

REFINEMENTS OF TWO DETERMINANTAL INEQUALITIES FOR POSITIVE SEMIDEFINITE MATRICES

YAN HONG AND FENG QI*

Dedicated to retired Professor Junesang Choi at Dongguk University in South Korea

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Abstract. Let $A, B, C \in \mathbb{C}^{n \times n}$ be positive semidefinite matrices and let $|A|, |B|, |C|$ be determinants of $A, B, C \in \mathbb{C}^{n \times n}$ respectively. In this paper, the authors prove two determinantal inequalities

$$|A + B + C| + |C| \geq |A + C| + |B + C| + (2^n - 2)|AB|^{1/2} + 3(3^{n-1} - 2^n + 1)|ABC|^{1/3}$$

and

$$|A + B + C| + |A| + |B| + |C| \geq |A + B| + |A + C| + |B + C| + 3(3^{n-1} - 2^n + 1)|ABC|^{1/3}.$$

These two inequalities refine known ones.

1. Introduction

Let $A \in \mathbb{C}^{n \times n}$ be a square matrix of order n . In this paper, we will denote the determinant of the matrix A by $|A|$ and denote eigenvalues of the matrix A by

$$\lambda_1(A), \lambda_2(A), \dots, \lambda_n(A).$$

If eigenvalues of the matrix A are real numbers, we specify

$$\lambda_1(A) \geq \lambda_2(A) \geq \dots \geq \lambda_n(A).$$

Let $A, B \in \mathbb{C}^{n \times n}$ be positive semidefinite matrices. In [1, p. 465, Corollary], Hartfiel obtained the determinantal inequality

$$|A + B| \geq |A| + |B| + (2^n - 2)(|A||B|)^{1/2}. \quad (1)$$

In [5, p. 215], Zhang established a determinantal inequality below.

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* Corresponding author.

THEOREM 1. ([5, p. 215]) *Let $A, B, C \in \mathbb{C}^{n \times n}$ be positive semidefinite matrices. Then*

$$|A + B + C| + |C| \geq |A + C| + |B + C|. \tag{2}$$

In [3], Lin established the following determinantal inequality.

THEOREM 2. ([3, Theorem 1.1]) *Let $A, B, C \in \mathbb{C}^{n \times n}$ be positive semidefinite matrices. Then*

$$|A + B + C| + |A| + |B| + |C| \geq |A + B| + |A + C| + |B + C|. \tag{3}$$

In this paper, we will refine determinantal inequalities (2) and (3) above.

2. Lemmas

To refine determinantal inequalities (2) and (3) in Theorems 1 and 2, we need the following lemmas.

LEMMA 1. ([4, p. 333]) *Let $A, B \in \mathbb{C}^{n \times n}$ be positive semidefinite matrices. Then*

$$\prod_{i=1}^n [\lambda_i(A) + \lambda_{n-i+1}(B)] \geq |A + B| \geq \prod_{i=1}^n [\lambda_i(A) + \lambda_i(B)].$$

By the proof of Theorem 1.1 in [3], we have the following inequality.

LEMMA 2. ([3, Theorem 1.1]) *Let $A, B \in \mathbb{C}^{n \times n}$ be positive semidefinite matrices. Then*

$$\prod_{i=1}^n \lambda_i(A + B + I_n) - \prod_{i=1}^n \lambda_i(A + B) \geq \prod_{i=1}^n [\lambda_i(A) + \lambda_i(B) + 1] - \prod_{i=1}^n [\lambda_i(A) + \lambda_i(B)].$$

LEMMA 3. *Let $a_i, b_i, c_i \geq 0$ for $i \in \mathbb{N}$. Then*

$$\begin{aligned} \prod_{i=1}^n (a_i + b_i + c_i) + \prod_{i=1}^n a_i + \prod_{i=1}^n b_i + \prod_{i=1}^n c_i - \prod_{i=1}^n (a_i + b_i) - \prod_{i=1}^n (a_i + c_i) - \prod_{i=1}^n (b_i + c_i) \\ \geq 3(3^{n-1} - 2^n + 1) \prod_{i=1}^n (a_i b_i c_i)^{1/3}. \end{aligned} \tag{4}$$

Proof. If $n = 1, 2$, the inequality (4) obviously holds. If $n \geq 3$, let $i_1, \dots, i_k, j_1, \dots, j_t, \ell_1, \dots, \ell_s$ be non-negative positive numbers such that

$$\{i_1, \dots, i_k, j_1, \dots, j_t, \ell_1, \dots, \ell_s\} \setminus \{0\} = \{1, 2, \dots, n\}.$$

Since

$$\prod_{i=1}^n (a_i + b_i + c_i) = \sum_{\substack{0 \leq i_1 < \dots < i_k, 0 \leq j_1 < \dots < j_t, \\ 0 \leq \ell_1 < \dots < \ell_s, i_k + j_t + \ell_s = n}} \prod_{m=1}^k a_{i_m} \prod_{p=1}^t b_{j_p} \prod_{q=1}^s c_{\ell_q},$$

by virtue of the inequality between the geometric and arithmetic means, we acquire

$$\begin{aligned} & \prod_{i=1}^n (a_i + b_i + c_i) + \prod_{i=1}^n a_i + \prod_{i=1}^n b_i + \prod_{i=1}^n c_i - \prod_{i=1}^n (a_i + b_i) - \prod_{i=1}^n (a_i + c_i) - \prod_{i=1}^n (b_i + c_i) \\ &= \sum_{\substack{1 \leq i_1 < \dots < i_k, 1 \leq j_1 < \dots < j_t, \\ 1 \leq \ell_1 < \dots < \ell_s, i_k + j_t + \ell_s = n}} \prod_{m=1}^k a_{i_m} \prod_{p=1}^t b_{j_p} \prod_{q=1}^s c_{\ell_q} \\ &\geq 3(3^{n-1} - 2^n + 1) \prod_{i=1}^n (a_i b_i c_i)^{\frac{3^{n-1} - 2^n + 1}{3(3^{n-1} - 2^n + 1)}} \\ &= 3(3^{n-1} - 2^n + 1) \prod_{i=1}^n (a_i b_i c_i)^{1/3}. \end{aligned}$$

The proof of Lemma 3 is complete. \square

REMARK 1. Taking $c_i = 1$ for $i = 1, 2, \dots, n$ in Lemma 3 results in

$$\begin{aligned} & \prod_{i=1}^n (a_i + b_i + 1) + \prod_{i=1}^n a_i + \prod_{i=1}^n b_i + 1 - \prod_{i=1}^n (a_i + b_i) - \prod_{i=1}^n (a_i + 1) - \prod_{i=1}^n (b_i + 1) \\ & \geq 3(3^{n-1} - 2^n + 1) \prod_{i=1}^n (a_i b_i)^{1/3}, \quad (5) \end{aligned}$$

where $a_i, b_i \geq 0$ for $i \in \mathbb{N}$.

LEMMA 4. Let $a_i, b_i, c_i \geq 0$ for $i \in \mathbb{N}$. Then

$$\begin{aligned} & \prod_{i=1}^n (a_i + b_i + c_i) + \prod_{i=1}^n c_i - \prod_{i=1}^n (a_i + c_i) - \prod_{i=1}^n (b_i + c_i) \\ & \geq (2^n - 2) \prod_{i=1}^n (a_i b_i)^{1/2} + 3(3^{n-1} - 2^n + 1) \prod_{i=1}^n (a_i b_i c_i)^{1/3}. \quad (6) \end{aligned}$$

Proof. Using the inequality (1), we obtain

$$\prod_{i=1}^n (a_i + b_i) - \prod_{i=1}^n a_i - \prod_{i=1}^n b_i \geq (2^n - 2) \prod_{i=1}^n (a_i b_i)^{1/2}.$$

Therefore, by the inequality (4), we arrive at

$$\begin{aligned} & \prod_{i=1}^n (a_i + b_i + c_i) + \prod_{i=1}^n c_i - \prod_{i=1}^n (a_i + c_i) - \prod_{i=1}^n (b_i + c_i) \\ & \geq \prod_{i=1}^n (a_i + b_i) - \prod_{i=1}^n a_i - \prod_{i=1}^n b_i + 3(3^{n-1} - 2^n + 1) \prod_{i=1}^n (a_i b_i c_i)^{1/3} \\ & \geq (2^n - 2) \prod_{i=1}^n (a_i b_i)^{1/2} + 3(3^{n-1} - 2^n + 1) \prod_{i=1}^n (a_i b_i c_i)^{1/3}. \end{aligned}$$

The proof of Lemma 4 is complete. \square

REMARK 2. Setting $c_i = 1$ for $i = 1, 2, \dots, n$ in the inequality (6) in Lemma 4 leads to

$$\prod_{i=1}^n (a_i + b_i + 1) + 1 - \prod_{i=1}^n (a_i + 1) - \prod_{i=1}^n (b_i + 1) \geq (2^n - 2) \prod_{i=1}^n (a_i b_i)^{1/2} + 3(3^{n-1} - 2^n + 1) \prod_{i=1}^n (a_i b_i)^{1/3}, \quad (7)$$

where $a_i, b_i \geq 0$ for $i \in \mathbb{N}$.

REMARK 3. From the inequality (7), it is easy to see that,

1. when $\prod_{i=1}^n a_i b_i \geq 1$, we have

$$\prod_{i=1}^n (a_i + b_i + 1) + 1 - \prod_{i=1}^n (a_i + 1) - \prod_{i=1}^n (b_i + 1) \geq (3^n - 2^{n+1} + 1) \prod_{i=1}^n (a_i b_i)^{1/3}.$$

2. when $0 \leq \prod_{i=1}^n a_i b_i \leq 1$, we have

$$\prod_{i=1}^n (a_i + b_i + 1) + 1 - \prod_{i=1}^n (a_i + 1) - \prod_{i=1}^n (b_i + 1) \geq (3^n - 2^{n+1} + 1) \prod_{i=1}^n (a_i b_i)^{1/2}.$$

3. Refinements of two determinantal inequalities

In this section, we refine determinantal inequalities (2) and (3).

THEOREM 3. Let $A, B, C \in \mathbb{C}^{n \times n}$ be positive semidefinite matrices. Then

$$|A + B + C| + |A| + |B| + |C| \geq |A + B| + |A + C| + |B + C| + 3(3^{n-1} - 2^n + 1) |ABC|^{1/3}. \quad (8)$$

Proof. If $|C| = 0$, the inequality (8) evidently holds.

If $|C| \neq 0$, putting

$$A_1 = C^{-1/2} A C^{-1/2} \quad \text{and} \quad B_1 = C^{-1/2} B C^{-1/2}.$$

Using Lemma 2 and the inequality (5), we deduce

$$\begin{aligned}
 & (|A+B+C| + |A| + |B| + |C| - |A+B| - |A+C| - |B+C|)|C|^{-1} \\
 &= |A_1 + B_1 + I_n| + |A_1| + |B_1| + 1 - |A_1 + B_1| - |A_1 + I_n| - |B_1 + I_n| \\
 &= \prod_{i=1}^n \lambda_i(A_1 + B_1 + I_n) + \prod_{i=1}^n \lambda_i(A_1) + \prod_{i=1}^n \lambda_i(B_1) + 1 \\
 &\quad - \prod_{i=1}^n \lambda_i(A_1 + B_1) - \prod_{i=1}^n \lambda_i(A_1 + I_n) - \prod_{i=1}^n \lambda_i(B_1 + I_n) \\
 &= \prod_{i=1}^n [\lambda_i(A_1 + B_1) + 1] + \prod_{i=1}^n \lambda_i(A_1) + \prod_{i=1}^n \lambda_i(B_1) + 1 \\
 &\quad - \prod_{i=1}^n [\lambda_i(A_1 + B_1)] - \prod_{i=1}^n [\lambda_i(A_1) + 1] - \prod_{i=1}^n [\lambda_i(B_1) + 1] \\
 &\geq \prod_{i=1}^n [\lambda_i(A_1) + \lambda_i(B_1) + 1] + \prod_{i=1}^n \lambda_i(A_1) + \prod_{i=1}^n \lambda_i(B_1) + 1 \\
 &\quad - \prod_{i=1}^n [\lambda_i(A_1) + \lambda_i(B_1)] - \prod_{i=1}^n [\lambda_i(A_1) + 1] - \prod_{i=1}^n [\lambda_i(B_1) + 1] \\
 &\geq 3(3^{n-1} - 2^n + 1)|ABC|^{1/3}|C|^{-1}.
 \end{aligned}$$

The proof of Theorem 3 is complete. \square

THEOREM 4. *Let $A, B, C \in \mathbb{C}^{n \times n}$ be positive semidefinite matrices. Then*

$$|A+B+C| + |C| \geq |A+C| + |B+C| + (2^n - 2)|AB|^{1/2} + 3(3^{n-1} - 2^n + 1)|ABC|^{1/3}. \tag{9}$$

Proof. By the inequality (8) and the inequality (1), we obtain

$$\begin{aligned}
 & |A+B+C| + |C| - |A+C| - |B+C| \\
 &\geq |A+B| - |A| - |B| + 3(3^{n-1} - 2^n + 1)|ABC|^{1/3} \\
 &\geq (2^n - 2)|AB|^{1/2} + 3(3^{n-1} - 2^n + 1)|ABC|^{1/3}.
 \end{aligned}$$

The proof of Theorem 4 is complete. \square

REMARK 4. Theorem 4 can be proved by Lemma 1 and the inequality (7), as done in the proof of Theorem 3.

REMARK 5. This paper is a corrected and revised version of the preprint [2].

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Yan Hong
College of Mathematics and Physics
Inner Mongolia Minzu University
Tongliao 028043, Inner Mongolia, China
e-mail: nmhongyan@163.com
<https://orcid.org/0000-0002-9916-9097>

Feng Qi
Institute of Mathematics
Henan Polytechnic University
Jiaozuo 454003, Henan, China
and
School of Mathematical Sciences
Tiangong University
Tianjin 300387, China
e-mail: qifeng618@gmail.com
qifeng618@hotmail.com
qifeng618@qq.com
<https://qifeng618.wordpress.com>
<https://orcid.org/0000-0001-6239-2968>