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FURTHER PROPERTIES OF THE MINUS ORDER IN HILBERT SPACES

WENTING LIANG AND CHUNYUAN DENG*

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Abstract. In this paper, we present the geometric structures of minus order, demonstrating that the minus order is an extension of the star order. The properties of the common minus upper bound are investigated. Some equivalent characterizations of $(\mathcal{S},\mathcal{T})$-complementability are given, and several properties of the minus shorting operation are obtained.

1. Introduction and preliminaries

Let $\mathcal{H}$ and $\mathcal{K}$ be separable complex Hilbert spaces. We denote the set of all bounded linear operators from $\mathcal{H}$ into $\mathcal{K}$ by $\mathcal{B}(\mathcal{H},\mathcal{K})$ and by $\mathcal{B}(\mathcal{H})$ when $\mathcal{H}=\mathcal{K}$. For an operator A, $\mathcal{N}(A)$ and $\mathcal{R}(A)$ denote the null space and the range of A, respectively. $\overline{\mathcal{R}(A)}$ is the closure of $\mathcal{R}(A)$. An operator A is said to be positive, i.e., $A \geqslant 0$ if $\langle Ax,x\rangle$ is nonnegative for all $x \in \mathcal{H}$. The unique positive square root is denoted by $A^{\frac{1}{2}}$. An operator A is said to be selfadjoint if $A^* = A$ and is said to be idempotent if $A^2 = A$. The orthogonal projection onto a closed subspace $\mathcal{M} \subseteq \mathcal{H}$ is denoted by $P_{\mathcal{M}}$. Let $\mathcal{B}^{\mathcal{S}}(\mathcal{H})$, $\mathcal{B}^{+}(\mathcal{H})$, $\mathcal{Q}(\mathcal{H})$ and $\mathcal{P}(\mathcal{H})$ be sets of all selfadjoint operators, positive operators, idempotents and orthogonal projections on $\mathcal{H}$, respectively. For an operator $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$, if $\mathcal{N}(A) = \{0\}$ and $\overline{\mathcal{R}(A)} = \mathcal{K}$, we say that A is injective with dense range. If $U \in \mathcal{B}(\mathcal{H})$ satisfies $\|Ux\| = \|x\|$ for all $x \in \mathcal{N}(U)^{\perp}$, then U is called a partial isometry. The symbols "$\dotplus$" and "$\oplus$" denote algebra direct sum and orthogonal direct sum, respectively.

For A, $B \in \mathcal{B}^{\mathcal{S}}(\mathcal{H})$, $A \leqslant B$ means $\langle Ax,x\rangle \leqslant \langle Bx,x\rangle$ for all $x \in \mathcal{H}$. The relation "$\leqslant$" is the usual order on $\mathcal{B}^{\mathcal{S}}(\mathcal{H})$. For $A,B \in \mathcal{B}(\mathcal{H})$, $A \overset{*}{\leqslant} B$ means

$$A^*A = A^*B = B^*A \text{ and } AA^* = BA^* = AB^*.$$

The relation "$\overset{*}{\leqslant}$" is a star order on $\mathcal{B}(\mathcal{H})$ [11], which is a generalization of the usual order [7, 12]. The minus order $\overset{-}{\leqslant}$ is defined by Hartwig [13] and Nambooripad [18]. It is also called the rank subtractivity order in the finite matrix case because for matrices A and B of the same order $A \overset{-}{\leqslant} B$ if and only if $\operatorname{rank}(B-A) = \operatorname{rank}(B) - \operatorname{rank}(A)$ [12]. The

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[*] Corresponding author.

concept of minus order is extended to operators in infinite-dimensional Hilbert spaces by Antezana, Corach, Stonjanoff [3] and Šemrl [21]. In [9], a new characterization of the minus order for operators acting on infinite-dimensional Hilbert spaces is given in terms of the so-called range additivity properties. Namely, it is proved that $A\overset{-}{\leqslant}B$ if and only if $\mathcal{R}(B)=\mathcal{R}(A)\dot{+}\mathcal{R}(B-A)$ and $\mathcal{R}(B^*)=\mathcal{R}(A^*)\dot{+}\mathcal{R}(B^*-A^*)$, which generalizes previous results presented in the papers [12, 21]. In [17], Mosić and Cvetković-Ilić present some interesting properties of the diamond, (left, right) star and sharp orders. In [20], Rakić and Ljubenović study the star and the minus orders and extend some results from the matrix case to the case of general operators. The minus order is also present in papers [5, 6, 15, 16, 19]. As an application of the minus order, Antezana et al. define the shorting operation of bilateral complementary operators in $\mathcal{B}(\mathcal{H},\mathcal{K})$ [3]. Let $\mathcal{S}\subseteq\mathcal{H}$ and $\mathcal{T}\subseteq\mathcal{K}$ be closed subspaces. Define the set of operators $\mathcal{M}^-(A,\mathcal{S},\mathcal{T})$ by

$$\mathcal{M}^-(A,\mathcal{S},\mathcal{T}):=\{D\in\mathcal{B}(\mathcal{H},\mathcal{K}):D\overset{-}{\leqslant}A,\quad \mathcal{R}(D)\subseteq\mathcal{T},\quad \mathcal{R}(D^*)\subseteq\mathcal{S}\}. \tag{1}$$

In [3], the authors prove that the infimum of $\mathcal{M}^-(A,\mathcal{S},\mathcal{T})$ can be attained if $A\in\mathcal{B}(\mathcal{H},\mathcal{K})$ is complementable.

DEFINITION 1.1. [9, 12, 21] Let A, $B\in\mathcal{B}(\mathcal{H},\mathcal{K})$. We write $A\overset{-}{\leqslant}B$ if there exist idempotents $P\in\mathcal{Q}(\mathcal{H})$ and $Q\in\mathcal{Q}(\mathcal{K})$ satisfying $\mathcal{R}(P)=\overline{\mathcal{R}(A^*)}$ and $\mathcal{R}(Q)=\overline{\mathcal{R}(A)}$ such that $A=QB=BP^*$. The relation $\overset{-}{\leqslant}$ is called the minus order. In this case, we say $A\overset{-}{\leqslant}B$ with respect to P and Q.

For the usual order, Xu, Du and Fang [22] use the invariant subspaces of A and B to characterize the common infimum $A\wedge B$. As for the star order, Xu et al. [23] obtain some necessary and sufficient conditions for which the common supremum $A\overset{*}{\vee}B$ exists and present an explicit representation for $A\overset{*}{\vee}B$. As we know, $(\mathcal{B}(\mathcal{H}),\overset{-}{\leqslant})$ is a partially ordered set and not a lattice [3, Corlllary 4.14], [21, Corollary 3]. This leads to considering the natural questions: what are the conditions for the existence of upper and lower bounds with respect to the minus order and what are the relationships between the minus order and the star order? In this note, we mainly consider how to solve these problems. Some properties of the minus order and minus shorting operation are investigated. In Section 2, we present some geometric structures of minus order. In Section 3, the relationships between the minus order and the star order are given by using the technique of operator matrix decompositions. In Section 4, the common minus upper bound is given. In Section 5, as an application of the minus order, an equivalent characterization of $(\mathcal{S},\mathcal{T})$-complementability is obtained.

2. Some lemmas and basic properties

In this section, we consider the matrix structures of operators with respect to minus order. In order to give proofs of our results, we begin with some lemmas.

LEMMA 2.1. [10] *If A, $B\in\mathcal{B}(\mathcal{H})$, then the following statements are equivalent.*

(i) *$A=BC$ for some operator $C\in\mathcal{B}(\mathcal{H})$;*

(ii) $AA^* \leqslant kBB^*$ *for some* $k > 0$*;*

(iii) $\mathcal{R}(A) \subseteq \mathcal{R}(B)$.

If one of these conditions holds, then there is a unique solution $C_0 \in \mathcal{B}(\mathcal{H})$ *such that* $BC_0 = A$ *with* $\mathcal{R}(C_0) \subseteq \mathcal{N}(B)^{\perp}$ *and* $\mathcal{N}(C_0) = \mathcal{N}(A)$*. This solution is called the Douglas reduced solution of the operator equation* $BX = A$*.*

LEMMA 2.2. [14, Proposition 2.7] *Let* A*,* B *and* $C \in \mathcal{B}(\mathcal{H})$ *be such that* $\overline{\mathcal{R}(B)} = \overline{\mathcal{R}(C)}$*. Then* $\overline{\mathcal{R}(AB)} = \overline{\mathcal{R}(AC)}$*.*

The next lemma presents the relationship between an idempotent P and the orthogonal projection with the same range. Note that $(P^* + P - I)^2 = I + (P^* - P)^*(P^* - P) > 0$. Then $P^* + P - I$ is invertible. Define

$$W_P := (I - P^* - PP^*)(I - P - P^*)^{-1} = (I - P - P^*)^{-1}(I - P - P^*P). \tag{2}$$

Then, we have the following observations.

LEMMA 2.3. *Let* $P \in \mathcal{Q}(\mathcal{H})$ *and* W_P *be defined by* (2)*. Then the following statements hold.*

(i) W_P *is invertible.*

(ii) $P_{\mathcal{N}(P^*)} = W_P(I - P) = (I - P^*)W_P^*$.

(iii) $P_{\mathcal{R}(P)} = W_P P W_P^{-1}$.

(iv) *Let* $A \in \mathcal{B}(\mathcal{H})$ *and* $Q \in \mathcal{Q}(\mathcal{H})$ *be such that* $\mathcal{R}(P) = \overline{\mathcal{R}(A^*)}$ *and* $\mathcal{R}(Q) = \overline{\mathcal{R}(A)}$*. Then*

$$A = W_Q A W_P^*, \quad P^* = W_P^* P_{\overline{\mathcal{R}(A^*)}} (W_P^*)^{-1}, \quad Q = W_Q^{-1} P_{\overline{\mathcal{R}(A)}} W_Q.$$

Proof. (i) $\sim$ (iii) For every $P \in \mathcal{Q}(\mathcal{H})$, it holds that

$$P = \begin{pmatrix} I & P_0 \\ 0 & 0 \end{pmatrix}, \quad P_0 \in \mathcal{B}(\mathcal{N}(P^*), \mathcal{R}(P)).$$

By a direct calculation, one has

$$(P^* + P - I)^{-1} = \begin{pmatrix} I & P_0 \\ P_0^* & -I \end{pmatrix}^{-1} = \begin{pmatrix} I - P_0(I + P_0^*P_0)^{-1}P_0^* & P_0(I + P_0^*P_0)^{-1} \\ (I + P_0^*P_0)^{-1}P_0^* & -(I + P_0^*P_0)^{-1} \end{pmatrix}.$$

So,

$$\begin{aligned} P_{\mathcal{N}(P^*)} &= \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ P_0^* & -I \end{pmatrix} \begin{pmatrix} I - P_0(I + P_0^*P_0)^{-1}P_0^* & P_0(I + P_0^*P_0)^{-1} \\ (I + P_0^*P_0)^{-1}P_0^* & -(I + P_0^*P_0)^{-1} \end{pmatrix} \\ &= (I - P^*)(I - P - P^*)^{-1} = (I - P - P^*)^{-1}(I - P). \end{aligned}$$

Note that $P_{\mathcal{N}(P^*)}+P$ is invertible. One has

$$\begin{aligned} W_P &= (I-P^*-PP^*)(I-P-P^*)^{-1} \\ &= (I-P^*)(I-P-P^*)^{-1} - PP^*(I-P-P^*)^{-1} \\ &= (I-P^*)(I-P-P^*)^{-1} + P(I-P-P^*)(I-P-P^*)^{-1} \\ &= (I-P^*)(I-P-P^*)^{-1} + P \\ &= P_{\mathcal{N}(P^*)}+P \end{aligned}$$

is invertible and $W_P = P_{\mathcal{N}(P^*)}+P = \begin{pmatrix} I & P_0 \\ 0 & I \end{pmatrix}$. Hence,

$$P_{\mathcal{N}(P^*)} = P_{\mathcal{N}(P^*)}(I-P) = (W_P-P)(I-P) = W_P(I-P) = (I-P^*)W_P^*$$

and

$$W_PPW_P^{-1} = \begin{pmatrix} I & P_0 \\ 0 & I \end{pmatrix}\begin{pmatrix} I & P_0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} I & -P_0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} = P_{\mathcal{R}(P)}.$$

Items (i)$\sim$(iii) hold.

(iv) If $\mathcal{R}(P)=\overline{\mathcal{R}(A^*)}$ and $\mathcal{R}(Q)=\overline{\mathcal{R}(A)}$, then

$$W_Q^{-1}P_{\overline{\mathcal{R}(A)}}W_Q = \begin{pmatrix} I & -Q_0 \\ 0 & I \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} I & Q_0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & Q_0 \\ 0 & 0 \end{pmatrix} = Q.$$

Similarly, by a direct computation, we obtain

$$A = W_QAW_P^* \text{ and } P^* = W_P^*P_{\overline{\mathcal{R}(A^*)}}(W_P^*)^{-1}.$$

The proof is complete. $\square$

We first consider a special case that P and Q are orthogonal projections in Definition 1.1.

THEOREM 2.1. *For $A,\ B\in\mathcal{B}(\mathcal{H})$, if there exist $P,\ Q\in\mathcal{P}(\mathcal{H})$ satisfying $\mathcal{R}(P)=\overline{\mathcal{R}(A^*)}$ and $\mathcal{R}(Q)=\overline{\mathcal{R}(A)}$ such that $A=QB=BP$, then $B=A+P_{\mathcal{N}(A^*)}VP_{\mathcal{N}(A)}$ for some $V\in\mathcal{B}(\mathcal{H})$.*

Proof. One can write P and Q as operator matrix forms

$$P=\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{R}(P) \\ \mathcal{N}(P) \end{pmatrix}, \quad Q=\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{R}(Q) \\ \mathcal{N}(Q) \end{pmatrix},$$

respectively. Here, we write $\begin{pmatrix} \mathcal{R}(P) \\ \mathcal{N}(P) \end{pmatrix} \to \begin{pmatrix} \mathcal{R}(P) \\ \mathcal{N}(P) \end{pmatrix}$ as $\begin{pmatrix} \mathcal{R}(P) \\ \mathcal{N}(P) \end{pmatrix}$ for simplicity, and so is $\begin{pmatrix} \mathcal{R}(Q) \\ \mathcal{N}(Q) \end{pmatrix}$. If $A\in\mathcal{B}(\mathcal{H})$ with $\overline{\mathcal{R}(A^*)}=\mathcal{R}(P)$ and $\overline{\mathcal{R}(A)}=\mathcal{R}(Q)$, then A has the operator matrix form

$$A=\begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{R}(P) \\ \mathcal{N}(P) \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{R}(Q) \\ \mathcal{N}(Q) \end{pmatrix},$$

where $A_1 \in \mathcal{B}(\mathcal{R}(P),\mathcal{R}(Q))$ is injective with dense range. If $A = QB = BP$, then B has the corresponding matrix form

$$B = \begin{pmatrix} A_1 & 0 \\ 0 & * \end{pmatrix} = A + P_{\mathcal{N}(Q)}VP_{\mathcal{N}(P)} = A + P_{\mathcal{N}(A^*)}VP_{\mathcal{N}(A)}$$

for some $V \in \mathcal{B}(\mathcal{H})$. This completes the proof. ☐

REMARK 2.1. In [8, Corollary 2.1], Deng and Yu prove that $A \overset{*}{\leqslant} B$ if and only if $B = A + P_{\mathcal{N}(A^*)}VP_{\mathcal{N}(A)}$ for some $V \in \mathcal{B}(\mathcal{H})$. By Theorem 2.1, it follows that the minus order is an extension of star order. In particular, if there exist P, $Q \in \mathcal{P}(\mathcal{H})$ satisfying $\mathcal{R}(P) = \overline{\mathcal{R}(A^*)}$ and $\mathcal{R}(Q) = \overline{\mathcal{R}(A)}$ such that $A = QB = BP$, then $A\overline{\leqslant}B$ is reduced as $A \overset{*}{\leqslant} B$.

If $A\overline{\leqslant}B$, then there exist idempotents P and Q satisfying $\mathcal{R}(P) = \overline{\mathcal{R}(A^*)}$ and $\mathcal{R}(Q) = \overline{\mathcal{R}(A)}$ such that $A = QB = BP^*$ by Definition 1.1. Notice that $T^*P(T^*)^{-1}$ and SQS^{-1} are idempotents for arbitrary invertible operators S and T. By Lemma 2.2,

$$\mathcal{R}(T^*P(T^*)^{-1}) = \mathcal{R}(T^*P) = \overline{\mathcal{R}(T^*A^*)} = \overline{\mathcal{R}(T^*A^*S^*)} = \overline{\mathcal{R}((SAT)^*)},$$

$$\mathcal{R}(SQS^{-1}) = \mathcal{R}(SQ) = \overline{\mathcal{R}(SA)} = \overline{\mathcal{R}(SAT)}$$

and

$$SAT = (SQS^{-1})SBT = SBT(T^{-1}P^*T) = SBT\left[T^*P(T^*)^{-1}\right]^*.$$

Hence, $SAT\overline{\leqslant}SBT$.

The following observations follow immediately by Definition 1.1.

THEOREM 2.2. *Let $A,B \in \mathcal{B}(\mathcal{H})$. Then the following statements hold.*

(i) $A\overline{\leqslant}B \Longleftrightarrow A^*\overline{\leqslant}B^* \Longleftrightarrow SAT\overline{\leqslant}SBT$ *for arbitrary invertible operators* $S,T \in \mathcal{B}(\mathcal{H})$.

(ii) *If A is injective or surjective, then* $A\overline{\leqslant}B \Longleftrightarrow A = B$.

(iii) [3, Corollary 4.14] *If $A\overline{\leqslant}B$, then $\mathcal{R}(A) \subseteq \mathcal{R}(B)$ and $\mathcal{R}(A^*) \subseteq \mathcal{R}(B^*)$.*

(iv) *If $A\overline{\leqslant}B$ and $B\overline{\leqslant}C$, then $A\overline{\leqslant}C$.*

(v) *If $A\overline{\leqslant}B$, then $BX = BY$ (resp. $XB = YB$) implies that $AX = AY$ (resp. $XA = YA$) for some $X,Y \in \mathcal{B}(\mathcal{H})$.*

Proof. (i), (ii) and (v) are obvious by Definition 1.1. (iii) follows immediately by Lemma 2.1.

(iv) By Definition 1.1, if $A\overline{\leqslant}B$ and $B\overline{\leqslant}C$, then there exist idempotents P_i and Q_i ($i = 1,2$) satisfying

$$\mathcal{R}(P_1) = \overline{\mathcal{R}(A^*)}, \quad \mathcal{R}(P_2) = \overline{\mathcal{R}(B^*)}, \quad \mathcal{R}(Q_1) = \overline{\mathcal{R}(A)}, \quad \mathcal{R}(Q_2) = \overline{\mathcal{R}(B)}$$

such that

$$A = Q_1B = BP_1^*, \quad B = Q_2C = CP_2^*.$$

By item (iii) we know $\mathcal{R}(Q_1)\subseteq\mathcal{R}(Q_2)$ and $\mathcal{R}(P_1)\subseteq\mathcal{R}(P_2)$. Hence, Q_1Q_2 and P_1P_2 are idempotents satisfying $\mathcal{R}(P_1P_2)=\mathcal{R}(P_1)=\overline{\mathcal{R}(A^*)}$ and $\mathcal{R}(Q_1Q_2)=\mathcal{R}(Q_1)=\overline{\mathcal{R}(A)}$ such that $A=Q_1Q_2C=CP_2^*P_1^*$, i.e., $A\overset{-}{\leqslant}C$. $\square$

In the sequel, we present matrix structures of operators with respect to the minus order. Let $A\in\mathcal{B}(\mathcal{H},\mathcal{K})$ have the operator matrix representation

$$A=\begin{pmatrix}A_{11}&0\\0&0\end{pmatrix}:\quad\begin{pmatrix}\overline{\mathcal{R}(A^*)}\\\mathcal{N}(A)\end{pmatrix}\longrightarrow\begin{pmatrix}\overline{\mathcal{R}(A)}\\\mathcal{N}(A^*)\end{pmatrix},\tag{3}$$

where $A_{11}\in\mathcal{B}(\overline{\mathcal{R}(A^*)},\overline{\mathcal{R}(A)})$ is injective with dense range. Let $B\in\mathcal{B}(\mathcal{H},\mathcal{K})$, $P\in\mathcal{Q}(\mathcal{H})$ with $\mathcal{R}(P)=\overline{\mathcal{R}(A^*)}$ and $Q\in\mathcal{Q}(\mathcal{K})$ with $\mathcal{R}(Q)=\overline{\mathcal{R}(A)}$ have the corresponding operator matrix forms

$$B=\begin{pmatrix}B_{11}&B_{12}\\B_{21}&B_{22}\end{pmatrix}:\quad\begin{pmatrix}\overline{\mathcal{R}(A^*)}\\\mathcal{N}(A)\end{pmatrix}\longrightarrow\begin{pmatrix}\overline{\mathcal{R}(A)}\\\mathcal{N}(A^*)\end{pmatrix},\tag{4}$$

$$P=\begin{pmatrix}I&P_0\\0&0\end{pmatrix}:\quad\begin{pmatrix}\overline{\mathcal{R}(A^*)}\\\mathcal{N}(A)\end{pmatrix}\longrightarrow\begin{pmatrix}\overline{\mathcal{R}(A^*)}\\\mathcal{N}(A)\end{pmatrix}\tag{5}$$

and

$$Q=\begin{pmatrix}I&Q_0\\0&0\end{pmatrix}:\quad\begin{pmatrix}\overline{\mathcal{R}(A)}\\\mathcal{N}(A^*)\end{pmatrix}\longrightarrow\begin{pmatrix}\overline{\mathcal{R}(A)}\\\mathcal{N}(A^*)\end{pmatrix},\tag{6}$$

respectively. Then we have the following observations.

THEOREM 2.3. *For given $A\in\mathcal{B}(\mathcal{H})$, there exists $B\in\mathcal{B}(\mathcal{H})$ such that $A\overset{-}{\leqslant}B$ if and only if there exist P and $Q\in\mathcal{Q}(\mathcal{H})$ satisfying $\mathcal{R}(P)=\overline{\mathcal{R}(A^*)}$ and $\mathcal{R}(Q)=\overline{\mathcal{R}(A)}$ such that*

$$B=A+(I-Q)V(I-P)^*\ \textit{for some}\ V\in\mathcal{B}(\mathcal{H}).$$

Proof. The sufficiency is obvious and it suffices to prove the converse. Assume that $A\overset{-}{\leqslant}B$. Then there exist $P,Q\in\mathcal{Q}(\mathcal{H})$ such that $A=QB=BP^*$ with $\mathcal{R}(P)=\overline{\mathcal{R}(A^*)}$ and $\mathcal{R}(Q)=\overline{\mathcal{R}(A)}$ by Definition 1.1. Let A,B,P and Q have the operator matrix forms as (3)$\sim$(6), respectively. From $A=QB$, one has

$$\begin{pmatrix}A_{11}&0\\0&0\end{pmatrix}=\begin{pmatrix}I&Q_0\\0&0\end{pmatrix}\begin{pmatrix}B_{11}&B_{12}\\B_{21}&B_{22}\end{pmatrix}\Longrightarrow\begin{cases}B_{11}+Q_0B_{21}=A_{11},\\B_{12}+Q_0B_{22}=0.\end{cases}$$

From $A=BP^*$, one obtains

$$\begin{pmatrix}A_{11}&0\\0&0\end{pmatrix}=\begin{pmatrix}B_{11}&B_{12}\\B_{21}&B_{22}\end{pmatrix}\begin{pmatrix}I&0\\P_0^*&0\end{pmatrix}\Longrightarrow\begin{cases}B_{11}+B_{12}P_0^*=A_{11},\\B_{21}+B_{22}P_0^*=0.\end{cases}$$

Then,

$$B_{11}=A_{11}+Q_0B_{22}P_0^*,\quad B_{12}=-Q_0B_{22},\quad B_{21}=-B_{22}P_0^*.$$

Hence,

$$\begin{aligned} B &= \begin{pmatrix} A_{11}+Q_0B_{22}P_0^* & -Q_0B_{22} \\ -B_{22}P_0^* & B_{22} \end{pmatrix} \\ &= \begin{pmatrix} A_{11} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} Q_0B_{22}P_0^* & -Q_0B_{22} \\ -B_{22}P_0^* & B_{22} \end{pmatrix} \\ &= A+(I-Q)V(I-P)^* \quad (\text{where } V \in \mathcal{B}(\mathcal{H})), \end{aligned} \tag{7}$$

which completes the proof. □

REMARK 2.2. Let $A,B \in \mathcal{B}(\mathcal{H})$. We have the following results.

(i) From the proof of Theorem 2.3, it is easy to see that if there exists $Q \in \mathcal{Q}(\mathcal{H})$ such that $\mathcal{R}(Q) = \overline{\mathcal{R}(A)}$ and $A = QB$, then $B = A + \underline{(I-Q)}V$ for some $V \in \mathcal{B}(\mathcal{H})$. Similarly, if there exists $P \in \mathcal{Q}(\mathcal{H})$ such that $\mathcal{R}(P) = \overline{\mathcal{R}(A^*)}$ and $A = BP^*$, then $B = A + W(I-P)^*$ for some $W \in \mathcal{B}(\mathcal{H})$.

(ii) Here several counterexamples are presented to illustrate the properties of minus order.

(a) $A\overline{\leqslant}B \nRightarrow A \overset{*}{\leqslant} B$. For example, put

$$A = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 2I & -I \\ -I & I \end{pmatrix}, \quad P = \begin{pmatrix} I & I \\ 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} I & I \\ 0 & 0 \end{pmatrix}.$$

Then $A\overline{\leqslant}B$ since $A = QB = BP^*$ with $\mathcal{R}(P) = \mathcal{R}(A^*)$ and $\mathcal{R}(Q) = \mathcal{R}(A)$. But $A^*A \neq A^*B$ and $AA^* \neq BA^*$.

(b) $A\overline{\leqslant}B$ and $B = B^* \nRightarrow A = A^*$. For example, put

$$A = \begin{pmatrix} I & I \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} I & I \\ I & -I \end{pmatrix}, \quad P = \frac{1}{2}\begin{pmatrix} I & I \\ I & I \end{pmatrix}, \quad Q = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}.$$

Then $A\overline{\leqslant}B$ since $A = QB = BP^*$ with $\mathcal{R}(P) = \mathcal{R}(A^*)$ and $\mathcal{R}(Q) = \mathcal{R}(A)$. But A is not selfadjoint.

(c) $A\overline{\leqslant}B \nRightarrow A^2\overline{\leqslant}B^2$. For example, put

$$A = \begin{pmatrix} I & I \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} I & I \\ 2I & -2I \end{pmatrix}, \quad P = \frac{1}{2}\begin{pmatrix} I & I \\ I & I \end{pmatrix}, \quad Q = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}.$$

Clearly, $A\overline{\leqslant}B$ since $A = QB = BP^*$ with $\mathcal{R}(P) = \mathcal{R}(A^*)$ and $\mathcal{R}(Q) = \mathcal{R}(A)$. But $A^2 = A$ and

$$B^2 = \begin{pmatrix} 3I & -I \\ -2I & 6I \end{pmatrix}.$$

It is easy to see that there is no idempotent Q' with $\mathcal{R}(Q') = \mathcal{R}(A^2)$ $(i.e., \mathcal{R}(Q') = \mathcal{R}(A))$ such that $A^2 = Q'B^2$.

(d) $A^2\overline{\leqslant}B^2 \nRightarrow A\overline{\leqslant}B$. For example, put

$$A=\begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}, \quad B=\begin{pmatrix} I & 0 \\ 0 & -2I \end{pmatrix}.$$

Clearly, $A^2=0\overline{\leqslant}B^2$ while there is no idempotent Q with $\mathcal{R}(Q)=\overline{\mathcal{R}(A)}$ such that $A=QB$.

As we know, if $T^n=T$ for $T\in\mathcal{B}(\mathcal{H})$ and $2\leqslant n\in\mathbb{N}$, then T is n-idempotent. The following result shows that the lower bound of an n-idempotent with respect to the minus order has a closed range. More precisely, we have the following theorem.

THEOREM 2.4. *Let A and $B\in\mathcal{B}(\mathcal{H})$ be such that $A\overline{\leqslant}B$. If B is n-idempotent $(2\leqslant n\in\mathbb{N})$, then $\mathcal{R}(A)$ is closed and AB^{n-2} is idempotent, and $AB^{n-1}=B^{n-1}A=A$.*

Proof. Since $A\overline{\leqslant}B$, there are P and $Q\in\mathcal{Q}(\mathcal{H})$ satisfying $\mathcal{R}(P)=\overline{\mathcal{R}(A^*)}$ and $\mathcal{R}(Q)=\overline{\mathcal{R}(A)}$ such that $A=QB=BP^*$ by Definition 1.1. Since $B^n=B$, one has

$$AB^{n-1}=QB^n=QB=A \text{ and } B^{n-1}A=B^nP^*=BP^*=A.$$

Note that

$$A=QA=Q(B^{n-1}A)=(QB)B^{n-2}A=AB^{n-2}A.$$

It follows that $(AB^{n-2})^2=AB^{n-2}AB^{n-2}=AB^{n-2}$. Hence, $\mathcal{R}(AB^{n-2})$ is closed. From $\mathcal{R}(A)=\mathcal{R}(AB^{n-2}A)\subseteq\mathcal{R}(AB^{n-2})\subseteq\mathcal{R}(A)$, we conclude that $\mathcal{R}(A)=\mathcal{R}(AB^{n-2})$ is closed. $\square$

Next we give a property of 2×2 operator matrices with respect to minus order.

THEOREM 2.5. *For a given 2×2 diagonal operator matrix $N=N_1\oplus 0\in\mathcal{B}(\mathcal{H}_1\oplus\mathcal{H}_2,\mathcal{K}_1\oplus\mathcal{K}_2)$, there exists*

$$M=\begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}\in\mathcal{B}(\mathcal{H}_1\oplus\mathcal{H}_2,\mathcal{K}_1\oplus\mathcal{K}_2)$$

such that $M\overline{\leqslant}N$ if and only if $M_i=0$, $i=2,3,4$ and $M_1\overline{\leqslant}N_1$. In addition, if $M\overline{\leqslant}N$ with respect to some $P\in\mathcal{Q}(\mathcal{H}_1\oplus\mathcal{H}_2)$ and $Q\in\mathcal{Q}(\mathcal{K}_1\oplus\mathcal{K}_2)$, then

$$M=NP^*+QN-QNP^*.$$

Proof. The sufficiency is obvious. We shall prove the necessity. If $M\overline{\leqslant}N$, then there exist $P\in\mathcal{Q}(\mathcal{H}_1\oplus\mathcal{H}_2)$ and $Q\in\mathcal{Q}(\mathcal{K}_1\oplus\mathcal{K}_2)$ such that $M=QN=NP^*$, $\mathcal{R}(P)=\overline{\mathcal{R}(M^*)}\subseteq\overline{\mathcal{R}(N^*)}\subseteq\mathcal{H}_1$ and $\mathcal{R}(Q)=\overline{\mathcal{R}(M)}\subseteq\overline{\mathcal{R}(N)}\subseteq\mathcal{K}_1$. It follows that $M_i=0$, $i=2,3,4$, $\mathcal{R}(P)=\overline{\mathcal{R}(M_1^*)}$ and $\mathcal{R}(Q)=\overline{\mathcal{R}(M_1)}$. Now, M, N, P and Q, as operators from

$$\mathcal{H}_1\oplus\mathcal{H}_2=\overline{\mathcal{R}(M_1^*)}\oplus[\mathcal{H}_1\ominus\overline{\mathcal{R}(M_1^*)}]\oplus\mathcal{H}_2$$

into

$$\mathcal{K}_1 \oplus \mathcal{K}_2 = \overline{\mathcal{R}(M_1)} \oplus [\mathcal{K}_1 \ominus \overline{\mathcal{R}(M_1)}] \oplus \mathcal{K}_2$$

have the matrix forms $M = M_{11} \oplus 0 \oplus 0$,

$$N = \begin{pmatrix} N_{11} & N_{12} & 0 \\ N_{21} & N_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} I & P_{12} & P_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} I & Q_{12} & Q_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

respectively, where $M_{11} \in \mathcal{B}(\mathcal{R}(P), \mathcal{R}(Q))$ is injective with dense range. The relations $M = QN = NP^*$ imply that

$$M_{11} = N_{11} + Q_{12}N_{21} = N_{11} + N_{12}P_{12}^*, \quad N_{12} = -Q_{12}N_{22}, \quad N_{21} = -N_{22}P_{12}^*.$$

Thus,

$$N = \begin{pmatrix} M_{11} + Q_{12}N_{22}P_{12}^* & -Q_{12}N_{22} & 0 \\ -N_{22}P_{12}^* & N_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} = M + (I-Q)N(I-P)^*.$$

So, $M = NP^* + QN - QNP^*$. By Theorem 2.3, we know

$$\begin{pmatrix} M_{11} & 0 \\ 0 & 0 \end{pmatrix} = M_1 \overset{-}{\leqslant} N_1 = \begin{pmatrix} M_{11} + Q_{12}N_{22}P_{12}^* & -Q_{12}N_{22} \\ -N_{22}P_{12}^* & N_{22} \end{pmatrix}$$

with respect to

$$P_1 := P|_{\mathcal{H}_1} = \begin{pmatrix} I & P_{12} \\ 0 & 0 \end{pmatrix}, \quad Q_1 := Q|_{\mathcal{K}_1} = \begin{pmatrix} I & Q_{12} \\ 0 & 0 \end{pmatrix},$$

which completes the proof. $\square$

3. Relationships between the minus order and the star order

In this section, we generalize several conclusions of star order to minus order. We show a connection between the minus order and the star order in Theorem 3.3, which also shows that the minus order is an extension of star order.

We first recall several known characterizations of star order. For operators $A, B \in \mathcal{B}(\mathcal{H})$, it has been proved that

$$AA^* = BA^* \Longleftrightarrow A = BP_{\overline{\mathcal{R}(A^*)}} \Longleftrightarrow A = BP \text{ for some } P \in \mathcal{P}(\mathcal{H}) \tag{8}$$

and

$$A^*A = A^*B \Longleftrightarrow A = P_{\overline{\mathcal{R}(A)}}B \Longleftrightarrow A = P'B \text{ for some } P' \in \mathcal{P}(\mathcal{H}) \tag{9}$$

in [4, Proposition 2.3]. Therefore, it holds that

$$A \overset{*}{\leqslant} B \Longleftrightarrow A = P_{\overline{\mathcal{R}(A)}}B = BP_{\overline{\mathcal{R}(A^*)}} = P_{\overline{\mathcal{R}(A)}}BP_{\overline{\mathcal{R}(A^*)}} \tag{10}$$

and

$$A \overset{*}{\leqslant} B \Longleftrightarrow A = PB = BP' = PBP' \text{ for some } P \in \mathcal{P}(\mathcal{H}) \text{ and some } P' \in \mathcal{P}(\mathcal{H}). \quad (11)$$

Both (10) and (11) are equivalent characterizations of the relation $A \overset{*}{\leqslant} B$ in terms of orthogonal projections. Here, we shall generalize (8) and (9) to the minus order.

THEOREM 3.1. *Let $A,\ B \in \mathcal{B}(\mathcal{H})$. The following statements hold.*

(i) *There exists some $Q \in \mathcal{Q}(\mathcal{H})$ such that $A = QB$ if and only if there exists $Q' \in \mathcal{Q}(\mathcal{H})$ with $\mathcal{R}(Q') = \overline{\mathcal{R}(A)}$ such that $A = Q'B$.*

(ii) *There exists some $P \in \mathcal{Q}(\mathcal{H})$ such that $A = BP^*$ if and only if there exists $P' \in \mathcal{Q}(\mathcal{H})$ with $\mathcal{R}(P') = \overline{\mathcal{R}(A^*)}$ such that $A = BP'^*$.*

Proof. (i) The sufficiency is obvious. We shall prove the necessity. Let $Q_0 \in \mathcal{Q}(\mathcal{H})$ be such that $\mathcal{R}(Q_0) = \overline{\mathcal{R}(A)}$. If $A = QB$, then $\overline{\mathcal{R}(A)} = \mathcal{R}(Q_0) \subseteq \mathcal{R}(Q)$ and $A = Q_0A = Q_0QB$. Put $Q' = Q_0Q$. Then

$$Q'^2 = Q_0QQ_0Q = Q_0^2Q = Q_0Q = Q'$$

and

$$\mathcal{R}(Q') = \mathcal{R}(Q_0Q) \subseteq \mathcal{R}(Q_0) = \mathcal{R}(Q_0QQ_0) \subseteq \mathcal{R}(Q_0Q) = \mathcal{R}(Q').$$

Therefore, we get that $\mathcal{R}(Q') = \overline{\mathcal{R}(A)}$ and $A = Q'B$.

(ii) By taking the adjoint and then applying item (i), one gets the result. □

Theorem 3.1 has been proved by Antezana et al. in [3, Proposition 4.13] by a different definition of the minus order introduced in [1, Definition 4.11]. According to Theorem 3.1, $A \overset{-}{\leqslant} B$ if and only if $A = QB = BP^*$ for some $P, Q \in \mathcal{Q}(\mathcal{H})$. Now we are in the position to present an equivalent characterization of the minus order in terms of invertible operators W_P and W_Q defined by (2).

THEOREM 3.2. *Let $A,\ B \in \mathcal{B}(\mathcal{H})$. Then $A \overset{-}{\leqslant} B$ if and only if there exist $P,\ Q \in \mathcal{Q}(\mathcal{H})$ satisfying $\mathcal{R}(P) = \overline{\mathcal{R}(A^*)}$ and $\mathcal{R}(Q) = \overline{\mathcal{R}(A)}$ such that*

$$A^*W_QA = A^*W_QB \text{ and } AW_P^*A^* = BW_P^*A^*, \quad (12)$$

where invertible operators W_P and W_Q are defined by (2).

Proof. Assume $A \overset{-}{\leqslant} B$. It follows from Definition 1.1 that there are P and $Q \in \mathcal{Q}(\mathcal{H})$ satisfying $\mathcal{R}(P) = \overline{\mathcal{R}(A^*)}$ and $\mathcal{R}(Q) = \overline{\mathcal{R}(A)}$ such that $A = BP^* = QB$. By Lemma 2.3 (ii), one has

$$B - A = B(I - P^*) = BP_{\mathcal{N}(P^*)}(W_P^*)^{-1}$$

and

$$B - A = (I - Q)B = W_Q^{-1}P_{\mathcal{N}(Q^*)}B.$$

Since $\mathcal{R}(P)=\overline{\mathcal{R}(A^*)}$ and $\mathcal{R}(Q)=\overline{\mathcal{R}(A)}$, we get

$$(B-A)W_P^*A^*=BP_{\mathcal{N}(P^*)}A^*=0 \quad\text{and}\quad A^*W_Q(B-A)=A^*P_{\mathcal{N}(Q^*)}B=0.$$

Therefore, we obtain

$$AW_P^*A^*=BW_P^*A^* \quad\text{and}\quad A^*W_QA=A^*W_QB.$$

Conversely, if there exist P and $Q\in\mathcal{Q}(\mathcal{H})$ such that $\mathcal{R}(P)=\overline{\mathcal{R}(A^*)}$ and $\mathcal{R}(Q)=\overline{\mathcal{R}(A)}$, then A, B, P and Q have operator matrix forms as (3) $\sim$ (6), respectively. The relation $AW_P^*A^*=BW_P^*A^*$ implies that

$$\begin{pmatrix} A_{11} & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix} I & 0\\ P_0^* & I\end{pmatrix}\begin{pmatrix} A_{11}^* & 0\\ 0 & 0\end{pmatrix}=\begin{pmatrix} B_{11} & B_{12}\\ B_{21} & B_{22}\end{pmatrix}\begin{pmatrix} I & 0\\ P_0^* & I\end{pmatrix}\begin{pmatrix} A_{11}^* & 0\\ 0 & 0\end{pmatrix}.$$

A direct computation shows $B_{11}+B_{12}P_0^*=A_{11}$ and $B_{21}+B_{22}P_0^*=0$. Therefore, we have $A=BP^*$. Analogously, it follows from $A^*W_QA=A^*W_QB$ that $A=QB$, which completes the proof. ☐

By Theorem 3.2, if $P,Q\in\mathcal{P}(\mathcal{H})$ are orthogonal projections, then $W_P=W_Q=I$. Clearly, the relations in (12) are reduced as $A^*A=A^*B$ and $AA^*=BA^*$, which equals to $A\overset{*}{\leqslant}B$. More precisely, the next theorem also presents a relationship between the minus order and star order. For simplification, we denote orthogonal projections by $P_A:=P_{\overline{\mathcal{R}(A)}}$ and $P_{A^*}:=P_{\overline{\mathcal{R}(A^*)}}$ in the sequel.

THEOREM 3.3. *Let $A,B\in\mathcal{B}(\mathcal{H})$. Then $A\overset{-}{\leqslant}B$ if and only if there exist P and $Q\in\mathcal{Q}(\mathcal{H})$ satisfying $\mathcal{R}(P)=\overline{\mathcal{R}(A^*)}$ and $\mathcal{R}(Q)=\overline{\mathcal{R}(A)}$ such that*

$$A\overset{*}{\leqslant}W_QBW_P^*,$$

where W_P and W_Q are defined by (2).

Proof. If $A\overset{-}{\leqslant}B$, by Definition 1.1, there exist P and $Q\in\mathcal{Q}(\mathcal{H})$ satisfying $\mathcal{R}(P)=\overline{\mathcal{R}(A^*)}$ and $\mathcal{R}(Q)=\overline{\mathcal{R}(A)}$ such that $A=QB=BP^*$. By Lemma 2.3 (iii), we have

$$W_QQW_Q^{-1}=P_{\mathcal{R}(Q)}=P_A \text{ and } W_PPW_P^{-1}=P_{\mathcal{R}(P)}=P_{A^*},$$

that is,

$$W_QQ=P_AW_Q \quad\text{and}\quad P^*W_P^*=W_P^*P_{A^*}.$$

Therefore,

$$W_QA=W_QQB=P_AW_QB \quad\text{and}\quad AW_P^*=BP^*W_P^*=BW_P^*P_{A^*},$$

which leads to

$$W_QAW_P^*=P_AW_QBW_P^* \text{ and } W_QAW_P^*=W_QBW_P^*P_{A^*}.$$

By Lemma 2.3 (iv),

$$A = W_Q A W_P^* = P_A W_Q B W_P^* = W_Q B W_P^* P_{A^*}.$$

Thus, $A \overset{*}{\leqslant} W_Q B W_P^*$ by (10).

For the converse, if there exist P and $Q \in \mathcal{Q}(\mathcal{H})$ satisfying $\mathcal{R}(P) = \overline{\mathcal{R}(A^*)}$ and $\mathcal{R}(Q) = \overline{\mathcal{R}(A)}$ such that $A \overset{*}{\leqslant} W_Q B W_P^*$, by Lemma 2.3 (iv) and (10),

$$A = W_Q A W_P^* = P_A W_Q B W_P^* = W_Q B W_P^* P_{A^*}.$$

Note that W_Q and W_P are invertible operators. Then it follows from the above equalities that

$$A = W_Q^{-1} P_A W_Q B = B W_P^* P_{A^*} (W_P^*)^{-1}.$$

Since $Q = W_Q^{-1} P_A W_Q$ and $P^* = W_P^* P_{A^*} (W_P^*)^{-1}$ by Lemma 2.3 (iv), we obtain $A = QB = BP^*$, i.e., $A \overset{-}{\leqslant} B$. $\square$

4. Common minus upper bound

Let A and $B \in \mathcal{B}(\mathcal{H})$. If there exists $C \in \mathcal{B}(\mathcal{H})$ such that $A \overset{-}{\leqslant} C$ and $B \overset{-}{\leqslant} C$, we say that A and B have a common minus upper bound. In this section, conditions in which bounded operators have a common minus upper bound are investigated. In order to simplify, we denote $Q_{\overline{\mathcal{R}(A)}}$ and $Q_{\overline{\mathcal{R}(A^*)}}$ by Q_A and Q_{A^*}, respectively, where $Q_{\overline{\mathcal{R}(A)}}$ and $Q_{\overline{\mathcal{R}(A^*)}}$ are idempotents with ranges $\overline{\mathcal{R}(A)}$ and $\overline{\mathcal{R}(A^*)}$, respectively.

THEOREM 4.1. *Let A, B and $C \in \mathcal{B}(\mathcal{H})$ be such that $A \overset{-}{\leqslant} B$ and $C \overset{-}{\leqslant} B$. Then $A \overset{-}{\leqslant} C$ if and only if $\mathcal{R}(A) \subseteq \mathcal{R}(C)$ and $\mathcal{R}(A^*) \subseteq \mathcal{R}(C^*)$.*

Proof. The necessity follows immediately by Definition 1.1. It suffices to prove the converse. Since $A \overset{-}{\leqslant} B$, there exist Q_A and $Q_{A^*} \in \mathcal{Q}(\mathcal{H})$ such that $A = Q_A B = B Q_{A^*}^*$. Similarly, $C \overset{-}{\leqslant} B$ implies that there are Q_C and $Q_{C^*} \in \mathcal{Q}(\mathcal{H})$ such that $C = Q_C B = B Q_{C^*}^*$. Hence,

$$\mathcal{R}(A^*) \subseteq \mathcal{R}(C^*) \Longrightarrow Q_A C = Q_A B Q_{C^*}^* = A Q_{C^*}^* = A$$

and

$$\mathcal{R}(A) \subseteq \mathcal{R}(C) \Longrightarrow Q_{A^*} C^* = Q_{A^*} B^* Q_C^* = A^* Q_C^* = A^*.$$

As a consequence, we obtain $A \overset{-}{\leqslant} C$. $\square$

It should be mentioned that a parallel result with regard to the star order has been proved in [4, Theorem 2.7]. The following result follows by Theorem 4.1.

COROLLARY 4.1. *Let $A, B \in \mathcal{B}(\mathcal{H})$ with $\mathcal{R}(A) = \mathcal{R}(B)$ and $\mathcal{R}(A^*) = \mathcal{R}(B^*)$. Then A and B have a common minus upper bound if and only if $A = B$.*

THEOREM 4.2. *Let* $A,B \in \mathcal{B}(\mathcal{H})$. *If* A *is injective or has a dense range, then* A *and* B *have a common minus upper bound if and only if* $B \overset{-}{\leqslant} A$. *In this case,* $A \triangledown B = A$, *where* $A \triangledown B$ *means the common minus supremum of* A *and* B.

Proof. The sufficiency is obvious and it suffices to prove the converse. Let $C \in \mathcal{B}(\mathcal{H})$ be such that $A \overset{-}{\leqslant} C$ and $B \overset{-}{\leqslant} C$. Since $A \overset{-}{\leqslant} C$, by Theorem 2.3, there exist $V \in \mathcal{B}(\mathcal{H})$, P and $Q \in \mathcal{Q}(\mathcal{H})$ with $\mathcal{R}(P) = \overline{\mathcal{R}(A^*)}$ and $\mathcal{R}(Q) = \overline{\mathcal{R}(A)}$ such that

$$C = A + (I - Q)V(I - P)^*.$$

If $\mathcal{N}(A^*) = \{0\}$ or $\mathcal{N}(A) = \{0\}$, then $Q = I$ or $P = I$. So, $C = A$, $B \overset{-}{\leqslant} A$ and $A \triangledown B = A$. $\square$

By Theorem 4.2 (resp. Theorem 2.2, item (ii)) one has the following corollary.

COROLLARY 4.2. *Let* A *and* $B \in \mathcal{B}(\mathcal{H})$. *If* A *and* B *are injective or have dense ranges, then* A *and* B *have a common minus upper bound if and only if* $A = B$.

THEOREM 4.3. *Let* $A \in \mathcal{B}(\mathcal{H})$. *If there exists* $C \in \mathcal{Q}(\mathcal{H})$ *such that* $A \overset{-}{\leqslant} C$ *and* $P_A \overset{-}{\leqslant} C$, *then there are* P, Q *and* $Q' \in \mathcal{Q}(\mathcal{H})$ *with* $\mathcal{R}(Q) = \mathcal{R}(P) = \mathcal{R}(Q') = \overline{\mathcal{R}(A)}$ *such that*

$$A = P_A + (Q - Q')C(P^* - I),$$

where P_A *denotes the orthogonal projection onto* $\overline{\mathcal{R}(A)}$.

Proof. By Definition 1.1, if $P_A \overset{-}{\leqslant} C$, then there exist P and $Q \in \mathcal{Q}(\mathcal{H})$ with $\mathcal{R}(P) = \mathcal{R}(Q) = \overline{\mathcal{R}(A)}$ such that $P_A = QC = CP^*$. Let P_A, P, Q and C, as operators from $\overline{\mathcal{R}(A)} \oplus \mathcal{N}(A^*)$ into $\overline{\mathcal{R}(A)} \oplus \mathcal{N}(A^*)$, have the operator matrix forms

$$P_A = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} I & P_0 \\ 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} I & Q_0 \\ 0 & 0 \end{pmatrix} \text{ and } C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix},$$

respectively. By (7) in the proof of Theorem 2.3,

$$C = \begin{pmatrix} I + Q_0C_{22}P_0^* & -Q_0C_{22} \\ -C_{22}P_0^* & C_{22} \end{pmatrix},$$

where $C_{22} \in \mathcal{B}(\mathcal{N}(A^*))$. Let

$$A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & 0 \end{pmatrix}, \quad A_{11} \in \mathcal{B}(\overline{\mathcal{R}(A)}), \quad A_{12} \in \mathcal{B}(\mathcal{N}(A^*), \overline{\mathcal{R}(A)}).$$

Since $A \overset{-}{\leqslant} C$, there exists $Q' \in \mathcal{Q}(\mathcal{H})$ with $\mathcal{R}(Q') = \overline{\mathcal{R}(A)}$ such that $A = Q'C$. Put $Q' = \begin{pmatrix} I & Q_0' \\ 0 & 0 \end{pmatrix}$. Then

$$\begin{pmatrix} A_{11} & A_{12} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I & Q_0' \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I + Q_0C_{22}P_0^* & -Q_0C_{22} \\ -C_{22}P_0^* & C_{22} \end{pmatrix},$$

which implies that

$$A_{11} = I + (Q_0 - Q_0')C_{22}P_0^* \ \text{ and } \ A_{12} = -(Q_0 - Q_0')C_{22}.$$

Therefore,

$$\begin{aligned} A &= \begin{pmatrix} I + (Q_0 - Q_0')C_{22}P_0^* & -(Q_0 - Q_0')C_{22} \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} (Q_0 - Q_0')C_{22}P_0^* & -(Q_0 - Q_0')C_{22} \\ 0 & 0 \end{pmatrix} \\ &= P_A + \begin{pmatrix} 0 & Q_0 - Q_0' \\ 0 & 0 \end{pmatrix} \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ P_0^* & -I \end{pmatrix} \\ &= P_A + (Q - Q')C(P^* - I), \end{aligned}$$

which completes the proof. □

Similar to Theorem 4.3, we have the following result.

THEOREM 4.4. *Let* $A \in \mathcal{B}(\mathcal{H})$. *If there exists* $C \in \mathcal{B}(\mathcal{H})$ *such that* $A \overline{\leqslant} C$ *and* $P_{A^*} \overline{\leqslant} C$, *then there exist* P, Q *and* $P' \in \mathcal{Q}(\mathcal{H})$ *with* $\mathcal{R}(P) = \mathcal{R}(P') = \mathcal{R}(Q) = \overline{\mathcal{R}(A^*)}$ *such that*

$$A = P_{A^*} + (Q - I)C(P - P')^*,$$

where P_{A^*} *denotes the orthogonal projection onto* $\overline{\mathcal{R}(A^*)}$.

Proof. Note that $A \overline{\leqslant} C$ and $P_{A^*} \overline{\leqslant} C$ if and only if $A^* \overline{\leqslant} C^*$ and $P_{A^*} \overline{\leqslant} C^*$ by Theorem 2.2. It follows from Theorem 4.3 that there exist Q, P and $P' \in \mathcal{Q}(\mathcal{H})$ with $\mathcal{R}(P) = \mathcal{R}(Q) = \mathcal{R}(P') = \overline{\mathcal{R}(A^*)}$ such that

$$A^* = P_{A^*} + (P - P')C^*(Q^* - I).$$

Hence, $A = P_{A^*} + (Q - I)C(P - P')^*$. □

5. Applying to $(\mathcal{S}, \mathcal{T})$-complementable operators

As an application of the usual order for positive operators, Anderson, Duffin and Trapp show how the positive operators can be expressed as a sum of operators whose ranges are restricted by given subspaces [2, Theorem 2]. Shorting operation for positive operators is introduced in [2]. Later, the conception of shorted operator is extended to $(\mathcal{S}, \mathcal{T})$-weakly complementable operators by Antezana, Corach and Stojanoff [3], where $\mathcal{S}$ and $\mathcal{T}$ are two closed subspaces of $\mathcal{H}$.

An operator

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in \mathcal{B}(\mathcal{S} \oplus \mathcal{S}^\perp, \mathcal{T} \oplus \mathcal{T}^\perp) \tag{13}$$

is called $(\mathcal{S}, \mathcal{T})$-complementable if

$$\mathcal{R}(A_{21}) \subseteq \mathcal{R}(A_{22}) \quad \text{and} \quad \mathcal{R}(A_{12}^*) \subseteq \mathcal{R}(A_{22}^*).$$

If $A \in \mathcal{B}(\mathcal{H})$ is $(\mathcal{S},\mathcal{T})$-complementable, then

$$A_{/(\mathcal{S},\mathcal{T})} := \begin{pmatrix} A_{11} - F^*E & 0 \\ 0 & 0 \end{pmatrix} \tag{14}$$

is called the bilateral shorted operator of A with respect to subspaces $\mathcal{S}$ and $\mathcal{T}$, where E and F are reduced solutions of the operator equations

$$A_{21} = |A_{22}^*|^{1/2}UX \text{ and } A_{12}^* = |A_{22}|^{1/2}X, \tag{15}$$

respectively, and U is a partial isometry of the polar decomposition of A_{22} ($A_{22} = U|A_{22}|$).

Clearly, if $\mathcal{R}(A) \subseteq \mathcal{T}$ and $\mathcal{R}(A^*) \subseteq \mathcal{S}$, then $A_{/(\mathcal{S},\mathcal{T})}$ is $(\mathcal{S},\mathcal{T})$-complementable and $A = A_{/(\mathcal{S},\mathcal{T})}$. If A is $(\mathcal{S},\mathcal{T})$-complementable, then $A_{/(\mathcal{S},\mathcal{T})}$ is the maximal element of the operators set defined in (1) [3, Theorem 4.15]. For simplification, $Q_\mathcal{S}$ and $Q_\mathcal{T}$ denote idempotents with ranges $\mathcal{S}$ and $\mathcal{T}$, respectively. Firstly, we provide an equivalent characterization of $(\mathcal{S},\mathcal{T})$-complementable operators by using idempotents.

THEOREM 5.1. *Let $A \in \mathcal{B}(\mathcal{H})$. Then A is $(\mathcal{S},\mathcal{T})$-complementable if and only if there exist $Q_\mathcal{S}$ and $Q_\mathcal{T} \in \mathcal{Q}(\mathcal{H})$ such that $Q_\mathcal{T}A = AQ_\mathcal{S}^*$. In this case,*

$$A_{/(\mathcal{S},\mathcal{T})} = Q_\mathcal{T}A = AQ_\mathcal{S}^* = Q_\mathcal{T}AQ_\mathcal{S}^*.$$

Proof. Let A have the operator matrix form (13). A is $(\mathcal{S},\mathcal{T})$-complementable if and only if $\mathcal{R}(A_{12}^*) \subseteq \mathcal{R}(A_{22}^*)$ and $\mathcal{R}(A_{21}) \subseteq \mathcal{R}(A_{22})$ by the definition in (14) if and only if there exist $Q_0 \in \mathcal{B}(\mathcal{T}^\perp,\mathcal{T})$ and $Q_1 \in \mathcal{B}(\mathcal{S}^\perp,\mathcal{S})$ such that $A_{12} = Q_0A_{22}$ and $A_{21} = A_{22}Q_1^*$ by Lemma 2.1. Then

$$Q_0A_{21} = A_{12}Q_1^* = Q_0A_{22}Q_1^*.$$

Define operators $Q_\mathcal{T}$ and $Q_\mathcal{S}$ on $\mathcal{T} \oplus \mathcal{T}^\perp$ and $\mathcal{S} \oplus \mathcal{S}^\perp$, respectively, as follows

$$Q_\mathcal{T} = \begin{pmatrix} I & -Q_0 \\ 0 & 0 \end{pmatrix} \text{ and } Q_\mathcal{S} = \begin{pmatrix} I & -Q_1 \\ 0 & 0 \end{pmatrix}.$$

It holds that

$$Q_\mathcal{T}A = \begin{pmatrix} A_{11} - Q_0A_{21} & A_{12} - Q_0A_{22} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A_{11} - Q_0A_{22}Q_1^* & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$AQ_\mathcal{S}^* = \begin{pmatrix} A_{11} - A_{12}Q_1^* & 0 \\ A_{21} - A_{22}Q_1^* & 0 \end{pmatrix} = \begin{pmatrix} A_{11} - Q_0A_{22}Q_1^* & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence, A is $(\mathcal{S},\mathcal{T})$-complementable if and only if there exist $Q_\mathcal{S}$ and $Q_\mathcal{T} \in \mathcal{Q}(\mathcal{H})$ such that $Q_\mathcal{T}A = AQ_\mathcal{S}^*$. Therefore, it also holds that $Q_\mathcal{T}A = AQ_\mathcal{S}^* = Q_\mathcal{T}AQ_\mathcal{S}^*$.

Next, we show

$$A_{/(\mathcal{S},\mathcal{T})} = \begin{pmatrix} A_{11} - Q_0A_{22}Q_1^* & 0 \\ 0 & 0 \end{pmatrix}.$$

In fact, by the uniqueness of reduced solutions, we have $E = |A_{22}|^{1/2}Q_1^*$ and $F = |A_{22}|^{1/2}U^*Q_0^*$. Therefore, a straightforward computation shows that

$$F^*E = Q_0U|A_{22}|^{1/2}|A_{22}|^{1/2}Q_1^* = Q_0A_{22}Q_1^*.$$

Thus, we complete the proof. $\square$

Theorem 5.1 presents a relationship between $(\mathcal{S},\mathcal{T})$-complementable operator A and the shorted operator $A_{/(\mathcal{S},\mathcal{T})}$. We refer the reader to [3, Proposition 4.7] and [3, Proposition 4.13] for other proof of the following corollary.

COROLLARY 5.1. [3, Theorem 4.15] *Let $A \in \mathcal{B}(\mathcal{H})$. If A is $(\mathcal{S},\mathcal{T})$-complementable, then $A_{/(\mathcal{S},\mathcal{T})} \overset{-}{\leqslant} A$.*

The following corollary shows that the shorting operation preserves the minus order with respect to closed subspaces.

COROLLARY 5.2. *Let $A \in \mathcal{B}(\mathcal{H})$, $\mathcal{U}$ and $\mathcal{V} \subseteq \mathcal{H}$ be closed subspaces such that $\mathcal{S} \subseteq \mathcal{U}$ and $\mathcal{T} \subseteq \mathcal{V}$. If A is both $(\mathcal{U},\mathcal{V})$-complementable and $(\mathcal{S},\mathcal{T})$-complementable, then $A_{/(\mathcal{S},\mathcal{T})} \overset{-}{\leqslant} A_{/(\mathcal{U},\mathcal{V})}$.*

Proof. Assume that A is $(\mathcal{U},\mathcal{V})$-complementable. It follows from Theorem 5.1 that there exist $Q_{\mathcal{U}}$ and $Q_{\mathcal{V}} \in \mathcal{Q}(\mathcal{H})$ such that

$$A_{/(\mathcal{U},\mathcal{V})} = Q_{\mathcal{V}}A = AQ_{\mathcal{U}}^*.$$

Similarly, the assumption that A is $(\mathcal{S},\mathcal{T})$-complementable implies that there exist $Q_{\mathcal{S}}$ and $Q_{\mathcal{T}}$ such that

$$A_{/(\mathcal{S},\mathcal{T})} = Q_{\mathcal{T}}A = AQ_{\mathcal{S}}^*.$$

In addition, $\mathcal{R}(A^*_{/(\mathcal{S},\mathcal{T})}) \subseteq \mathcal{S} \subseteq \mathcal{U}$ implies

$$A_{/(\mathcal{S},\mathcal{T})} = A_{/(\mathcal{S},\mathcal{T})}Q_{\mathcal{U}}^* = Q_{\mathcal{T}}AQ_{\mathcal{U}}^* = Q_{\mathcal{T}}A_{/(\mathcal{U},\mathcal{V})}.$$

Analogously, $\mathcal{R}(A_{/(\mathcal{S},\mathcal{T})}) \subseteq \mathcal{T} \subseteq \mathcal{V}$ implies

$$A_{/(\mathcal{S},\mathcal{T})} = Q_{\mathcal{V}}A_{/(\mathcal{S},\mathcal{T})} = Q_{\mathcal{V}}AQ_{\mathcal{S}}^* = A_{/(\mathcal{U},\mathcal{V})}Q_{\mathcal{S}}^*.$$

By Theorem 3.1, we get $A_{/(\mathcal{S},\mathcal{T})} \overset{-}{\leqslant} A_{/(\mathcal{U},\mathcal{V})}$. $\square$

For a given operator $A \in \mathcal{B}(\mathcal{H})$ and closed subspaces $\mathcal{S}$, $\mathcal{T} \subseteq \mathcal{H}$, if A is $(\mathcal{S},\mathcal{T})$-complementable, it follows from the uniqueness of reduced solutions to operators equations (15) that $A_{/(\mathcal{S},\mathcal{T})}$ is unique. However, it is clear that operators $D \in \mathcal{B}(\mathcal{H})$ with conditions that

$$(1)\quad D \overset{-}{\leqslant} A, \quad (2)\quad \mathcal{R}(D) \subseteq \mathcal{T}, \quad (3)\quad \mathcal{R}(D^*) \subseteq \mathcal{S}$$

are non-unique. In addition, if D also satisfies

$$(4)\quad \mathcal{R}(A-D) \cap \mathcal{T} = \{0\}, \quad (5)\quad \mathcal{R}(A^*-D^*) \cap \mathcal{S} = \{0\},$$

then it is unique and $D = A_{/(\mathcal{S},\mathcal{T})}$. More accurate, we have the following theorem.

THEOREM 5.2. *Let $A \in \mathcal{B}(\mathcal{H})$ be $(\mathcal{S},\mathcal{T})$-complementable. There is unique operator $F := A_{/(\mathcal{S},\mathcal{T})}$ satisfying*

$$\mathcal{R}(F) \subseteq \mathcal{T}, \quad \mathcal{R}(F^*) \subseteq \mathcal{S}, \quad \mathcal{R}(A-F) \cap \mathcal{T} = \{0\}, \quad \mathcal{R}(A^*-F^*) \cap \mathcal{S} = \{0\} \tag{16}$$

such that $F \overset{-}{\leqslant} A$.

Proof. Assume that A is $(\mathcal{S},\mathcal{T})$-complementable. By (14) and Corollary 5.1, $\mathcal{R}(F) \subseteq \mathcal{T}$, $\mathcal{R}(F^*) \subseteq \mathcal{S}$ and $F \overset{-}{\leqslant} A$.

Next, put $G := A - F = A - A_{/(\mathcal{S},\mathcal{T})}$. We prove $\mathcal{R}(G) \cap \mathcal{T} = \{0\}$. In fact, for every $(w,0) \in \mathcal{R}(G) \cap \mathcal{T}$, there exist $x \in \mathcal{S}$ and $y \in \mathcal{S}^{\perp}$ such that

$$\begin{pmatrix} w \\ 0 \end{pmatrix} = \left(A - A_{/(\mathcal{S},\mathcal{T})}\right)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} F^*E & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \tag{17}$$

where E and F are reduced solutions to operators equations $A_{21} = |A_{22}^*|^{1/2}UX$ and $A_{12}^* = |A_{22}|^{1/2}X$, respectively. $U : \overline{\mathcal{R}(A_{22}^*)} \longrightarrow \overline{\mathcal{R}(A_{22})}$ is a partial isometry of the polar decomposition of A_{22}. It follows from (17) that

$$A_{21}x + A_{22}y = 0 \text{ and } F^*Ex + A_{12}y = w.$$

Substituting $A_{21} = |A_{22}^*|^{1/2}UE$ into the first equation above, it holds that

$$|A_{22}^*|^{1/2}UEx + U|A_{22}|y = U|A_{22}|^{1/2}Ex + U|A_{22}|y = U|A_{22}|^{1/2}(Ex + |A_{22}|^{1/2}y) = 0.$$

Put $z = Ex + |A_{22}|^{1/2}y$. Then $z \in \mathcal{N}(U|A_{22}|^{1/2}) = \mathcal{N}(|A_{22}|^{1/2})$. Since F is the reduced solution to operator equation $A_{12}^* = |A_{22}|^{1/2}X$, we obtain $\mathcal{R}(F) \subseteq \mathcal{N}(|A_{22}|^{1/2})^{\perp}$ by Lemma 2.1. Therefore, we get $z \in \mathcal{N}(|A_{22}|^{1/2}) \subseteq \mathcal{N}(F^*)$. Hence, it holds that

$$0 = F^*z = F^*(Ex + |A_{22}|^{1/2}y) = F^*Ex + F^*|A_{22}|^{1/2}y = F^*Ex + A_{12}y = w.$$

Thus, $\mathcal{R}(G) \cap \mathcal{T} = \{0\}$.

As for $\mathcal{R}(G^*) \cap \mathcal{S} = \{0\}$, let $(v,0) \in \mathcal{R}(G^*) \cap \mathcal{S}$, then there exist $x' \in \mathcal{S}$ and $y' \in \mathcal{S}^{\perp}$ such that

$$\begin{pmatrix} v \\ 0 \end{pmatrix} = \left(A - A_{/(\mathcal{S},\mathcal{T})}\right)^*\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} E^*F & A_{21}^* \\ A_{12}^* & A_{22}^* \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix}.$$

It follows from the above equality that

$$A_{12}^*x' + A_{22}^*y' = 0 \text{ and } E^*Fx' + A_{21}^*y' = v.$$

Since F is the reduced solution to operator equation $A_{12}^* = |A_{22}|^{1/2}X$ and U is a partial isometry of the polar decomposition of A_{22}, we have

$$|A_{22}^*|^{1/2}Fx' + A_{22}^*y' = |A_{22}|^{1/2}(Fx' + |A_{22}|^{1/2}U^*y') = 0.$$

Put $z' = Fx' + |A_{22}|^{1/2}U^*y'$. Then $z' \in \mathcal{N}(|A_{22}|^{1/2})$. Since E is the reduced solution to operator equation $A_{21} = |A_{22}^*|^{1/2}UX$, we get $\mathcal{R}(E) \subseteq \mathcal{N}(|A_{22}^*|^{1/2}U)^{\perp}$ by Lemma 2.1 again. Hence, $z' \in \mathcal{N}(|A_{22}|^{1/2}) = \mathcal{N}(U|A_{22}|^{1/2}) = \mathcal{N}(|A_{22}^*|^{1/2}U) \subseteq \mathcal{N}(E^*)$. Therefore,

$$0 = E^*z' = E^*(Fx' + |A_{22}|^{1/2}U^*y') = E^*Fx' + E^*|A_{22}|^{1/2}U^*y' = E^*Fx' + A_{21}^*y' = v,$$

which implies that $\mathcal{R}(G^*) \cap \mathcal{S} = \{0\}$.

For uniqueness, suppose that there exists $F_0 \in \mathcal{B}(\mathcal{H})$ with $F_0 \overset{-}{\leqslant} A$ and $G_0 = A - F_0$ satisfying (16). Let $G_0 = \left(A - A_{/(\mathcal{S},\mathcal{T})}\right) + \left(A_{/(\mathcal{S},\mathcal{T})} - F_0\right)$. It follows from the existence proved above that $\mathcal{R}\left(A - A_{/(\mathcal{S},\mathcal{T})}\right) \cap \mathcal{T} = \mathcal{R}(G) \cap \mathcal{T} = \{0\}$ and from hypothesis that $\mathcal{R}(G_0) \cap \mathcal{T} = \{0\}$. However, $\mathcal{R}\left(A_{/(\mathcal{S},\mathcal{T})} - F_0\right) \subseteq \mathcal{R}(A_{/(\mathcal{S},\mathcal{T})}) + \mathcal{R}(F_0) \subseteq \mathcal{T}$, which is a contradiction. Therefore, it holds that $A_{/(\mathcal{S},\mathcal{T})} - F_0 = 0$, i.e., $F_0 = A_{/(\mathcal{S},\mathcal{T})}$, which completes the proof. ☐

By Theorem 5.2 and [3, Theorem 4.15],

$$\begin{aligned} A_{/(\mathcal{S},\mathcal{T})} &= \max_{\overset{-}{\leqslant}} \mathcal{M}^-(A,\mathcal{S},\mathcal{T}) \\ &= \max_{\overset{-}{\leqslant}} \{D \in \mathcal{B}(\mathcal{H}) : D \overset{-}{\leqslant} A, \quad \mathcal{R}(D) \subseteq \mathcal{T}, \quad \mathcal{R}(D^*) \subseteq \mathcal{S}\} \\ &= \{F \in \mathcal{B}(\mathcal{H}) : F \overset{-}{\leqslant} A, \quad \mathcal{R}(F) \subseteq \mathcal{T}, \quad \mathcal{R}(F^*) \subseteq \mathcal{S}, \\ &\qquad \mathcal{R}(A - F) \cap \mathcal{T} = \{0\}, \quad \mathcal{R}(A^* - F^*) \cap \mathcal{S} = \{0\}\}, \end{aligned}$$

which presents a new characterization of the minus shorted operator $A_{/(\mathcal{S},\mathcal{T})}$. In addition, we give the following corollary.

COROLLARY 5.3. *Let A and $B \in \mathcal{B}(\mathcal{H})$ be such that $A \overset{-}{\leqslant} B$, $\mathcal{U}$ and $\mathcal{V} \subseteq \mathcal{H}$ be closed subspaces satisfying $\mathcal{S} \subseteq \mathcal{U}$ and $\mathcal{T} \subseteq \mathcal{V}$. If A is $(\mathcal{S},\mathcal{T})$-complementable and B is $(\mathcal{U},\mathcal{V})$-complementable, then $A_{/(\mathcal{S},\mathcal{T})} \overset{-}{\leqslant} B_{/(\mathcal{U},\mathcal{V})}$.*

Proof. By Corollary 5.1, $A_{/(\mathcal{S},\mathcal{T})} \overset{-}{\leqslant} A$. By Theorem 2.2, $A_{/(\mathcal{S},\mathcal{T})} \overset{-}{\leqslant} B$. Define the set of operators

$$\mathcal{M}^-(B,\mathcal{U},\mathcal{V}) := \{D \in \mathcal{B}(\mathcal{H}) : D \overset{-}{\leqslant} B, \quad \mathcal{R}(D) \subseteq \mathcal{V}, \quad \mathcal{R}(D^*) \subseteq \mathcal{U}\}.$$

Note that

$$\mathcal{R}\left(A^*_{/(\mathcal{S},\mathcal{T})}\right) \subseteq \mathcal{S} \subseteq \mathcal{U} \text{ and } \mathcal{R}\left(A_{/(\mathcal{S},\mathcal{T})}\right) \subseteq \mathcal{T} \subseteq \mathcal{V}.$$

We get $A_{/(\mathcal{S},\mathcal{T})} \in \mathcal{M}^-(B,\mathcal{U},\mathcal{V})$. By [3, Theorem 4.15], $A_{/(\mathcal{S},\mathcal{T})} \overset{-}{\leqslant} B_{/(\mathcal{U},\mathcal{V})}$. ☐

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Wenting Liang
School of Mathematics and Statistics
Taiyuan Normal University
Jinzhong 030619, China
e-mail: `1032173303@qq.com`

Chunyuan Deng
School of Mathematics Science
South China Normal University
Guangzhou 510631, China
e-mail: `cydeng@scnu.edu.cn`