $\omega(E) := \int_E \omega(x)\, dx$. As usual we use B_k to denote the ball $\{x \in \mathbb{R}^n : |x| < 2^k\}$ with $k \in \mathbb{Z}$.

2. Preliminaries

In this section, we first recall some notations and definitions. For any $\beta \in (0, 1]$, let $\mathscr{C}_\beta$ be the family of functions φ defined on $\mathbb{R}^n$ such that supp $\varphi \subset \{x \in \mathbb{R}^n : |x| \leqslant 1\}$,

$$\int_{\mathbb{R}^n} \varphi(x)\, dx = 0,$$

and for all $x_1, x_2 \in \mathbb{R}^n$,

$$|\varphi(x_1) - \varphi(x_2)| \leqslant |x_1 - x_2|^\beta.$$

For all $f \in L^1_{\mathrm{loc}}$ and $(y,t) \in \mathbb{R}^{n+1}_+ := \mathbb{R}^n \times (0, \infty)$, let

$$A_\beta(f)(y,t) := \sup_{\varphi \in \mathscr{C}_\beta} |f * \varphi_t(y)|,$$

where, for all $t \in (0, \infty)$, $\varphi_t(\cdot) := \frac{1}{t^n}\varphi\left(\frac{\cdot}{t}\right)$. Then the intrinsic square function of f defined by setting, for any $x \in \mathbb{R}^n$,

$$\mathscr{S}_\beta(f)(x) := \left\{ \int\int_{\Gamma(x)} [A_\beta(f)(y,t)]^2 \frac{dydt}{t^{n+1}} \right\}^{1/2},$$

where $\Gamma(x) := \{(y,t) \in \mathbb{R}^{n+1}_+ : |y - x| < t\}$. Also, define varying-aperture versions of $\mathscr{S}_\beta(f)$ by the formula

$$\mathscr{S}_{\beta,\gamma}(f)(x) := \left\{ \int\int_{\Gamma_\gamma(x)} [A_\beta(f)(y,t)]^2 \frac{dy\,dt}{t^{n+1}} \right\}^{1/2},$$

where $\Gamma_\gamma(x) := \{(y,t) \in \mathbb{R}^{n+1}_+ : |y-x| < \gamma t\}$.

The intrinsic Littlewood-Paley $\mathscr{G}$-function and the intrinsic $\mathscr{G}^*_\lambda$-function of f defined by setting, for any $x \in \mathbb{R}^n$,

$$\mathscr{G}_\beta(f)(x) := \left\{ \int_0^\infty [A_\beta(f)(x,t)]^2 \frac{dt}{t} \right\}^{1/2}$$

and

$$\mathscr{G}^*_{\lambda,\beta}(f)(x) := \left\{ \int_0^\infty \int_{\mathbb{R}^n} \left(\frac{t}{t+|x-y|} \right)^{\lambda n} [A_\beta(f)(y,t)]^2 \frac{dy\,dt}{t^{n+1}} \right\}^{1/2}.$$

The following definition is from [5, 24]. Let b be a locally integrable function on $\mathbb{R}^n$. Define the commutators of intrinsic square functions generated by b

$$[b,\mathscr{S}_\beta](f)(x) := \left[\int\int_{\Gamma(x)} \sup_{\varphi \in \mathscr{C}_\beta} \left| \int_{\mathbb{R}^n} (b(x)-b(z))\varphi_t(y-z) f(z)\,dz \right|^2 \frac{dydt}{t^{n+1}} \right]^{1/2},$$

$$[b,\mathscr{G}_\beta](f)(x) := \left[\int_0^\infty \sup_{\varphi \in \mathscr{C}_\beta} \left| \int_{\mathbb{R}^n} (b(x)-b(y))\varphi_t(x-y) f(y)\,dy \right|^2 \frac{dt}{t} \right]^{1/2}$$

and

$$[b,\mathscr{G}^*_{\lambda,\beta}](f)(x)$$
$$:= \left[\int\int_{\mathbb{R}^{n+1}_+} \left(\frac{t}{t+|x-y|} \right)^{\lambda n} \sup_{\varphi \in \mathscr{C}_\beta} \left| \int_{\mathbb{R}^n} (b(x)-b(z))\varphi_t(y-z) f(z)\,dz \right|^2 \frac{dy\,dt}{t^{n+1}} \right]^{1/2}.$$

The classical $\mathbb{A}_p$ weight was first introduced by Muckenhoupt in the study of weighted L^p boundedness of Hardy-Littlewood maximal functions in [18]. Let ω be a nonnegative, locally integrable function defined on $\mathbb{R}^n$. All cubes are assumed to have their sides parallel to the coordinate axes. We say that $\omega \in \mathbb{A}_p$ with $1 < p < \infty$ if

$$\frac{1}{|B|} \int_B \omega(x)\,dx \left\{ \frac{1}{|B|} \int_B [\omega(x)]^{-\frac{1}{p-1}}\,dx \right\}^{p-1} \leqslant C \text{ for every cube } B \subset \mathbb{R}^n,$$

where C is a positive constant which is independent of the choice of B.

For the case $p = 1$, $\omega \in \mathbb{A}_1$, if

$$\frac{1}{|B|} \int_B \omega(x)\,dx \leqslant C \operatorname*{ess\,inf}_{x \in B} [\omega(x)]^{-1} \text{ for every cube } B \subset \mathbb{R}^n.$$

A weight function ω is said to belong to the reverse Hölder class $\mathbb{RH}_r$ if there exist two constants $r > 1$ and $C > 0$ such that the following reverse Hölder inequality holds

$$\left\{ \frac{1}{|B|} \int_B [\omega(x)]^r\,dx \right\}^{1/r} \leqslant C \frac{1}{|B|} \int_B \omega(x)\,dx \text{ for every cube } B \subset \mathbb{R}^n.$$

It is well known that if $\omega \in \mathbb{A}_p$ with $1 < p < \infty$, then $\omega \in \mathbb{A}_r$ for all $r > p$, and $\omega \in \mathbb{A}_q$ for some $1 < q < p$. Moreover, if $\omega \in \mathbb{A}_p$ with $1 \leqslant p < \infty$, then there exists $r > 1$ such that $\omega \in \mathbb{RH}_r$. For any given weight function ω on $\mathbb{R}^n$, and $q \in (0, \infty)$, we denote by L^q_ω the space of all functions f satisfying

$$\|f\|_{L^q_\omega} := \left(\int_{\mathbb{R}^n} |f(x)|^q \omega(x)\, dx \right)^{1/q}.$$

A locally integrable function b is said to be BMO if

$$\|b\|_{\mathrm{BMO}} := \sup_{B \subset \mathbb{R}^n} \frac{1}{|B|} \int_B |b(x) - b_B|\, dx < \infty,$$

where $b_B := \frac{1}{|B|} \int_B b(y)\, dy$.

Next we recall the definitions of the weighted Herz spaces and weighted Herz-type Hardy spaces. In what follows, we denote by $\mathscr{S}$ the *set of all Schwartz functions* and by $\mathscr{S}'$ its *dual space* (namely, the *set of all tempered distributions*). For any $f \in \mathscr{S}'$, then the grand maximal function of f is defined by

$$G_N(f) := \sup_{\varphi \in \mathscr{A}_N} \sup_{|y-x|<t} |\varphi_t * f(y)|,$$

where $\mathscr{A}_N := \{\varphi \in \mathscr{S} : \sup_{|\alpha|, |\beta| \leqslant N} |x^\alpha \partial^\beta \varphi(x)| < \infty\}$, $N \in \mathbb{N}$ is sufficiently large, and for any $t \in (0, \infty)$, $\varphi_t(\cdot) := t^{-n} \varphi(\frac{\cdot}{t})$.

In this paper, we denote $C_k = B_k \setminus B_{k-1}$ and denote briefly the characteristic function $\chi_{B_k \setminus B_{k-1}}$ by χ_k, where $B_k := \{x \in \mathbb{R}^n : |x| \leqslant 2^k\}$.

DEFINITION 2.1. [13] Let $\alpha \in \mathbb{R}$, $0 < q \leqslant \infty$, $0 < p < \infty$ and ω_1, ω_2 be two weight function on $\mathbb{R}^n$. The *homogeneous weighted Herz space* $\dot{K}^{\alpha,p}_q(\omega_1, \omega_2)$ and the *non-homogeneous weighted Herz space* $K^{\alpha,p}_q(\omega_1, \omega_2)$ are defined respectively by setting,

$$\dot{K}^{\alpha,p}_q(\omega_1, \omega_2) := \left\{ f \in L^q_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\}, \omega_2) : \|f\|_{\dot{K}^{\alpha,p}_q(\omega_1, \omega_2)} < \infty \right\}$$

and

$$K^{\alpha,p}_q(\omega_1, \omega_2) := \left\{ f \in L^q_{\mathrm{loc}}(\mathbb{R}^n, \omega_2) : \|f\|_{K^{\alpha,p}_q(\omega_1, \omega_2)} < \infty \right\}$$

where

$$\|f\|_{\dot{K}^{\alpha,p}_q(\omega_1, \omega_2)} = \left\{ \sum_{k \in \mathbb{Z}} [\omega_1(B_k)]^{\alpha p/n} \|f\chi_k\|^p_{L^q_{\omega_2}} \right\}^{1/p}$$

and

$$\|f\|_{K^{\alpha,p}_q(\omega_1, \omega_2)} = \left\{ \sum_{k \in \mathbb{Z}_+} [\omega_1(B_k)]^{\alpha p/n} \|f\chi_k\|^p_{L^q_{\omega_2}} \right\}^{1/p}$$

Here, there is the usual modification when $q = \infty$.

DEFINITION 2.2. [14] Let $\alpha \in \mathbb{R}$, $0 < q \leqslant \infty$, $0 < p < \infty$ and ω_1, ω_2 be two weight function on $\mathbb{R}^n$, $N \in \mathbb{N}$ be sufficiently large. The *homogeneous weghted Herz-type Hardy space* $H\dot{K}_q^{\alpha,p}(\omega_1, \omega_2)$ and the *non-homogeneous weighted Herz-type Hardy space* $HK_q^{\alpha,p}(\omega_1, \omega_2)$ are defined respectively by setting,

$$H\dot{K}_q^{\alpha,p}(\omega_1, \omega_2) := \left\{ f \in \mathscr{S}' : \|G_N(f)\| \in \dot{K}_q^{\alpha,p}(\omega_1, \omega_2) \right\}$$

and

$$HK_q^{\alpha,p}(\omega_1, \omega_2) := \left\{ f \in \mathscr{S}' : \|G_N(f)\| \in K_q^{\alpha,p}(\omega_1, \omega_2) \right\},$$

where

$$\|f\|_{H\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)} := \|G_N(f)\|_{\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)} \text{ and } \|f\|_{HK_q^{\alpha,p}(\omega_1,\omega_2)} := \|G_N(f)\|_{K_q^{\alpha,p}(\omega_1,\omega_2)}.$$

Here, there is the usual modification when $q = \infty$.

DEFINITION 2.3. [14] Let $1 < q < \infty$, $n(1-1/q) \leqslant \alpha < \infty$, s be a non-negative integer greater than or equal to $\alpha + n(1/q-1)$ and $b \in \mathrm{BMO}$.

(1) A real-valued function a is called a central $(\alpha, q, s; b)$-atom with respect to (ω_1, ω_2) if it satisfes

  (i) (support) $\operatorname{supp} a \subset B_r$, where $B_r = \{x \in \mathbb{R}^n : |x| \leqslant r\}$, $r > 0$;

  (ii) (size) $\|a\|_{L^q_{\omega_2}} \leqslant [\omega_1(B_r)]^{-\alpha/n}$;

  (iii) (vanishing moment) $\int_{\mathbb{R}^n} a(x)x^\beta dx = \int_{\mathbb{R}^n} a(x)b(x)x^\beta dx = 0$ for any $\beta \in \mathbb{Z}^n_+$ with $|\beta| \leqslant s$.

(2) A real-valued function $a(x)$ is called a central $(\alpha, q, s; b)$-atom of restricted type with respect to (ω_1, ω_2) if it satisfies

  (i) (support) $\operatorname{supp} a \subset B_r$, where $B_r = \{x \in \mathbb{R}^n : |x| \leqslant r\}$, $r > 1$;

  (ii) (size) $\|a\|_{L^q_{\omega_2}} \leqslant [\omega_1(B_r)]^{-\alpha/n}$;

  (iii) (vanishing moment) $\int_{\mathbb{R}^n} a(x)x^\beta dx = \int_{\mathbb{R}^n} a(x)b(x)x^\beta dx = 0$ for any $\beta \in \mathbb{Z}^n_+$ with $|\beta| \leqslant s$.

The main results of this paper are as follows.

THEOREM 2.4. *Let* $0 < p < \infty$, $1 < q < \infty$, $0 < \beta \leqslant 1$ *and* $n(1-1/q) \leqslant \alpha < n(1-1/q) + \beta$. *If* ω_1, $\omega_2 \in \mathbb{A}_1$ *and* $b \in \mathrm{BMO}$, *then* $\left[b, \mathscr{S}_\beta\right]$ *is bounded from* $H\dot{K}_q^{\alpha,p}(\omega_1, \omega_2)$ *to* $\dot{K}_q^{\alpha,p}(\omega_1, \omega_2)$. *Moreover, there exists a positive constant C such that, for any* $f \in H\dot{K}_q^{\alpha,p}(\omega_1, \omega_2)$,

$$\left\| \left[b, \mathscr{S}_\beta\right](f) \right\|_{\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)} \leqslant C\|b\|_{\mathrm{BMO}} \|f\|_{H\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)}.$$

THEOREM 2.5. *Let* $0 < p < \infty$, $1 < q < \infty$, $0 < \beta \leqslant 1$ *and* $n(1-1/q) \leqslant \alpha < n(1-1/q)+\beta$. *If* $\omega_1, \omega_2 \in \mathbb{A}_1$ *and* $b \in \mathrm{BMO}$, *then* $[b, \mathscr{G}_\beta]$ *is bounded from* $H\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)$ *to* $\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)$. *Moreover, there exists a positive constant* C *such that, for any* $f \in H\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)$,

$$\left\| [b, \mathscr{G}_\beta](f) \right\|_{\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)} \leqslant C\|b\|_{\mathrm{BMO}}\|f\|_{H\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)}.$$

THEOREM 2.6. *Let* $0<p<\infty$, $1<q<\infty$, $0<\beta\leqslant 1$, $n(1-1/q)\leqslant\alpha<n(1-1/q)+\beta$ *and* $\lambda > 3+2\beta/n$. *If* $\omega_1, \omega_2 \in \mathbb{A}_1$ *and* $b \in \mathrm{BMO}$, *then* $[b, \mathscr{G}^*_{\lambda,\beta}]$ *is bounded from* $H\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)$ *to* $\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)$. *Moreover, there exists a positive constant* C *such that, for any* $f \in H\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)$,

$$\left\| \left[b, \mathscr{G}^*_{\lambda,\beta} \right](f) \right\|_{\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)} \leqslant C\|b\|_{\mathrm{BMO}}\|f\|_{H\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)}.$$

REMARK 2.7. All these results for non-homogeneous weighted Herz-type Hardy spaces can also be proved by atomic decomposition theory. The arguments are similar, so the details are omitted here.

3. Proofs of Theorems 2.4, 2.5 and 2.6

To prove the main results, we need the following technical lemmas.

LEMMA 3.1. [14] *Let* $\omega_1, \omega_2 \in \mathbb{A}_1$, $0<p<\infty$, $1<q<\infty$ *and* $n-n/q \leqslant \alpha < \infty$, s *be a non-negative integer greater than or equal to* $\alpha + n(1/q-1)$ *and* $b \in \mathrm{BMO}$. *Then we have that* $f \in H\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)$ *if and only if*

$$f = \sum_{j\in\mathbb{Z}} \lambda_j a_j \ \text{ in } \mathscr{S}',$$

where each a_j *is a central* $(\alpha, q, s; b)$*-atom with support contained in* B_j *and*

$$\left(\sum_{j\in\mathbb{Z}} |\lambda_j|^p \right)^{1/p} < \infty.$$

Moreover,

$$\|f\|_{H\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)} \sim \inf \left(\sum_{j\in\mathbb{Z}} |\lambda_j|^p \right)^{1/p},$$

where the infimum is taken over all above decompositions of f.

LEMMA 3.2. [24] *Let* $0<\beta\leqslant 1$, $1<p<\infty$, $\omega \in \mathbb{A}_p$ *and* $b \in \mathrm{BMO}$. *Then the commutators* $[b, \mathscr{S}_\beta]$ *and* $[b, \mathscr{G}_\beta]$ *are bounded on* L^p_ω. *If* $\lambda > \max\{p, 3\}$, *then the* *commutator* $[b, \mathscr{G}^*_{\lambda,\beta}]$ *is also bounded on* L^p_ω. *Moreover, there exist constants* C_1, C_2 *and* C_3 *such that, for any* $f \in L^p_\omega$,

$$\big\|[b,\mathscr{S}_\beta](f)\big\|_{L^p_\omega} \leqslant C_1\|b\|_{\mathrm{BMO}}\|f\|_{L^p_\omega},$$

$$\big\|[b,\mathscr{G}_\beta](f)\big\|_{L^p_\omega} \leqslant C_1\|b\|_{\mathrm{BMO}}\|f\|_{L^p_\omega}$$

and

$$\Big\|[b,\mathscr{G}^*_{\lambda,\beta}](f)\Big\|_{L^p_\omega} \leqslant C_3\|b\|_{\mathrm{BMO}}\|f\|_{L^p_\omega}.$$

LEMMA 3.3. [18] *Let* $\omega \in \mathbb{A}_1 \cap \mathbb{RH}_r$ *with* $r > 1$. *Then there exist two positive constants* C_1 *and* C_2 *such that*

$$C_1\frac{|E|}{|F|} \leqslant \frac{\omega(E)}{\omega(F)} \leqslant C_2\left(\frac{|E|}{|F|}\right)^{(r-1)/r},$$

where for any measurable subsets $E, F \subset \mathbb{R}^n$ *satisfying* $E \subset F$.

The following lemma is from [9] or [8, 11].

LEMMA 3.4. (John-Nirenberg inequality) *Let* $b \in \mathrm{BMO}$. *Then for any ball* $B \subset \mathbb{R}^n$, *there exist positive constants* C_1 *and* C_2 *such that for all* $\lambda > 0$,

$$|\{x \in B : |b(x) - b_B| > \lambda\}| \leqslant C_1|B|\exp\Big(-\frac{C_2\lambda}{\|b\|_{\mathrm{BMO}}}\Big).$$

Proof of Theorem 2.4. According to Lemma 3.1, for any $f \in H\dot{K}^{\alpha,p}_q(\omega_1,\omega_2)$, we have the decomposition $f = \sum_{j\in\mathbb{Z}}\lambda_j a_j$, where $(\sum_{j\in\mathbb{Z}}|\lambda_j|^p)^{1/p} < \infty$ and a_j is a central $(\alpha, q, s; b;\ \omega_1, \omega_2)$-atom with supp $a_j \subset B_j$. To prove Theorem 2.4, we only need to show that

$$\big\|[b,\mathscr{S}_\beta](f)\big\|_{\dot{K}^{\alpha,p}_q(\omega_1,\omega_2)} \lesssim \|b\|_{\mathrm{BMO}}\left(\sum_{j\in\mathbb{Z}}|\lambda_j|^p\right)^{1/p}.$$

Now we prove Theorem 2.4 by two steps: $p \in (0, 1]$ and $p \in (1, \infty)$.

Step 1. If $p \in (0, 1]$, then

$$\begin{aligned}\big\|[b,\mathscr{S}_\beta](f)\big\|^p_{\dot{K}^{\alpha,p}_q(\omega_1,\omega_2)} &\lesssim \sum_{k\in\mathbb{Z}}[\omega_1(B_k)]^{\alpha p/n}\left(\sum_{j\in\mathbb{Z}}|\lambda_j|\|[b,\mathscr{S}_\beta](a_j)\chi_k\|_{L^q_{\omega_2}}\right)^p\\ &\lesssim \sum_{k\in\mathbb{Z}}[\omega_1(B_k)]^{\alpha p/n}\sum_{j\in\mathbb{Z}}|\lambda_j|^p\|[b,\mathscr{S}_\beta](a_j)\chi_k\|^p_{L^q_{\omega_2}}\\ &\sim \sum_{j\in\mathbb{Z}}|\lambda_j|^p\sum_{k\in\mathbb{Z}}[\omega_1(B_k)]^{\alpha p/n}\|[b,\mathscr{S}_\beta](a_j)\chi_k\|^p_{L^q_{\omega_2}}.\end{aligned}$$

Therefore, to complete the proof of *Step* 1, we only need to show that, for any central $(\alpha, q, s; b; \omega_1, \omega_2)$-atom a_j,

$$\sum_{k\in\mathbb{Z}}[\omega_1(B_k)]^{\alpha p/n}\|[b,\mathscr{S}_\beta](a_j)\chi_k\|^p_{L^q_{\omega_2}} \lesssim \|b\|^p_{\mathrm{BMO}}. \tag{3.1}$$

We can write

$$\begin{aligned}\sum_{k\in\mathbb{Z}}[\omega_1(B_k)]^{\alpha p/n}\|[b,\mathscr{S}_\beta](a_j)\chi_k\|^p_{L^q_{\omega_2}} &= \sum_{k=-\infty}^{j+1}[\omega_1(B_k)]^{\alpha p/n}\|[b,\mathscr{S}_\beta](a_j)\chi_k\|^p_{L^q_{\omega_2}}\\ &\quad+\sum_{k=j+2}^{\infty}[\omega_1(B_k)]^{\alpha p/n}\|[b,\mathscr{S}_\beta](a_j)\chi_k\|^p_{L^q_{\omega_2}}\\ &=: \mathrm{I}_1+\mathrm{I}_2.\end{aligned}$$

For I_1, by the fact that $\omega_2\in\mathbb{A}_1\subset\mathbb{A}_q$ with $1<q<\infty$, we know that $\omega\in\mathbb{RH}_r$ for some $r>1$. From this and the boundedness of $[b,\mathscr{S}_\beta]$ on $L^q_{\omega_2}$, the size condition of a_j, and Lemma 3.3, we have

$$\begin{aligned}\mathrm{I}_1 &= \sum_{k=-\infty}^{j+1}[\omega_1(B_k)]^{\alpha p/n}\|[b,\mathscr{S}_\beta](a_j)\|^p_{L^q_{\omega_2}}\\ &\lesssim \sum_{k=-\infty}^{j+1}[\omega_1(B_k)]^{\alpha p/n}\|a_j\|^p_{L^q_{\omega_2}}\|b\|^p_{\mathrm{BMO}}\\ &\lesssim \sum_{k=-\infty}^{j+1}\left[\frac{\omega_1(B_k)}{\omega_1(B_j)}\right]^{\alpha p/n}\|b\|^p_{\mathrm{BMO}}\\ &\lesssim \sum_{k=-\infty}^{j+1}\left[\frac{|B_k|}{|B_{j+1}|}\right]^{\alpha p(r-1)/nr}\|b\|^p_{\mathrm{BMO}}\\ &\sim \sum_{k=-\infty}^{j+1}2^{(k-j-1)\alpha p(r-1)/r}\|b\|^p_{\mathrm{BMO}}\lesssim\|b\|^p_{\mathrm{BMO}}.\end{aligned}$$

To deal with the term I_2, we have

$$\begin{aligned}\mathrm{I}_2 &= \sum_{k=j+2}^{\infty}[\omega_1(B_k)]^{\alpha p/n}\|[b,\mathscr{S}_\beta](a_j)\chi_k\|^p_{L^q_{\omega_2}}\\ &\leqslant C\sum_{k=j+2}^{\infty}[\omega_1(B_k)]^{\alpha p/n}\left(\int_{C_k}\left|\left(b(x)-b_{B_j}\right)\mathscr{S}_\beta\left(a_j\right)(x)\right|^q\omega_2(x)dx\right)^{p/q}\\ &\quad+\sum_{k=j+2}^{\infty}[\omega_1(B_k)]^{\alpha p/n}\left\{\int_{C_k}\left|\mathscr{S}_\beta\left[\left(b(\cdot)-b_{B_j}\right)a_j\right](x)\right|^q\omega_2(x)dx\right\}^{p/q}\\ &=: C(\mathrm{I}_{21}+\mathrm{I}_{22}).\end{aligned}$$

For I_{21}, let $\varphi \in \mathscr{C}_\beta$ with $0<\beta \leqslant 1$. From the vanishing moment condition of a_j, the size condition of a_j, Hölder's inequality and $\omega_2 \in \mathbb{A}_1$, we deduce that

$$\begin{aligned}
\left|a_j * \varphi_t(y)\right| &= \left|\int_{B_j} a_j(z)\left[\varphi_t(y-z)-\varphi_t(y)\right] dz\right| \\
&\lesssim \frac{1}{t^{n+\beta}} \int_{B_j}\left|a_j(z)\right||z|^\beta dz \\
&\lesssim \frac{2^{j\beta}}{t^{n+\beta}}\left\|a_j\right\|_{L_{\omega_2}^q}\left[\int_{B_j}\left(\omega_2(z)\right)^{-q'/q} dz\right]^{1/q'} \\
&\lesssim \frac{2^{j(\beta+n)}}{t^{n+\beta}}\left\|a_j\right\|_{L_{\omega_2}^q}\left[\omega_2\left(B_j\right)\right]^{-1/q} \\
&\lesssim \frac{2^{j(\beta+n)}}{t^{n+\beta}}\left[\omega_1\left(B_j\right)\right]^{-\alpha/n}\left[\omega_2\left(B_j\right)\right]^{-1/q}.
\end{aligned}$$

Thus, for any $(y, t) \in \mathbb{R}_+^{n+1}$, we have

$$A_\beta\left(a_j\right)(y, t)=\sup_{\varphi \in \mathscr{C}_\beta}\left|a_j * \varphi_t(y)\right| \lesssim \frac{2^{j(\beta+n)}}{t^{n+\beta}}\left[\omega_1\left(B_j\right)\right]^{-\alpha/n}\left[\omega_2\left(B_j\right)\right]^{-1/q}. \tag{3.2}$$

If $x \in C_k$ with $k \geqslant j+2$, and $z \in B_j$, then we have $|z| \leqslant \frac{|x|}{2}$. By the assumption condition of φ, we know that

$$2t>|x-y|+|y-z| \geqslant|x-z| \geqslant|x|-|z| \geqslant \frac{|x|}{4}.$$

From this and (3.2), we conclude that, for any $x \in C_k$ with $k \geqslant j+2$,

$$\begin{aligned}
\left|\mathscr{S}_\beta(a_j)(x)\right| &= \left\{\int\int_{\Gamma(x)}\left[A_\beta(f)(y, t)\right]^2 \frac{dydt}{t^{n+1}}\right\}^{1/2} \qquad (3.3)\\
&\lesssim 2^{j(\beta+n)}\left[\omega_1(B_j)\right]^{-\alpha/n}\left[\omega_2(B_j)\right]^{-1/q}\left(\int_{\frac{|x|}{8}}^{\infty} \int_{|y-x|<t} \frac{dydt}{t^{2(n+\beta)+n+1}}\right)^{1/2} \\
&\lesssim 2^{j(\beta+n)}\left[\omega_1(B_j)\right]^{-\alpha/n}\left[\omega_2(B_j)\right]^{-1/q} \frac{1}{|x|^{n+\beta}}.
\end{aligned}$$

Substituting the above inequality into the term I_{21}, we get

$$\begin{aligned}
\mathrm{I}_{21} &= \sum_{k=j+2}^{\infty}\left[\omega_1(B_k)\right]^{\alpha p/n}\left(\int_{C_k}\left|\left(b(x)-b_{B_j}\right) \mathscr{S}_\beta\left(a_j\right)(x)\right|^q \omega_2(x) dx\right)^{p/q} \\
&\lesssim 2^{pj(\beta+n)}\left[\omega_1(B_j)\right]^{-\alpha p/n}\left[\omega_2(B_j)\right]^{-p/q} \\
&\quad \times \sum_{k=j+2}^{\infty}\left[\omega_1(B_k)\right]^{\alpha p/n}\left(\int_{C_k} \frac{\left|b(x)-b_{B_j}\right|^q}{|x|^{q(n+\beta)}} \omega_2(x) dx\right)^{p/q} \\
&\sim 2^{pj(\beta+n)}\left[\omega_2(B_j)\right]^{-p/q} \sum_{k=j+2}^{\infty}\left[\frac{\omega_1(B_k)}{\omega_1(B_j)}\right]^{\alpha p/n}\left(\int_{C_k} \frac{\left|b(x)-b_{B_j}\right|^q}{|x|^{q(n+\beta)}} \omega_2(x) dx\right)^{p/q}.
\end{aligned}$$

For the last integral, by the John-Nirenberg inequality, it is easy to see that

$$
\begin{aligned}
&\left(\int_{C_k} \frac{\left|b(x)-b_{B_j}\right|^q}{|x|^{q(n+\beta)}}\omega_2(x)dx\right)^{p/q} \qquad (3.4)\\
&\leqslant \frac{1}{2^{kp(n+\beta)}}\left[\left(\int_{C_k}\left|b(x)-b_{B_k,\omega_2}\right|^q\omega_2(x)dx\right)^{p/q}\right.\\
&\qquad \left.+\left(\left|b_{B_k}-b_{B_k,\omega_2}\right|+\left|b_{B_k}-b_{B_j}\right|\right)^p\left(\int_{C_k}\omega_2(x)dx\right)^{p/q}\right]\\
&\lesssim \frac{1}{2^{kp(n+\beta)}}\left[\|b\|^p_{\mathrm{BMO}}\left(\int_{B_k}\omega_2(x)dx\right)^{p/q}+|j-k|^p\|b\|^p_{\mathrm{BMO}}\left(\int_{B_k}\omega_2(x)dx\right)^{p/q}\right]\\
&\lesssim \frac{1}{2^{kp(n+\beta)}}|j-k|^p\|b\|^p_{\mathrm{BMO}}[\omega_2(B_k)]^{p/q},
\end{aligned}
$$

where $b_{B_k,\omega_2} := \frac{1}{\omega_2(B_k)}\int_{B_k} b(x)\omega_2(x)\,dx$. By the above inequality, the fact that $\alpha < n+\beta-n/q$, $b\in\mathrm{BMO}$ and Lemma 3.3, we obtain

$$
\begin{aligned}
\mathrm{I}_{21} &\lesssim \sum_{k=j+2}^{\infty}\frac{2^{jp(n+\beta)}}{2^{kp(n+\beta)}}\left[\frac{\omega_1(B_k)}{\omega_1(B_j)}\right]^{\alpha p/n}\left[\frac{\omega_2(B_k)}{\omega_2(B_j)}\right]^{p/q}|j-k|^p\|b\|^p_{\mathrm{BMO}}\\
&\lesssim \sum_{k=j+2}^{\infty}2^{(j-k)p(n+\beta)}2^{(k-j)\alpha p}2^{(k-j)np/q}|j-k|^p\|b\|^p_{\mathrm{BMO}}\\
&\sim \sum_{k=j+2}^{\infty}2^{j[p(n+\beta)-\alpha p-np/q]}2^{-k[p(n+\beta)-\alpha p-np/q]}|j-k|^p\|b\|^p_{\mathrm{BMO}}\\
&\lesssim \|b\|^p_{\mathrm{BMO}}.
\end{aligned}
$$

To estimate the term I_{22}, since $\omega_2\in\mathbb{A}_1\subset\mathbb{A}_q$, by the property of $\mathbb{A}_q$ weight, we see that $\omega_2^{1-q'}\in\mathbb{A}_{q'}$. If $x\in C_k$ with $k\geqslant j+2$ and $z\in B_j$, then

$$
2t>|x-y|+|y-z|\geqslant|x-z|\geqslant|x|-|z|\geqslant\frac{|x|}{4}.
$$

Therefore, using the vanishing moment of a_j and a_jb, and the assumption condition φ, it follows that

$$
\begin{aligned}
&\mathscr{S}_\beta\left[\left(b(\cdot)-b_{B_j}\right)a_j\right](x) \qquad (3.5)\\
&=\left[\int\int_{\Gamma_x}\sup_{\varphi\in\mathscr{C}_\beta}\left|\int_{B_j}\left(b(z)-b_{B_j}\right)\varphi_t(y-z)a_j(z)dz\right|^2\frac{dy\,dt}{t^{n+1}}\right]^{\frac{1}{2}}\\
&\lesssim\left[\int_{\frac{|x|}{8}}^{\infty}\int_{|y-x|<t}\frac{2^{2j\beta}}{t^{2(n+\beta)}}\left(\int_{B_j}\left|b(z)-b_{B_j}\right||a_j(z)|dz\right)^2\frac{dy\,dt}{t^{n+1}}\right]^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&\sim 2^{j\beta}\left[\int_{\frac{|x|}{8}}^{\infty}\int_{|y-x|<t}\left(\int_{B_j}\left|b(z)-b_{B_j}\right||a_j(z)|dz\right)^2\frac{dy\,dt}{t^{2(n+\beta)+n+1}}\right]^{\frac{1}{2}}\\
&\sim 2^{j\beta}\left[\int_{\frac{|x|}{8}}^{\infty}\int_{|y-x|<t}\frac{dy\,dt}{t^{2(n+\beta)+n+1}}\right]^{\frac{1}{2}}\int_{B_j}\left|b(z)-b_{B_j}\right||a_j(z)|dz\\
&\lesssim \frac{2^{j\beta}}{|x|^{n+\beta}}\int_{B_j}\left|b(z)-b_{B_j}\right||a_j(z)|dz.
\end{aligned}
$$

From this, we further conclude that

$$
\begin{aligned}
\mathrm{I}_{22} &= \sum_{k=j+2}^{\infty}[\omega_1(B_k)]^{\alpha p/n}\left[\int_{C_k}\left|\mathscr{S}_\beta\left[\left(b(\cdot)-b_{B_j}\right)a_j\right](x)\right|^q\omega_2(x)dx\right]^{p/q} \qquad (3.6)\\
&\lesssim \sum_{k=j+2}^{\infty}[\omega_1(B_k)]^{\alpha p/n}\left[\int_{C_k}\frac{2^{qj\beta}}{|x|^{n+\beta}}\left(\int_{B_j}\left|b(z)-b_{B_j}\right||a_j(z)|dz\right)^q\omega_2(x)dx\right]^{p/q}\\
&\sim \sum_{k=j+2}^{\infty}[\omega_1(B_k)]^{\alpha p/n}\frac{2^{jp\beta}}{2^{kp(n+\beta)}}\left[\int_{C_k}\left(\int_{B_j}\left|b(z)-b_{B_j}\right||a_j(z)|dz\right)^q\omega_2(x)dx\right]^{p/q}\\
&\lesssim \sum_{k=j+2}^{\infty}[\omega_1(B_k)]^{\alpha p/n}\frac{2^{jp\beta}}{2^{kp(n+\beta)}}\left(\int_{B_j}\left|b(z)-b_{B_j}\right||a_j(z)|dz\right)^p\left(\int_{C_k}\omega_2(x)dx\right)^{p/q}.
\end{aligned}
$$

By Hölder's inequality, the John-Nirenberg inequality (see Lemma 3.4) and the size condition of a_j, we obtain

$$
\begin{aligned}
\left(\int_{B_j}\left|b(z)-b_{B_j}\right||a_j(z)|dz\right)^p &\leqslant \left(\int_{B_j}\left|b(z)-b_{B_j}\right|^{q'}[\omega_2(z)]^{1-q'}dz\right)^{p/q'}\|a_j\|_{L^q_{\omega_2}}^p\\
&\lesssim \|b\|_{\mathrm{BMO}}^p\left(\int_{B_j}[\omega_2(z)]^{1-q'}dz\right)^{p/q'}[\omega_1(B_j)]^{-\alpha p/n}.
\end{aligned}
$$

Substituting the above inequality into (3.6), by $\omega_2 \in \mathbb{A}_1 \subset \mathbb{A}_q$, the fact that $k \geqslant j+2$ and Lemma 3.3, we obtain

$$
\begin{aligned}
\mathrm{I}_{22} &\lesssim \sum_{k=j+2}^{\infty}\frac{2^{jp(n+\beta)}}{2^{kp(n+\beta)}}\left[\frac{\omega_1(B_k)}{\omega_1(B_j)}\right]^{\alpha p/n}\left[\frac{\omega_2(B_k)}{\omega_2(B_j)}\right]^{p/q}\|b\|_{\mathrm{BMO}}^p\\
&\lesssim \sum_{k=j+2}^{\infty}2^{(j-k)p(n+\beta)}2^{(k-j)\alpha p}2^{(k-j)np/q}\|b\|_{\mathrm{BMO}}^p\\
&\sim \|b\|_{\mathrm{BMO}}^p.
\end{aligned}
$$

Combining the estimates of I_1 and I_2, we obtain (3.1) and hence complete the proof of *Step* 1.

Step 2. If $p\in(1,\infty)$, then we write

$$\begin{aligned}\big\|[b,\mathscr{S}_\beta](f)\big\|^p_{\dot{K}^{\alpha,p}_q(\omega_1,\omega_2)} &\leqslant C\sum_{k\in\mathbb{Z}}[\omega_1(B_k)]^{\alpha p/n}\left(\sum_{j=k-1}^{\infty}|\lambda_j|\|[b,\mathscr{S}_\beta](a_j)\chi_k\|_{L^q_{\omega_2}}\right)^p\\ &\quad+C\sum_{k\in\mathbb{Z}}[\omega_1(B_k)]^{\alpha p/n}\left(\sum_{j=-\infty}^{k-2}|\lambda_j|\|[b,\mathscr{S}_\beta](a_j)\chi_k\|_{L^q_{\omega_2}}\right)^p\\ &=:\mathrm{J}_1+\mathrm{J}_2.\end{aligned}$$

For the term J_1, by the fact that $\omega_2\in\mathbb{A}_1\subset\mathbb{A}_q$ with $1<q<\infty$, the boundedness of $[b,\mathscr{S}_\beta]$ on $L^q_{\omega_2}$, and Hölder's inequality, we obtain

$$\begin{aligned}\mathrm{J}_1&\lesssim\sum_{k\in\mathbb{Z}}[\omega_1(B_k)]^{\alpha p/n}\left(\sum_{j=k-1}^{\infty}|\lambda_j|\|a_j\|_{L^q_{\omega_2}}\|b\|_{\mathrm{BMO}}\right)^p\\ &\lesssim\sum_{k\in\mathbb{Z}}[\omega_1(B_k)]^{\alpha p/n}\left\{\sum_{j=k-1}^{\infty}|\lambda_j|[\omega_1(B_j)]^{-\alpha/n}\|b\|_{\mathrm{BMO}}\right\}^p\\ &\lesssim\sum_{k\in\mathbb{Z}}[\omega_1(B_k)]^{\alpha p/n}\|b\|^p_{\mathrm{BMO}}\left\{\sum_{j=k-1}^{\infty}|\lambda_j|^p[\omega_1(B_j)]^{-\alpha p/2n}\right\}\left\{\sum_{j=k-1}^{\infty}[\omega_1(B_j)]^{-\alpha p'/2n}\right\}^{p/p'}.\end{aligned}$$

If $j\geqslant k-1$, then $B_{k-1}\subset B_j$. By the condition that $\omega_1\in\mathbb{A}_1$, we know that there exists $r>1$ such that $\omega_1\in\mathbb{RH}_r$. Therefore,

$$\begin{aligned}\left\{\sum_{j=k-1}^{\infty}[\omega_1(B_j)]^{-\alpha p'/2n}\right\}^{p/p'}&=\left\{[\omega_1(B_{k-1})]^{-\alpha p'/2n}\sum_{j=k-1}^{\infty}\left[\frac{\omega_1(B_{k-1})}{\omega_1(B_j)}\right]^{\alpha p'/2n}\right\}^{p/p'}\\ &\lesssim[\omega_1(B_{k-1})]^{-\alpha p/2n}.\end{aligned}$$

From this and Lemma 3.3, we further conclude that

$$\begin{aligned}\mathrm{J}_1&\lesssim\|b\|^p_{\mathrm{BMO}}\sum_{k\in\mathbb{Z}}[\omega_1(B_{k-1})]^{\alpha p/2n}\left\{\sum_{j=k-1}^{\infty}|\lambda_j|^p[\omega_1(B_j)]^{-\alpha p/2n}\right\}\\ &\sim\|b\|^p_{\mathrm{BMO}}\sum_{j\in\mathbb{Z}}|\lambda_j|^p\sum_{k=-\infty}^{j+1}[\omega_1(B_{k-1})]^{\alpha p/2n}[\omega_1(B_j)]^{-\alpha p/2n}\\ &\lesssim\|b\|^p_{\mathrm{BMO}}\sum_{j\in\mathbb{Z}}|\lambda_j|^p.\end{aligned}$$

Next we estimate the term J_2. It is easy to see that, for any $k\geqslant j+2$,

$$\begin{aligned}\|[b,\mathscr{S}_\beta](a_j)\chi_k\|_{L^q_{\omega_2}}&\leqslant C\left[\int_{C_k}\left|\left(b(x)-b_{B_j}\right)\mathscr{S}_\beta(a_j)(x)\right|^q\omega_2(x)dx\right]^{1/q}\\ &\quad+C\left\{\int_{C_k}\left|\mathscr{S}_\beta\left[\left(b(\cdot)-b_{B_j}\right)a_j\right](x)\right|^q\omega_2(x)dx\right\}^{1/q}\\ &=:\mathrm{J}_{21}+\mathrm{J}_{22}.\end{aligned}$$

To deal with J_{21}, for $x \in C_k$ with $k \geqslant j+2$, repeating the proof of (3.3), it follows that

$$\mathscr{S}_\beta(a_j)(x) \lesssim 2^{j(n+\beta)}[\omega_1(B_j)]^{-\alpha/n}[\omega_2(B_j)]^{-1/q}\frac{1}{|x|^{n+\beta}}. \tag{3.7}$$

From the above inequality, (3.4) and Lemma 3.3, we conclude that

$$\begin{aligned}
\mathrm{J}_{21} &\lesssim 2^{j(n+\beta)}[\omega_1(B_j)]^{-\alpha/n}[\omega_2(B_j)]^{-1/q}\left(\int_{C_k}\frac{|b(x)-b_{B_j}|^q}{|x|^{q(n+\beta)}}\omega_2(x)dx\right)^{1/q}\\
&\lesssim 2^{j(n+\beta)}[\omega_1(B_j)]^{-\alpha/n}[\omega_2(B_j)]^{-1/q}2^{-k(n+\beta)}|j-k|\|b\|_{\mathrm{BMO}}[\omega_2(B_k)]^{1/q}\\
&\sim 2^{(j-k)(n+\beta)}[\omega_1(B_j)]^{-\alpha/n}\left[\frac{\omega_2(B_k)}{\omega_2(B_j)}\right]^{1/q}|j-k|\|b\|_{\mathrm{BMO}}\\
&\lesssim 2^{(j-k)(n+\beta-n/q)}[\omega_1(B_j)]^{-\alpha/n}|j-k|\|b\|_{\mathrm{BMO}}.
\end{aligned}$$

By (3.5), Hölder's inequality and the size condition of a_j, we obtain

$$\begin{aligned}
\mathrm{J}_{22} &\lesssim \left[\int_{C_k}\left|\frac{2^{j\beta}}{|x|^{n+\beta}}\int_{B_j}\left|\left(b(z)-b_{B_j}\right)a_j(z)\right|dz\right|^q\omega_2(x)dx\right]^{1/q}\\
&\lesssim \frac{2^{j\beta}}{2^{k(n+\beta)}}\int_{B_j}\left|\left(b(z)-b_{B_j}\right)a_j(z)\right|dz\left(\int_{B_k}\omega_2(x)dx\right)^{1/q}\\
&\lesssim \frac{2^{j\beta}}{2^{k(n+\beta)}}\left\{\int_{B_j}\left|b(z)-b_{B_j}\right|^{q'}[\omega_2(z)]^{1-q'}dz\right\}^{1/q'}\|a_j\|_{L^q_{\omega_2}}[\omega_2(B_k)]^{1/q}\\
&\lesssim \frac{2^{j\beta}}{2^{k(n+\beta)}}\|b\|_{\mathrm{BMO}}\left\{\int_{B_j}[\omega_2(z)]^{1-q'}dz\right\}^{1/q'}[\omega_1(B_j)]^{-\alpha/n}[\omega_2(B_k)]^{1/q}\\
&\lesssim 2^{(j-k)(n+\beta-n/q)}[\omega_1(B_j)]^{-\alpha/n}\|b\|_{\mathrm{BMO}}.
\end{aligned}$$

Combined with the estimates of J_{21} and J_{22}, we have that, for any $k \geqslant j+2$,

$$\|[b,\mathscr{S}_\beta](a_j)\chi_k\|_{L^q_{\omega_2}} \lesssim 2^{(j-k)(n+\beta-n/q)}[\omega_1(B_j)]^{-\alpha/n}|j-k|\|b\|_{\mathrm{BMO}}.$$

From this, Hölder's inequality, Lemma 3.3 and the fact that $\alpha < n+\beta-n/q$, we deduce that

$$\begin{aligned}
\mathrm{J}_2 &\lesssim \sum_{k\in\mathbb{Z}}[\omega_1(B_k)]^{\alpha p/2n}\left\{\sum_{j=-\infty}^{k-2}|\lambda_j|2^{(j-k)(n+\beta-n/q)}[\omega_1(B_j)]^{-\alpha/n}|j-k|\|b\|_{\mathrm{BMO}}\right\}^p\\
&\lesssim \|b\|^p_{\mathrm{BMO}}\sum_{k\in\mathbb{Z}}[\omega_1(B_k)]^{\alpha p/2n}\left\{\sum_{j=-\infty}^{k-2}|\lambda_j|^p 2^{(j-k)(n+\beta-n/q)p/2}[\omega_1(B_j)]^{-\alpha p/2n}|j-k|^{p/2}\right\}\\
&\quad\times\left\{\sum_{j=-\infty}^{k-2}2^{(j-k)(n+\beta-n/q)p'/2}[\omega_1(B_j)]^{-\alpha p'/2n}[\omega_1(B_k)]^{\alpha p'/2n}|j-k|^{p'/2}\right\}^{p/p'}
\end{aligned}$$

$$\begin{aligned}
&\lesssim \|b\|_{\mathrm{BMO}}^p \sum_{k\in\mathbb{Z}} [\omega_1(B_k)]^{\alpha p/2n} \left\{ \sum_{j=-\infty}^{k-2} |\lambda_j|^p 2^{(j-k)(n+\beta-n/q)p/2} [\omega_1(B_j)]^{-\alpha p/2n} |j-k|^{p/2} \right\} \\
&\quad \times \left[\sum_{j=-\infty}^{k-2} 2^{(j-k)(n+\beta-n/q-\alpha)p'/2} |j-k|^{p'/2} \right] \\
&\sim \|b\|_{\mathrm{BMO}}^p \sum_{j=-\infty}^{\infty} |\lambda_j|^p \sum_{k=j+2}^{\infty} 2^{(j-k)(n+\beta-n/q-\alpha)p/2} |j-k|^{p/2} \\
&\lesssim \|b\|_{\mathrm{BMO}}^p \sum_{j\in\mathbb{Z}} |\lambda_j|^p.
\end{aligned}$$

Combined with the estimates of J_1 and J_2, we complete the proof of *Step* 2 and hence the Theorem 2.4. □

Proof of Theorem 2.5. From [27], we know that, for any $0<\beta\leqslant 1$ and $x\in\mathbb{R}^n$, the square functions $\mathscr{S}_\beta(f)(x)$ and $\mathscr{G}_\beta(f)(x)$ are pointwise comparable, with comparability constants depending only on β and n. Therefore, repeating the proof of Theorem 2.4, we obtain that, for any $f\in H\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)$,

$$\left\| [b,\mathscr{G}_\beta](f) \right\|_{\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)} \leqslant C\|b\|_{\mathrm{BMO}} \|f\|_{H\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)}.$$

This finishes the proof of Theorem 2.5. □

Proof of Theorem 2.6. By Lemma 3.1, we know that, for any $f\in H\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)$, there exists a decomposition $f=\sum_{j\in\mathbb{Z}}\lambda_j a_j$, where $(\sum_{j\in\mathbb{Z}}|\lambda_j|^p)^{1/p}<\infty$ and a_j is a central $(\alpha,q,s;b;\omega_1,\omega_2)$-atom with supp $a_j\subset B_j$. It's easy to see that, for any $x\in\mathbb{R}^n$, $b\in\mathrm{BMO}$,

$$\begin{aligned}
&[b,\mathscr{G}_{\lambda,\beta}^*](f)(x) \\
&=\left[\int\int_{\mathbb{R}_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \sup_{\varphi\in\mathscr{C}_\beta} \left| \int_{\mathbb{R}^n} (b(x)-b(z))\varphi_t(y-z)f(z)\,dz \right|^2 \frac{dy\,dt}{t^{n+1}} \right]^{1/2} \\
&\leqslant \left[\int_0^\infty \int_{|x-y|<t} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \sup_{\varphi\in\mathscr{C}_\beta} \left| \int_{\mathbb{R}^n} (b(x)-b(z))\varphi_t(y-z)f(z)\,dz \right|^2 \frac{dy\,dt}{t^{n+1}} \right]^{1/2} \\
&\quad + \sum_{i=1}^{\infty} 2^{-\lambda in/2} \left[\int_0^\infty \int_{2^{i-1}t\leqslant|x-y|<2^i t} \sup_{\varphi\in\mathscr{C}_\beta} \left| \int_{\mathbb{R}^n} (b(x)-b(z))\varphi_t(y-z)f(z)\,dz \right|^2 \frac{dy\,dt}{t^{n+1}} \right]^{1/2} \\
&\leqslant [b,\mathscr{S}_\beta](f)(x) + \sum_{i=1}^{\infty} 2^{-\lambda in/2} [b,\mathscr{S}_{\beta,2^i}](f)(x).
\end{aligned}$$

From this and a similar proof of Theorem 2.4, we easily deduce that, for any $f\in H\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)$,

$$\left\| \left[b,\mathscr{G}_{\lambda,\beta}^*\right](f) \right\|_{\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)} \lesssim \|f\|_{H\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)}.$$

To limit the length of this article, we leave the details to the interested readers. This finishes the proof of Theorem 2.6. □

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