

LOG-CONVEXITY OF GENERALIZED KANTOROVICH FUNCTION

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Abstract. We aim to derive some important properties about the generalized Kantorovich constant $K(h, p) := \frac{h^p - h}{(p-1)(h-1)} \left(\frac{p-1}{p} \frac{h^p - 1}{h^p - h} \right)^p$ for $h > 0$ and $p \in \mathbb{R}$. In particular, we point out that $K(h, p)$ is a log-convex function for p . As applications, we show the monotonicity of $K(h, p)^{\frac{1}{p}}$.

1. Introduction

Let $\mathbb{B}(\mathcal{H})$ denote the algebra of all bounded linear operators on a complex Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ with the identity operator $I_{\mathcal{H}}$, and let $\mathbb{B}^+(\mathcal{H})$ be the set of all positive operators in $\mathbb{B}(\mathcal{H})$. Let $P[\mathbb{B}(\mathcal{H}), \mathbb{B}(\mathcal{H})]$ denote a set of all normalized positive linear maps $\Phi: \mathbb{B}(\mathcal{H}) \rightarrow \mathbb{B}(\mathcal{H})$ such that $A \in \mathbb{B}^+(\mathcal{H}) \rightarrow \Phi(A) \in \mathbb{B}^+(\mathcal{H})$ with $\Phi(I_{\mathcal{H}}) = I_{\mathcal{H}}$.

Greub and Rheinboldt [11] gave the following Kantorovich operator inequality: If a positive operator A fulfills the condition $0 < mI_{\mathcal{H}} \leq A \leq MI_{\mathcal{H}}$ for some scalars $m \leq M$, then

$$\langle x, x \rangle \leq \langle Ax, x \rangle \langle A^{-1}x, x \rangle \leq \frac{(M+m)^2}{4Mm} \langle x, x \rangle \quad (1.1)$$

for all $x \in \mathcal{H}$. The constant $\frac{(M+m)^2}{4Mm}$ in (1.1) is called the *Kantorovich constant*. Kantorovich represented (1.1) as a sequence of positive real numbers in [13, p.142].

Mond and Pečarić [18] generalized (1.1) as follows: Let Φ be a normalized positive linear map in $P[\mathbb{B}(\mathcal{H}), \mathbb{B}(\mathcal{H})]$. If A is a positive operator on \mathcal{H} satisfying $0 < mI_{\mathcal{H}} \leq A \leq MI_{\mathcal{H}}$ for some scalars $m \leq M$, then

$$\Phi(A^{-1}) \leq \frac{(M+m)^2}{4Mm} \Phi(A)^{-1}, \quad (1.2)$$

also see [10].

On the other hand, Furuta gave complementary inequalities to the Hölder-McCarthy inequality as an extension of the Kantorovich type one as follows [5], [6], [7], [9], [10]

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and [2]: Let A be a positive operator on \mathcal{H} such that $0 < mI_{\mathcal{H}} \leq A \leq MI_{\mathcal{H}}$ for some scalars $m < M$. Let $h = \frac{M}{m}$. If $p \notin [0, 1]$, then

$$\langle Ax, x \rangle^p \leq \langle A^p x, x \rangle \leq K(h, p) \langle Ax, x \rangle^p \tag{1.3}$$

for every unit vector $x \in \mathcal{H}$, where the *generalized Kantorovich constant* $K(h, p)$ is defined by

$$K(h, p) := \frac{h^p - h}{(p - 1)(h - 1)} \left(\frac{p - 1}{p} \frac{h^p - 1}{h^p - h} \right)^p \quad \text{for all } p \in \mathbb{R} \tag{1.4}$$

with $K(h, 1) = 1$. (See [5] in detail.) If $p \in [0, 1]$, then the reverse inequality is valid in (1.3).

In addition, we cite the following ratio inequality ([10] and [17]), where the estimation is given by the generalized Kantorovich constant $K(h, p)$. This inequality is an extension of the result of C.-K.Li and R.Mathias [16] considered for the matrix case: Let Φ be a normalized positive linear map in $P[\mathbb{B}(\mathcal{H}), \mathbb{B}(\mathcal{H})]$. If A is a positive operator on \mathcal{H} satisfying $0 < mI_{\mathcal{H}} \leq A \leq MI_{\mathcal{H}}$ for some scalars $m < M$, then for any $p \in \mathbb{R} \setminus \{0\}$

$$\gamma_2 \Phi(A)^p \leq \Phi(A^p) \leq \gamma_1 \Phi(A)^p \tag{1.5}$$

where

$$\gamma_1 := \begin{cases} K(h, p) & \text{if } p > 1 \text{ or } p < 0, \\ 1 & \text{if } 0 < p \leq 1. \end{cases}$$

$$\gamma_2 := \begin{cases} K(h, p)^{-1} & \text{if } p > 2 \text{ or } p < -1, \\ 1 & \text{if } -1 \leq p < 0 \text{ or } 1 \leq p \leq 2, \\ K(h, p) & \text{if } 0 < p < 1, \end{cases}$$

The inequality (1.5) can be considered as an extension of inequalities (1.2) and (1.3) for power functions, and the generalized Kantorovich constant $K(h, p)$ plays an important role in estimations of these inequalities.

Here we remark that Nakamura [20] gave a simple proof of (1.1) by using a convexity of $f(t) = t^{-1}$. The inequality (1.3) is generalized to arbitrary general convex functions in [19]. The formulated method which is reduced to solving a single variable maximization or minimization problem by using the concavity of a real valued function is called the *Mond-Pečarić method* [10]. Using it we can derive several operator inequalities for the difference and ratio including inequalities (1.1)–(1.3) and (1.5), see [5] and [10].

In this paper, we show some meaningful properties of the generalized Kantorovich constant $K(h, p)$. One of them, the symmetric property $K(h, p) = K(h, 1 - p)$ given by Furuta [5], [9] is noteworthy. In connection with this, we give some properties of the *generalized Kantorovich function* $K(h, p)$ for $p \in \mathbb{R}$. In particular, we prove a basic and important fact that $K(h, p)$ is log-convex on p , and so convex. For this, an inequality due to Takahasi et al. [23] and some properties of the Klein inequality play an essential role. One of applications, we show that $K(h, p)^{\frac{1}{p}}$ is strictly increasing for $p \in \mathbb{R}$ which is an extension of [14, Lemma 3.8].

2. Some properties of the Klein function and its applications

The following inequality is known as the *Klein inequality* [15] and [21]: For a positive real number $c > 0$ with $c \neq 1$

$$\log c < c - 1 \quad (\text{or } c \log c > c - 1). \tag{2.1}$$

We define the *Klein function*: For a fixed $c > 0$

$$\mathcal{K}_c(t) := c^t \log c - (c - 1) \quad \text{for } t \geq 0. \tag{2.2}$$

It has the following properties:

THEOREM 2.1. *Let $c > 0$. Then the Klein function $\mathcal{K}_c(t)$ for $t \geq 0$ has the following properties:*

- (1) $\mathcal{K}_c(t)$ is a strictly increasing function for $t \geq 0$ with

$$\mathcal{K}_c(0) (= \log c - (c - 1)) \leq 0 \leq \mathcal{K}_c(1) (= c \log c - (c - 1)).$$

In the above inequalities, equalities hold only the case of $c = 1$.

- (2) The equation $\mathcal{K}_c(t) = 0$ has a unique solution $t = t_0$ $\begin{cases} \in (0, \frac{1}{2}) & \text{if } 0 < c < 1 \\ \in (\frac{1}{2}, 1) & \text{if } c > 1. \end{cases}$

- (3) The inequality $\mathcal{K}_c(1) + \mathcal{K}_c(0) (= c \log c + \log c - 2(c - 1)) > 0$ holds for $c > 1$. If $0 < c < 1$, then the reverse inequality holds. Consequently, the inequality $(\mathcal{K}_c(1) + \mathcal{K}_c(0)) \log c > 0$ holds for $c > 0$ with $c \neq 1$.

Proof. It is easy to see (1) from $\frac{d}{dt} \mathcal{K}_c(t) = c^t (\log c)^2 > 0$ and (2.1).

The property (2) is given by $\mathcal{K}_1(\frac{1}{2}) = 0$, $\frac{d}{dc} \mathcal{K}_c(\frac{1}{2}) = c^{-\frac{1}{2}} (\log c^{\frac{1}{2}} - (c^{\frac{1}{2}} - 1)) < 0$ and (1).

Since $\mathcal{K}_1(1) + \mathcal{K}_1(0) = 0$ and $\frac{d}{dc} (\mathcal{K}_c(1) + \mathcal{K}_c(0)) = \log c + \frac{1}{c} - 1 = (\frac{1}{c} - 1) - \log \frac{1}{c} > 0$, we have (3). \square

Next we recall the generalized logarithmic function $\ln_p(c) := \frac{c^p - 1}{p}$. S.-E. Takahasi et al [23] treat the following function related to the generalized logarithmic function for $M \geq m > 0$

$$\sigma_p(m, M) = \frac{p}{p-1} \frac{mM^p - Mm^p}{M^p - m^p} \left(= \frac{\ln_{p-1}(h)}{\ln_p(h)} M \right) \quad \text{for any real number } p (\neq 0, 1)$$

where $h = \frac{M}{m}$. They proved that the function $p \mapsto \sigma_p(m, M)$ is strictly monotone decreasing. In the following lemma, we give a simplified proof.

LEMMA 2.2. *Let c be a positive real number with $c \neq 1$. Then the following properties hold:*

(1) The following inequality holds for $p \notin (0, 1)$

$$(c - 1) \log c \leq \frac{c^p - 1}{p} \cdot \frac{c^{1-p} - 1}{1 - p} \left(= \frac{\ln_p(h)}{\ln_{1-p}(h)} M \right). \tag{2.3}$$

If $p \in [0, 1]$, then the reverse inequality of (2.3) holds. The equality is attained if and only if $p \rightarrow 0, 1$.

(2) The function $f(p) := \frac{1 - c^{-p}}{p} \cdot \frac{p - 1}{1 - c^{-(p-1)}} \left(= \frac{\ln_{-p}(h)}{\ln_{1-p}(h)} \right)$ is strictly monotone increasing for $p \in \mathbb{R}$.

(3) Let $c > 1$. Then the function $g(p) := \frac{p}{1 - c^{-p}} - \frac{p - 1}{1 - c^{-(p-1)}} \left(= \ln_{-p}(h)^{-1} - \ln_{1-p}(h)^{-1} \right)$ is strictly monotone increasing for $p \in \mathbb{R}$.

If $0 < c < 1$, then $g(p)$ is strictly monotone decreasing for $p \in \mathbb{R}$.

Proof. (1) We easily see that $\frac{c^p-1}{p} \cdot \frac{c^{1-p}-1}{1-p}$ converges $(c - 1) \log c$ for $p \rightarrow 0, 1$. So we may prove that for $p \in \mathbb{R} \setminus \{0, 1\}$, the function $f_0(x) := p(1 - p)(x - 1) \log x - (x^p - 1)(x^{1-p} - 1)$ is positive for any positive real number $x (x \neq 1)$. Then we have

$$\begin{aligned} \frac{d}{dx} f_0(x) &= p(1 - p)(\log x - x^{-1} + 1) - \{px^{p-1}(x^{1-p} - 1) + (1 - p)(x^p - 1)x^{-p}\} \\ &= p(1 - p)(\log x - x^{-1} + 1) + px^{p-1} + (1 - p)x^{-p} - 1, \end{aligned}$$

and moreover

$$\begin{aligned} \frac{d^2}{dx^2} f_0(x) &= p(1 - p)(x^{-1} + x^{-2} - x^{p-2} - x^{-p-1}) \\ &= p(1 - p)x^{-\frac{3}{2}} \left((x^{\frac{1}{2}} + x^{-\frac{1}{2}}) - (x^{p-\frac{1}{2}} + x^{-p+\frac{1}{2}}) \right) > 0, \end{aligned}$$

because

$$(x^{\frac{1}{2}} + x^{-\frac{1}{2}}) - (x^{p-\frac{1}{2}} + x^{-p+\frac{1}{2}}) \begin{cases} < 0 & \text{if } p \notin [0, 1] \\ > 0 & \text{if } p \in (0, 1). \end{cases}$$

Moreover, it follows from $\lim_{x \rightarrow 1} f_0(x) = \lim_{x \rightarrow 1} \frac{d}{dx} f_0(x) = \lim_{x \rightarrow 1} \frac{d^2}{dx^2} f_0(x) = 0$ that $f_0(x) \geq 0$. So the desired property (2.3) holds.

(2) Since $\lim_{p \rightarrow 0} f(p) = \frac{\log c}{c-1}$ and $\lim_{p \rightarrow 1} f(p) = \frac{c-1}{c \log c}$, the function $f(p)$ is continuous on \mathbb{R} . So we may show this property only for $p \in \mathbb{R} \setminus \{0, 1\}$.

It follows from (1) that

$$\begin{aligned} \frac{d}{dp} \log f(p) &= \frac{1}{(p - 1)p} + \frac{(1 - c^{1-p}) - (c - c^{1-p})}{(1 - c^{-p})(1 - c^{1-p})} c^{-p} \log c \\ &= \frac{1}{(c^p - 1)(c^{1-p} - 1)} \left(-\frac{(c^p - 1)(c^{1-p} - 1)}{p(1 - p)} + (c - 1) \log c \right) > 0. \end{aligned}$$

In the above last inequality, we remark that $(c^p - 1)(c^{1-p} - 1) \begin{cases} < 0 & (p \notin [0, 1]) \\ > 0 & (p \in (0, 1)). \end{cases}$

Moreover, we have $\frac{d}{dp}f(p) = f(p) \cdot \frac{d}{dp} \log f(p) > 0$ by $f(p) > 0$, and so the property (2) holds.

(3) Define a function $G(t)$ by

$$G(t) := \begin{cases} \frac{t \log t}{(t-1) \log c} & (t \neq 1) \\ (\log c)^{-1} & (t = 1) \end{cases} \quad \text{for } t > 0.$$

Then we have for $t \neq 1$

$$\frac{d}{dt}G(t) = \frac{(t-1) - \log t}{(t-1)^2 \log c}, \quad \text{and} \quad \frac{d^2}{dt^2}G(t) = \frac{1 - t^2 + 2t \log t}{t(t-1)^3 \log c}.$$

Here we put $t = c^p (> 0)$. Then it follows from $G(c^p) = \frac{p}{1-c^{-p}}$ that

$$g(p) = G(c^p) - G(c^{p-1}) \quad \text{and} \quad \frac{d}{dp}g(p) = \frac{d}{dp}G(c^p) - \frac{d}{dp}G(c^{p-1}).$$

Moreover we have $\frac{dt}{dp} = t \log c$ and so

$$\begin{aligned} \frac{d^2}{dp^2}G(c^p) &= \frac{d}{dt} \left(\frac{d}{dt}G(t) \cdot \frac{dt}{dp} \right) \cdot \frac{dt}{dp} \\ &= \frac{d^2}{dt^2}G(t) \cdot (t \log c)^2 + \frac{d}{dt}G(t) \cdot t(\log c)^2 \\ &= \frac{t \cdot (t \log t + \log t - 2(t-1)) \log t}{(t-1)^3 \cdot p}. \end{aligned}$$

Hence we have $\frac{d^2}{dp^2}G(c^p) \begin{cases} > 0 & (c > 1) \\ < 0 & (0 < c < 1) \end{cases}$ by $(t-1)^3 \cdot p \begin{cases} > 0 & (c > 1) \\ < 0 & (0 < c < 1) \end{cases}$ and

Theorem 2.1 (3). If $c > 1$ (resp. $0 < c < 1$), then $\frac{d}{dp}G(c^p)$ is an increasing function (resp. a decreasing function) for p . So the property (3) is given by

$$\frac{d}{dp}g(p) \begin{cases} > 0 & (c > 1) \\ < 0 & (0 < c < 1). \end{cases} \quad \square$$

3. Log-convexity of the generalized Kantorovich function

We recall several *properties* of the generalized Kantorovich function $K(h, p)$ as follows [3], [5], [8], [9] and [10]: Let $h > 0$ be given. Then

(K-1) $K(h, p) = K(\frac{1}{h}, p)$ for all $p \in \mathbb{R}$.

(K-2) $K(h, p) = K(h, 1-p)$ (i.e., $K(h, \frac{1}{2} + p) = K(h, \frac{1}{2} - p)$) for all $p \in \mathbb{R}$, that is, $K(h, p)$ is symmetric with respect to $p = \frac{1}{2}$.

(K-3) $K(h, 0) = K(h, 1) = 1$ and $K(1, p) = 1$ for all $p \in \mathbb{R}$, where $K(h, 0) = \lim_{p \rightarrow 0} K(h, p)$,
 $K(h, 1) = \lim_{p \rightarrow 1} K(h, p)$ and $K(1, p) = \lim_{h \rightarrow 1} K(h, p)$.

(K-4) $K(h, p)$ is increasing for $p > \frac{1}{2}$ and decreasing for $p < \frac{1}{2}$, and

$$\min_{p \in \mathbb{R}} K(h, p) = K\left(h, \frac{1}{2}\right) = \frac{2h^{1/4}}{h^{1/2} + 1} \in (0, 1].$$

(K-5) $K\left(h^r, \frac{p}{r}\right)^{\frac{1}{p}} = K\left(h^p, \frac{r}{p}\right)^{-\frac{1}{r}}$ for $rp \neq 0$.

In particular, if $r = 1$, then $K(h, p)^{\frac{1}{p}} = K\left(h^p, \frac{1}{p}\right)^{-1}$ for $p \neq 0$.

(K-6) $K(h, p) < h^{p-1}$ for all $h > 1$ and $p > 1$.

Here it follows from (K-2), (K-3) and (K-4) that $K(h, p) > 0$ for any $p \in \mathbb{R}$ and

$$K(h, p) \begin{cases} \geq 1 & \text{if } p \notin (0, 1) \\ < 1 & \text{if } p \in (0, 1) \end{cases} \quad (\text{e.g. [5], [9]}).$$

Moreover, we mention the following properties [12]:

(K-7) Let $h > 1$. If $p > 1$ (resp. $0 < p < 1$), then $K(h^t, p)^{\frac{1}{t}}$ is increasing (resp. decreasing) for $t > 0$ ([4]), and $1 < K(h^t, p)^{\frac{1}{t}} < h^{p-1}$ for all $t > 0$.

(K-8) $\lim_{t \rightarrow 0} K(h^t, p)^{\frac{1}{t}} = 1$ for all $p \in \mathbb{R}$.

Here we provide a proof of (K-8) for the sake of convenience:

Proof of (K-8). We may assume that $h > 1$ and $t \downarrow 0$ by (K-1). By L'Hospital's rule, we have

$$\lim_{t \downarrow 0} \frac{h^t - h^{t^p}}{h^t - 1} = \lim_{t \downarrow 0} \frac{h^t \log h - h^{t^p} \log h^p}{h^t \log h} = 1 - p \quad \text{and} \quad \lim_{t \downarrow 0} \frac{h^{t^p} - 1}{h^t - 1} = p \quad (3.1)$$

and moreover

$$\lim_{t \downarrow 0} \frac{d}{dt} \frac{h^t - h^{t^p}}{h^t - 1} = \frac{p(1-p)}{2} \log h \quad \text{and} \quad \lim_{t \downarrow 0} \frac{d}{dt} \frac{h^{t^p} - 1}{h^t - 1} = \frac{p(p-1)}{2} \log h. \quad (3.2)$$

As a result, applying L'Hospital's rule by (3.1) and moreover using (3.2), we obtain

$$\begin{aligned} \lim_{t \downarrow 0} \log K(h^t, p)^{\frac{1}{t}} &= \lim_{t \downarrow 0} \log \left\{ \frac{h^{tp} - h^t}{(p-1)(h^t - 1)} \left(\frac{p-1}{p} \frac{h^{tp} - 1}{h^t - 1} \right)^p \right\}^{\frac{1}{t}} \\ &= \lim_{t \downarrow 0} \log \left\{ \left(\frac{1}{1-p} \frac{h^t - h^{tp}}{h^t - 1} \right)^{\frac{1-p}{t}} \left(\frac{1}{p} \frac{h^{tp} - 1}{h^t - 1} \right)^{\frac{p}{t}} \right\} \\ &= \lim_{t \downarrow 0} \frac{1}{t} \left\{ (1-p) \log \left(\frac{1}{1-p} \frac{h^t - h^{tp}}{h^t - 1} \right) + p \log \left(\frac{1}{p} \frac{h^{tp} - 1}{h^t - 1} \right) \right\} \\ &= (1-p) \frac{\frac{p(1-p)}{2} \log h}{1-p} + p \frac{\frac{p(p-1)}{2} \log h}{p} \\ &= 0. \end{aligned}$$

In the above equality, we remark that $\frac{h^t - h^{tp}}{1-p}$ and $\frac{h^{tp} - 1}{p}$ are positive. Hence we have the desired equality (K-8). \square

The following result represents the relation of the convex function and its secant line (cf. [10, Corollary 2.10]):

(K-9) Let $0 < m < M$ with $h := \frac{M}{m} > 1$. For $p > 1$, the convex function t^p ($t > 0$) has a secant line $\alpha_p t + \beta_p$ at $t = m, M$, where $\alpha_p := \frac{M^p - m^p}{M - m}$ and $\beta_p := \frac{Mm^p - mM^p}{M - m}$. Then it follows that

$$\max_{m \leq t \leq M} \frac{\alpha_p t + \beta_p}{t^p} = K(h, p).$$

Here we treat the Specht ratio $S(h) = \frac{h^{\frac{1}{h-1}}}{e^{\log h \frac{1}{h-1}}}$ [22] which is the best constant of the reverse arithmetic-geometric mean inequality. It has the following property (e.g. [3], [8]) related to the generalized Kantorovich function $K(h, p)$:

$$(K-10) \quad \lim_{p \rightarrow 1} \frac{\partial}{\partial p} \log K(h, p) = \lim_{p \rightarrow 1} \frac{\frac{\partial}{\partial p} K(h, p)}{K(h, p)} = \lim_{p \rightarrow 1} \frac{\partial}{\partial p} K(h, p) = \log S(h).$$

We mention some important properties of the generalized Kantorovich function $K(h, p)$ as our main result:

THEOREM 3.1. *Let $h > 0$ and $p \in \mathbb{R}$. The generalized Kantorovich function $K(h, p)$ has the following properties:*

(K-11) $\log K(h, p)$ is a convex function for $p \in \mathbb{R}$.

Consequently,

(K-12) $K(h, p)$ is a convex function for $p \in \mathbb{R}$.

Proof. (K-11) From the properties (K-1), (K-2), (K-3) and (K-4), we may show the properties (K-11) (and (K-12)) for the case of $h \geq 1$ and $p \geq \frac{1}{2}$. In particular, we only prove for the case $p > 1$. The case $\frac{1}{2} \leq p < 1$ is given by a similar method. Here we remark that $\lim_{p \rightarrow 1} \frac{\partial}{\partial p} \log K(h, p)$ exists by (K-10).

The generalized Kantorovich function is represented as follows:

$$K(h, p) = \frac{(p - 1)^{p-1}}{p^p} \frac{(h^p - 1)^p}{(h^p - h)^{p-1}(h - 1)},$$

and so

$$\begin{aligned} \log K(h, p) &= \log(h^p - h) - \log(h - 1) - \log(p - 1) \\ &\quad + p(\log(h^p - 1) - \log(h^p - h) + \log(p - 1) - \log p). \end{aligned}$$

Moreover, we have

$$\begin{aligned} \frac{\partial}{\partial p} \log K(h, p) &= \left(\frac{(1-p)h^p}{h^p - h} + \frac{ph^p}{h^p - 1} \right) \log h + \log \frac{(p-1)(h^p - 1)}{p(h^p - h)} \quad (3.3) \\ &= \left(\frac{p}{1 - h^{-p}} - \frac{p-1}{1 - h^{-(p-1)}} \right) \log h + \log \frac{(p-1)(1 - h^{-p})}{p(1 - h^{-(p-1)})}. \end{aligned}$$

By (2) and (3) in Lemma 2.2, the function $\frac{\partial}{\partial p} \log K(h, p)$ is strictly monotone increasing, and so we hold the property (K-11).

(K-12) This property is satisfied by (K-11). \square

Kian et al. obtained the following result [14, Lemma 3.8] by using the property (K-12):

LEMMA KMS. *Let $h \geq 1$. Then the generalized Kantorovich function $K(h, p)$ has the following property:*

$$K(h, -p) \leq K(h, -1)^p \quad \text{for } p \in (0, 1).$$

If $p \notin (0, 1)$, then the reverse inequality of above holds.

The above lemma is equivalent to the following result: For $p \leq 1$

$$K(h, -p)^{-\frac{1}{p}} \geq K(h, -1)^{-1}.$$

If $p \geq 1$, then the reverse inequality of above holds.

From this view point, we improve it as follows:

COROLLARY 3.2. *Let $h > 0$. The generalized Kantorovich function $K(h, p)$ has the following monotone property:*

(K-13) $K(h, p)^{\frac{1}{p}}$ is strictly increasing for $p \in \mathbb{R}$.

Proof. First of all, we have

$$\frac{\partial}{\partial p} \log K(h, p)^{\frac{1}{p}} = \frac{\partial}{\partial p} \frac{\log K(h, p)}{p} = \frac{p \frac{\partial}{\partial p} \log K(h, p) - \log K(h, p)}{p^2}. \tag{3.4}$$

Next, we consider the tangent line $\ell(p)$ of $\log K(h, p)$ at any $p = p_0 \in \mathbb{R}$. It is represented as follows:

$$\ell(p) = \left. \frac{\partial}{\partial p} \log K(h, p) \right|_{p=p_0} (p - p_0) + \log K(h, p_0).$$

By (K-11), we have

$$\log K(h, p) \geq \ell(p) = \left. \frac{\partial}{\partial p} \log K(h, p) \right|_{p=p_0} (p - p_0) + \log K(h, p_0). \tag{3.5}$$

If $p = 0$, then the inequation (3.5) implies

$$0 \geq -p_0 \left. \frac{\partial}{\partial p} \log K(h, p) \right|_{p=p_0} + \log K(h, p_0).$$

So the equation (3.4) is positive, and hence we have the property (K-13). \square

As another application, we have the following corollary:

COROLLARY 3.3. *The generalized Kantorovich function $K(h, p)$ for $p \in \mathbb{R}$ has the following properties:*

(K-14) *For a fixed $h > 0$, the following equation holds:*

$$\lim_{p \rightarrow \infty} \frac{\partial}{\partial p} \log K(h, p) = \lim_{p \rightarrow \infty} \frac{\frac{\partial}{\partial p} K(h, p)}{K(h, p)} = \log h.$$

Moreover, $\left| \frac{\partial}{\partial p} \log K(h, p) \right| < |\log h|$. Consequently, there is a unique solution $p = p_0 \in \mathbb{R}$ such that $\left. \frac{\partial}{\partial p} K(h, p) \right|_{p=p_0} = \log h_0$ for any $h_0 \in I_h$, where I_h is the open interval determined by $\frac{1}{h}$ and h .

(K-15) *Let $h \geq 1$ and $h_0 > 0$. Then the equation $K(h, p) = h_0^{p-1}$ has the following solutions $p \in \mathbb{R}$:*

$$p := \begin{cases} 1, p_0 \in (-\infty, 1) & \text{if } h^{-1} < h_0 < S(h) \\ 1, p_0 \in (1, \infty) & \text{if } S(h) < h_0 < h \\ 1 & \text{otherwise.} \end{cases}$$

Moreover, suppose that $h_0 \in (h^{-1}, h) \setminus \{S(h)\}$. Let I_0 be the closed interval determined by 1 and p_0 with $K(h, p_0) = h_0^{p_0-1}$. Then the following inequality holds

$$K(h, p) - h_0^{p-1} \begin{cases} \leq 0 & \text{if } p \in I_0 \\ \geq 0 & \text{otherwise.} \end{cases}$$

Proof. (K-14) By (3.3) and $\lim_{p \rightarrow \infty} ph^{-p} = 0$, we have

$$\begin{aligned} \lim_{p \rightarrow \infty} \frac{\partial}{\partial p} \log K(h, p) &= \lim_{p \rightarrow \infty} \left\{ \left(\frac{p}{1-h^{-p}} - \frac{p-1}{1-h^{-(p-1)}} \right) \log h + \log \frac{(p-1)(1-h^{-p})}{p(1-h^{-(p-1)})} \right\} \\ &= \lim_{p \rightarrow \infty} \frac{(1-h) \cdot ph^{-p} - h^{-p} + 1}{(1-h^{-p})(1-h^{-(p-1)})} \log h \\ &= \log h. \end{aligned}$$

(K-15) We consider

$$\log K(h, p) = (p-1) \log h_0 \tag{3.6}$$

instead of $K(h, p) = h_0^{p-1}$. Then the equation (3.6) has a solution $p = 1$.

We consider the tangent line $\ell(p)$ at $p = 1$ with respect to the function $\log K(h, p)$ for p . Then we have $\ell(p) = (p-1) \log S(h)$ by (K-3) and (K-10).

Let $h^{-1} < h_0 < h$. Then the equation (3.6) has only the following solution p where

$$p := \begin{cases} 1, p_0 \in (-\infty, 1) & \text{if } h^{-1} < h_0 < S(h) \\ 1, p_0 \in (1, \infty) & \text{if } S(h) < h_0 < h \\ 1 & \text{if } h_0 = S(h) \end{cases}$$

by (K-2), (K-10), (K-11) and (K-14).

Next, let $h_0 \leq h^{-1}$ or $h \leq h_0$. Then we see that the equation (3.6) has a solution $p = 1$ only. \square

4. Concluding remarks

In Theorem 3.1, we see that $K(h, p)$ and $\log K(h, p)$ are convex functions for $p \in \mathbb{R}$. In this section, we give some remarks of the function $K(h, p)$ for $h > 0$.

It is known that $K(h, 2)$ is convex for $h > 1$ as in [1]. But, $\log K(h, 2)$ is not a convex function for $h > 1$. As a matter of fact, since

$$\log K\left(\frac{2+4}{2}, 2\right) = \log \frac{4}{3} \quad \text{and} \quad \frac{\log K(2, 2) + \log K(4, 2)}{2} = \frac{\log \frac{9}{8} + \log \frac{25}{16}}{2} = \log \frac{15}{8\sqrt{2}},$$

we have $\log K\left(\frac{2+4}{2}, 2\right) > \frac{\log K(2, 2) + \log K(4, 2)}{2}$, i.e., $\log K(h, 2)$ is not a convex function.

In addition, that there exists $p = p_0 \in \mathbb{R}$ such that $K(h, p_0)$ is not convex for $h > 1$. Indeed, since

$$K\left(\frac{3+7}{2}, \frac{3}{2}\right) \doteq 1.25726 \quad \text{and} \quad \frac{K\left(3, \frac{3}{2}\right) + K\left(7, \frac{3}{2}\right)}{2} \doteq \frac{1.11626 + 1.38604}{2} = 1.25115,$$

we have $K\left(\frac{3+7}{2}, \frac{3}{2}\right) > \frac{K\left(3, \frac{3}{2}\right) + K\left(7, \frac{3}{2}\right)}{2}$.

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